# Edge Connectivity Problems in Telecommunication Networks

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**Abstract**. If a communication network $N$ with $n$ stations has every station linked with at least $\lfloor n/2 \rfloor$ other stations, then the edge-connectivity of $N$ equals its minimum degree. Also, in general, this limitation is stated to be the best possibility, as was proved by Chartrand in 1966. A more developed notion of edge-connectivity is introduced, which is called $k$-component order edge-connectivity. It is the minimum number of edges required to be removed so that the order of each disconnected component is less than $k$.

**Keywords**: *edge-connectivity; k-component edge-failure set; k-component edge-failure state; k-component order edge-connectivity; minimum degree*

## 1 Introduction

Any network, be it a ring fiber optic network, satellite communication network, terrestrial microwave network, or social relationships network, can be modeled using graph-theoretic methods. Network vulnerability is an important consideration in network design.

The utilization of communication networks has grown tremendously in the last decade, for example for transmitting voice, data, and images around the world. With the widespread dependence upon such networks, it becomes important to find networks that yield a high level of reliability and a low level of vulnerability to disruption.

It is desirable to consider the quantitative measures of a network's vulnerability. To obtain such measures, we can model the network by a graph in which the station terminals are represented by the nodes of the graph and the links are represented by the edges.

An important measure of a network's vulnerability is its edge-connectivity, the minimum number of edges whose removal from the network disconnects it into two or more components. We assume that telecommunication networks have

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stations that are perfectly reliable, but links that could fail in accordance with some known probabilistic model or due to purposeful attack.

In the classical model, a network is considered to be operating if every station can communicate with every other station through some path connecting stations and operating links. If any failure of a link results in a pair of stations no longer being able to communicate, then the network has failed. In this case, we say that the network has become disconnected.

The traditional vulnerability concept for this model is the minimum number of links whose failure results in a disconnected network. Therefore, if all links originating from a particular station fail, then that station is unable to communicate with other stations and the network has become disconnected. Equivalently, the value of the traditional edge-connectivity is at most the minimum degree of any station. In 1966, Chartrand [1] proved that if each station is linked to at least half of the other stations, the network cannot be disconnected if fewer than a minimum number of links fail, i.e. the network is invulnerable to failure if fewer than the minimum amount of links to any station fail.

Under the supervision of Frank Boesch, together with Charles Suffel, Daniel Gross, John Saccoman, and L.W. Kazmierczak, I had the opportunity to study a new network vulnerability model called *k*-component order edge-connectivity in which a network is considered operating as long as there is a predetermined number of stations, say *k* that can still communicate regardless of whether the network is connected [2],[3].

We introduced a new concept called *k*-component order edge-connectivity as a new vulnerability parameter, which is the minimum number of links whose failure results in a disconnected graph and the number of stations in each subnetwork containing less than the predetermined number *k*. Hence the value of the new vulnerability model of *k*-component order edge-connectivity depends on *k*, the minimum number of stations that is needed to communicate. It is clear that if *k* increases, then the value of parameter *k*-component order edge-connectivity decreases, i.e. fewer links need to fail in order for the network to fail.

When we need all stations to communicate, this *k*-component order edge-connectivity equals the edge-connectivity of the traditional parameter. Also, the failure of all the links to one station results in a disconnected network only if we need all stations to communicate. Thus, the value of *k*-component order edge-connectivity can be greater than the minimum degree of links to any station. In this paper, we study the relationship between the minimum number of links to any station, edge-connectivity, and *k*-component order edge-connectivity parameters [4].

## 2 Vulnerability Models

### 2.1 Traditional Edge-Connectivity $\lambda(G)$

Given a connected communication network *N* with *n* stations, $n \geq 2$, we can make *N* into a disconnected network by removing certain links between stations in *N*. The interest in this problem is usually maximized by minimizing the number of links whose removal will disconnect *N*. Chartrand [1] presents the result of a disconnected network by those links leading to the station having the fewest links. In particular, he proved the following theorem, where $\lfloor x \rfloor$ denotes the largest integer not exceeding *x*.

**Theorem 2.1** [1]. If a communications network *N* with *n* stations has every station linked with at least $\lfloor n/2 \rfloor$ other stations, then the minimum number of links whose removal will disconnect *N* is equal to the least number of links to any station in *N*. Also, the number $\lfloor n/2 \rfloor$ cannot, in general, be improved.

With every communication network *N* an ordinary graph *G* is associated, whose set *V* of nodes corresponds to the stations of *N* and whose set *E* of edges corresponds to the links in *N*.

Without a doubt, the problem of communications under discussion is equivalent to determining the minimum number of edges whose removal will disconnect the associated network. This leads to two definitions.

**Definition 1** [1]. A connected graph *G* is *m*-edge connected if the removal of any *k* edges from *G*, $0 \leq k < m$, results in a connected graph. A disconnected graph is defined to be 0-edge connected.

**Definition 2** [1]. The maximum value of *m* for which a graph *G* is *m*-edge connected is referred to as the *edge-connectivity* of *G* and is denoted by $\lambda(G)$. It follows immediately that the edge-connectivity of a graph is the minimal number of edges whose removal disconnects the graph.

**Theorem 2.2** [4]. Let $G$ be a connected graph of order *n*. If $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$ then $\lambda(G) = \delta(G)$. Furthermore, there possibly exists a connected graph $G'$ of order $n \geq 6$ with $\delta(G') < \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $\lambda(G') < \delta(G')$.

This theorem is illustrated by graphs $G_1$ and $G_2$ in Figure 1 [4],[5].

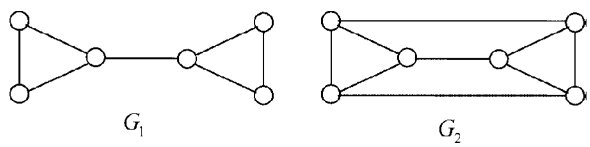


**Figure 1** Graphs $G_1$ and $G_2$ with $\lambda(G_1) < \delta(G_1)$ and $\lambda(G_2) = \delta(G_2) = \left\lfloor \frac{n}{2} \right\rfloor$.

## 2.2 *k*-Component Order Edge-Connectivity $\lambda_c^{(k)}(G)$

It is reasonable to consider a model of network in which it is not necessary that the surviving edges form a connected subgraph as long as they form a subgraph with a component of some predetermined order. Thus we introduce a new edge-failure model, the *k*-component order edge-failure model. In this model, when a set of edges $F$ fail, we refer to $F$ as a *k*-component edge-failure set and the surviving subgraph $G - F$ as a *k*-component edge-failure state if $G$ - $F$ contains no component of order at least *k*, where *k* is a predetermined threshold value.

**Definition 3** [3],[4]. Let $2 \leq k \leq n$ be a predetermined threshold value. The ***k*-component order edge-connectivity** or **component order edge connectivity** of $G$, denoted by $\lambda_c^{(k)}(G)$ or simply $\lambda_c^{(k)}$, is defined to be $\lambda_c^{(k)}(G) = \min\{|F| : F \subseteq E, F$ is *k*-component edge-failure set$\}$, i.e. all components of $G - F$ have order $\leq k-1$.

**Definition 4** [3],[4]. A set of edges $F$ of graph $G$ is $\lambda_c^{(k)}$ – edge set if and only if it is a *k*-component order edge-failure set and $|F| = \lambda_c^{(k)}$.

We give two examples for showing the above definition of this parameter in a cycle graph, $G(C_n)$, and a star graph, $G(K_{1,n-1})$, with $n$ nodes.

**Example 1** [1],[4],[5]. $\lambda_c^{(k)}(C_n)$: We start with removing one edge, then it becomes a path with $n$ nodes with $n$ – 1 edges. Let $F$ be the set of edges of the new graph (or a path) which is divisible by $k$ – 1. It will be an easy exercise to see that each component of $C_n - (F+1)$ has order no more than $k$ – 1 and $|F|$ is minimum. Therefore, $\lambda_c^{(k)}(C_n) = \left\lfloor \frac{n-1}{k-1} \right\rfloor + 1 = \left\lceil \frac{n}{k-1} \right\rceil$.

**Example 2** [1],[4],[5]. $\lambda_c^{(k)}(K_{1,n-1})$: Deletion of any set of $m$ edges results in a subgraph consisting of $m$+1 components, one isomorphic to $K_{1,n-m-1}$ and the remaining components isolated nodes. Therefore a $k$-component edge-failure state exists if the component $K_{1,n-m-1}$ contains at most $k$ – 1 nodes. Thus $n - m \le k - 1$ or $n - k + 1 \le m$. Since component order edge connectivity is the minimum number of edges whose removal results in a $k$-component edge-failure state, we obtain the following result: $\lambda_c^{(k)}(K_{1,n-1}) = n - k + 1$.

## 3 Bounds on Edge- and *k*-Component Order Edge-Connectivity

If we remove all edge incidents on a single node, this creates a $k$-component order edge-failure state only when $k = n$; therefore, if $k < n$ we cannot conclude that $\lambda_c^{(k)}(G) \le \delta(G)$, such that it may possible that

$$\lambda_c^{(k)}(G) < \delta(G), \lambda_c^{(k)} = \delta(G), \text{ or } \lambda_c^{(k)}(G) > \delta(G).$$

We now consider the graph $G_1$ from Figure 1,

$$\lambda_c^{(6)}(G_1) = \lambda_c^{(5)}(G_1) = \lambda_c^{(4)} = 1 < \delta(G_1), \ \lambda_c^{(3)}(G_1) > \delta(G_1),$$

$$\text{and } \lambda_c^{(2)}(G_1) = 7 > \delta(G_1).$$

Hence, it is a fact that if $G - F$ is a $k$-component order edge-failure state, then it is also a edge-failure state; therefore $\lambda \le \lambda_c^{(k)}$ for every $k$.

**Theorem 3.1** [4],[5]. For any graph $G$, $\lambda = \lambda_c^{(n)} \le \lambda_c^{(n-1)} \le \lambda_c^{(n-2)} \le \cdots \le \lambda_c^{(2)} = e$.

Since $\lambda \le \delta \le e$, we will find $k$ that fits into the string of this inequality, such that $\lambda_c^{(k+1)} < \delta \le \lambda_c^{(k)}$.

## 4 Preliminary Result

We will present two lemmas required for our main results. The first lemma establishes a lower bound for $\lambda_c^{(k)}(G)$, where $G$ is a connected graph of order $n$.

**Lemma 4.1** [4]. Given $G$ be a connected graph of order $n$ and let $2 \le k \le n$. Then $\lambda_c^{(k)}(G) \ge \left\lfloor \frac{n-1}{k-1} \right\rfloor$.

**Proof**. Assume $F \subseteq E$ is $k$-component edge-failure set and $G - F = \bigcup_{i=1}^{p} C_i$, where $C_1, C_2, \cdots, C_p$ are the component subgraphs of $G - F$, each of order no more than $k - 1$. Thus $n = \sum_{i=1}^{p} order(C_i) \le p(k-1)$, which implies $p \ge \left\lceil \frac{n}{k-1} \right\rceil$. Upon $G$ is connected, if $|F| = \lambda_c^{(k)}(G)$ then $\lambda_c^{(k)}(G) \ge p - 1 \left\lceil \frac{n}{k-1} \right\rceil - 1 = \left\lfloor \frac{n-1}{k-1} \right\rfloor$.

We conclude that the proof of the above lemma also establishes the fact that any $k$-component order edge-failure state contains at least $\left\lceil \frac{n}{k-1} \right\rceil$ components.

The second lemma gives a sufficient condition on $\delta(G)$ for the conclusion $\lambda_c^{(k)}(G) \geq \delta(G)$.

**Lemma 4.2** [5]. Given $G$ be a connected graph of order $n$ and let $2 \leq k \leq n$. If $\left\lceil \frac{n}{k-1} \right\rceil \left(\delta(G)+1\right) > n$, then $\lambda_c^{(k)}(G) \geq \delta(G)$.

**Proof**. Assume $F \subseteq E$ is a $k$-component edge-failure set and $G - F = \cup_{i=1}^{p} C_i$, where $C_1, C_2, \cdots, C_p$ are the component subgraphs of $G - F$, each of order no more than $k - 1$.

If $\lambda_c^{(k)}(G) < \delta(G)$, then $order(C_i) \geq \delta(G)+1$ for each $i$. For this, we consider an arbitrary component $C_i$ and let $u$ be a node of $C_i$. Since the degree ($u$) $\geq \delta(G) \geq \lambda_c^{(k)} = |F|$, $u$ must be adjacent to at least one additional node of $C_i$; thus order $C_i \geq 2$. Using the nodes in $C_i$, produces inequality

$$order(C_i)\left(order(C_i)-1\right) + \lambda_c^{(k)}(G) \geq order(C_i)\delta(G)\text{, which implies}$$

$$order(C_i)\left(order(C_i)-1\right) + \delta(G) \geq order(C_i)\delta(G)$$

Hence,

$$order(C_i)\left(order(C_i)-1\right) > \delta(G)\left(order(C_i)-1\right)$$

Dividing both sides by $\left(order(C_i)-1\right)$ yields the result. Finally, if $\lambda_c^{(k)} < \delta(G)$, then

$$n = \sum_{i=1}^{p} order(C_i) \geq p\left(\delta(G)+1\right) \geq \left\lceil \frac{n}{k-1} \right\rceil \left(\delta(G)+1\right)$$

In Figure 1, consider graph $G_2$. If $k$ = 6, $\lambda_c^{(6)}(G_2) = 3 = \delta(G_2)$, but $\left\lceil \frac{n}{k-1} \right\rceil \left(\delta(G_2)+1\right) = \left\lceil \frac{6}{6-1} \right\rceil (3+1) = 8 > n.$

We conclude from the proof of lemmas 4.1 and 4.2 that two other sufficient conditions for $\lambda_c^{(k)} \ge \delta(G)$ are given in the following lemma without proof.

**Lemma 4.3** [5]. Given $G$ be a connected graph of order $n$ and let $2 \le k \le n$.

If $\left\lceil \frac{n}{k-1} \right\rceil - 1 \ge \delta(G),$ then $\lambda_c^{(k)}(G) \ge \delta(G).$

If $\delta \ge k-1,$ then $\lambda_c^{(k)}(G) \ge \delta(G).$

## 5 Main Results

Now we establish the basis theorem for determining the best possible solution $k$, such that there exists a connected graph $G$ of order $n$ where in such inequality of $\lambda_c^{(k+1)}(G) < \delta(G) \le \lambda_c^{(k)}(G).$

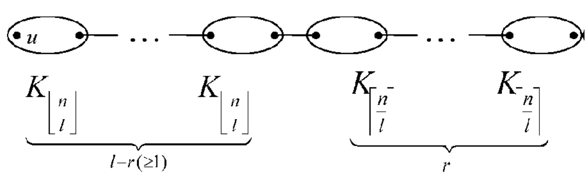


**Figure 2** A graph $G'$ with $\delta(G')$ and $\lambda_c^{\lceil n/l \rceil + 1}(G') < \delta(G') \le \lambda_c^{\lceil n/l \rceil}(G').$

**Theorem 5.1** [4,5]. Consider $G$ be a connected graph of order $n$. If $\delta(G) \ge \left\lfloor \frac{n}{l+1} \right\rfloor, 1 \le l \le n-1,$ then $\lambda_c^{\lceil n/l \rceil}(G) \ge \delta(G).$ Furthermore, if $n \ge l(l+1)$ this is the best possible solution in the sense that for all $\delta$ such that $\left\lfloor \frac{n}{l+1} \right\rfloor \le \delta \le \left\lfloor \frac{n}{l} \right\rfloor - 1,$

there exists a connected graph $G^{/}$ of order $n$ with $\delta(G^{/}) = \delta(G^{/})$ and $\lambda_c^{\left\lceil \frac{n}{l} \right\rceil + 1}(G^{/}) < \delta(G^{/})$.

**Proof**. We first show that if $1 \leq l \leq n-1$, then $\left\lceil \frac{n}{\left\lceil \frac{n}{l} \right\rceil - 1} \right\rceil \geq l+1$. If $l$ does not divide $n$, then $\left\lceil \frac{n}{l} \right\rceil - 1 = \left\lfloor \frac{n}{l} \right\rfloor$. But $n = \left\lfloor \frac{n}{l} \right\rfloor l + r, 0 < r \leq l-1$, so $\frac{n}{\left\lceil \frac{n}{l} \right\rceil - 1} = l + \frac{r}{\left\lfloor \frac{n}{l} \right\rfloor}$ and $\left\lceil \frac{n}{\left\lceil \frac{n}{l} \right\rceil - 1} \right\rceil \geq l+1.$ If $l$ divides $n$, then $n = \frac{n}{l} l = \left\lceil \frac{n}{l} \right\rceil l = \left( \left\lceil \frac{n}{l} \right\rceil - 1 \right) l + 1.$ Thus $\frac{n}{\left\lceil \frac{n}{l} \right\rceil - 1} = l + \frac{l}{\left\lfloor \frac{n}{l} \right\rfloor - 1}$ and as before $\left\lceil \frac{n}{\left\lceil \frac{n}{l} \right\rceil - 1} \right\rceil \geq l+1.$ Finally $\left\lceil \frac{n}{\left\lceil \frac{n}{l} \right\rceil - 1} \right\rceil (\delta + 1) \geq (l+1) \left( \left\lfloor \frac{n}{l+1} \right\rfloor + 1 \right) > n.$

The conclusion follows by applying Lemma 4.2 [4] with $k = \left\lceil \frac{n}{l} \right\rceil$. To demonstrate the best possible condition we construct graph $G^{/}$. Assume $n \geq l(l+1)$ and write $n = \left\lfloor \frac{n}{l} \right\rfloor l + r = \left\lfloor \frac{n}{l} \right\rfloor (l+r) + \left\lceil \frac{n}{l} \right\rceil r$, where $0 \leq r \leq l-1$. Let us start with $l$ distinct complete graphs, $l - r$ of order $\left\lfloor \frac{n}{l} \right\rfloor$ and $r$ of order $\left\lceil \frac{n}{l} \right\rceil$. These cliques are connected in a path-like manner as shown in Figure 2.

Finally, since $l \geq 1$ and $\left\lfloor \frac{n}{l} \right\rfloor \geq l + 1 \geq 2,$ there exists a node $u$ in a distinguished $K_{\left\lfloor \frac{n}{l} \right\rfloor}$ of degree $\left\lfloor \frac{n}{l} \right\rfloor - 1$ and edges are removed if necessary to obtain $\delta(G^{/}) = \delta.$ Since $n \geq l(l+1), \left\lfloor \frac{n}{l+1} \right\rfloor \geq l,$ we have $\left\lfloor \frac{n}{l+1} \right\rfloor \geq l.$

Thus,

$$\lambda_c^{\left(\left\lceil \frac{n}{l} \right\rceil + 1\right)}(G^{/}) = l - 1 < l \leq \left\lfloor \frac{n}{l+1} \right\rfloor \leq \delta .$$

**Corollary 5.1** [1]. Let $G$ be a connected graph of order $n$. If $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor,$ then $\lambda(G) \geq \delta(G).$

**Proof**. If we set $l$ = 1, then Theorem 5.1 becomes: Let $G$ be a connected graph on $n \geq 2$ nodes. If $\left\lfloor \frac{n}{2} \right\rfloor \leq \delta(G)$ then $\lambda_c^{(k)}(G) \geq \delta(G).$ Since $\lambda_c^{(k)}(G) = \delta(G),$ the conclusion follows.

Corollary 5.1 is the first part of Chartrand's Theorem (Theorem 2.2). Since any $\lambda_c^{\left\lceil \frac{n}{l} \right\rceil}(G) < \delta G),$ then $k = \left\lceil \frac{n}{l} \right\rceil$ is not the best possible condition. We need to do more work to discuss and further investigate application in telecommunication networks that mention the second part of Chartrand's Theorem.

Applications in telecommunication networks can take a sample of the terrestrial communications network connecting the various cities, which can be described in a graph as a network topology path $P_n(G).$ To provide reliability of local connections in every city, you need complete network topology $K_n(G).$

For example, $n$ = 10, and $l$ = 3 and 4. Then for $l$ = 3, the value of $\left\lfloor \frac{n}{l} \right\rfloor = \left\lfloor \frac{10}{3} \right\rfloor = 3,$ and $\left\lceil \frac{n}{l} \right\rceil = 4.$ Thus, for a number of nodes, $n$ = 10, the network topology for $l$ = 3 will consist two complete graphs of $K_3(G)$, and a complete graph of $K_4(G)$.

Similar to the value of $l$ = 4, the value - the value $\left\lfloor \frac{n}{l} \right\rfloor = \left\lfloor \frac{10}{4} \right\rfloor = 2$, and $\left\lceil \frac{n}{l} \right\rceil = 2$. Then, related to the number of nodes $n$ = 10, the shape network topology will consist of two complete graphs of $K_2(G)$, and two complete graphs of $K_3(G)$.

This example is illustrated in Figure 3 for the same number of nodes, $n$ = 10, and for different values of $l$, $l$ = 3 and $l$ = 4.

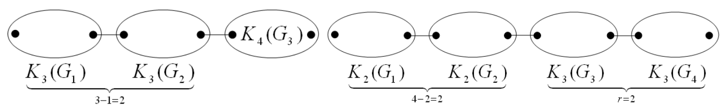


**Figure 3** (a) Path Graph $P_{10}(G)$ contains two $K_3(G)$ and one $K_4(G)$ with $\lambda_c^{(k)}(G) \geq \delta(G)$ for $2 \leq k \leq 5$ and $\lambda_c^{(k)}(G) < \delta(G)$ for $5 < k \leq n$. (b) Path Graph $P_{10}(G)$ contains two $K_2(G)$ and two $K_3(G)$ with $\lambda_c^{(k)}(G) \geq \delta(G)$ for $2 \leq k \leq 7$ and $\lambda_c^{(k)}(G) < \delta(G)$ for $7 < k \leq n$.

## 6 Conclusion

In 1966, Chartrand proved that if $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$, where $\delta(G)$ is the minimum degree of any node in the connected graph $G$ of order $n$, then $\lambda(G) = \delta(G)$, where $\lambda(G)$ is the edge connectivity, which is the minimum number of edges that must be removed in order to make the graph disconnected. We have demonstrated that the analogous result holds for $\lambda_c^{(k)}(G)$, which is the minimum number of edges that must be removed to disconnect the graph into components, each of order no more than $k$ – 1. Namely, for all connected graphs $G$ of order $n$ there exists a value of $k$ such that if $\delta(G)$ is sufficiently large then $\lambda_c^{(k)} \geq \delta(G)$. Moreover, the value of $k$ can be chosen such that $\lambda_c^{(k+1)} < \delta(G)$.

When $k = n$, $\lambda_c^{(k)}(G) = \lambda(G)$, and the lower bound on $\delta(G)$ is $\left\lfloor \frac{n}{2} \right\rfloor$; thus Chartrand's result follows from ours.

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