Boolean Algebra of C-Algebras

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Abstract. A C- algebra is the algebraic form of the 3-valued conditional logic, which was introduced by F. Guzman and C.C. Squier in 1990. In this paper, some equivalent conditions for a C- algebra to become a boolean algebra in terms of congruences are given. It is proved that the set of all central elements B(A) is isomorphic to the Boolean algebra $\mathfrak{B}_{S(A)}$ of all C-algebras S_a, where a $\in$ B(A). It is also proved that B(A) is isomorphic to the Boolean algebra $\mathfrak{B}_{R(A)}$ of all C-algebras A_a, where a $\in$ B(A).

Keywords: *Boolean algebra; C-algebra; central element; permutable congruences.*

1 Introduction

The concept of C-algebra was introduced by Guzman and Squier as the variety generated by the 3-element algebra C={T,F,U}. They proved that the only subdirectly irreducible C-algebras are either C or the 2-element Boolean algebra B ={T,F} [1,2].

For any universal algebra A, the set of all congruences on A (denoted by *Con A*) is a lattice with respect to set inclusion. We say that the congruences θ,ϕ are permutable if $\theta\circ\varphi=\varphi\circ\theta$. We say that *Con A* is permutable if $\theta\circ\varphi=\varphi\circ\theta$ for all $\theta,\phi\in Con\,A$. It is known that *Con A* need not be permutable for any C-algebra A.

In this paper, we give sufficient conditions for congruences on a C-algebra A to be permutable. Also we derive necessary and sufficient conditions for a C-algebra A to become a Boolean algebra in terms of the congruences on A. We also prove that the three Boolean algebras B(A), the set of C-algebras $\mathfrak{B}_{S(A)}$ and the set of C-algebras $\mathfrak{B}_{R(A)}$ are isomorphic to each other.

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2 Preliminaries

In this section we recall the definition of a C-algebra and some results from [1,3,5] which will be required later.

Definition 2.1. By a C-algebra we mean an algebra $< A, \wedge, \vee, ' >$ of type (2,2,1) satisfying the following identities [1].

(a) $x'' = x$;
(b) $(x \wedge y)' = x' \vee y'$;
(c) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
(d) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(e) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$;
(f) $x \vee (x \wedge y) = x$;
(g) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$.

Example 2.2. [1]:

The 3- element algebra C={T, F, U} is a C-algebra with the operations $\wedge, \vee$ and $'$ defined as in the following tables.

x	x'
T	F
F	T
U	U

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

Every Boolean algebra is a C-algebra.

Lemma 2.3. Every C-algebra satisfies the following laws [1,3,5].

(a) $x \wedge x = x$;
(b) $x \wedge y = x \wedge (x' \vee y) = (x' \vee y) \wedge x$;
(c) $x \vee (x' \wedge x) = x$;
(d) $(x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y)$;
(e) $(x \vee x') \wedge x = x$;
(f) $x \vee x' = x' \vee x$;
(g) $x \vee y \vee x = x \vee y$;
(h) $x \wedge x' \wedge y = x \wedge x'$;
(i) $x \wedge (y \vee x) = (x \wedge y) \vee x$.

The duals of all above statements are also true.

Definition 2.4. If A has identity T for $\wedge$ (that is, $x \wedge T = T \wedge x = x$ for all $x \in A$), then it is unique and in this case, we say that A is a C -algebra with T. We denote T' by F and this F is the identity for $\vee$ [1].

Lemma 2.5 [1]: Let A be a C -algebra with T and $x, y \in$ A. Then

(i) $x \vee y = F$ if and only if $x = y = F$
(ii) if $x \vee y = T$ then $x \vee x' = T$.
(iii) $x \vee T = x \vee x'$
(iv) $x \wedge F = x \wedge x'$.

Theorem 2.6. Let $< A, \wedge, \vee, ' >$ be a C-algebra. Then the following are equivalent [6]:

(i) A is a Boolean algebra.
(ii) $x \vee (y \wedge x) = x,$ for all $x, y \in A$.
(iii) $x \wedge y = y \wedge x,$ for all $x, y \in A$.
(iv) $(x \vee y) \wedge y = y,$ for all $x, y \in A$.
(v) $x \vee x'$ is the identity for $\wedge,$ for every $x \in A$.
(vi) $x \vee x' = y \vee y',$ for all $x, y \in A$.
(vii) A has a right zero for $\wedge$.
(viii) for any $x, y \in A$, there exists $a \in A$ such that $x \vee a = y \vee a = a$.
(ix) for any $x, y \in A$, if $x \vee y = y$, then $y \wedge x = x$.

Definition 2.7. Let A be a C-algebra with T. An element $x \in A$ is called a central element of A if $x \vee x' = T$. The set of all central elements of A is called the Centre of A and is denoted by B(A) [5].

Theorem 2.8. Let A be a C-algebra with T .Then $< B(A), \wedge, \vee, ' >$ is a Boolean Algebra [5].

3 Some Properties of a C -algebra and Its Congruences

In this section we prove some important properties of a C-algebra and we give sufficient conditions for two congruences on a C-algebra A to be permutable. Also we derive necessary and sufficient conditions for a C-algebra A to become a Boolean algebra in terms of the congruences on A.

Lemma 3.1. Every C-algebra satisfies the following identities:

(i) $x \vee y = x \vee (y \wedge x')$;
(ii) $x \wedge y = x \wedge (y \vee x')$.

Proof. Let A be a C-algebra and $x, y \in A$. Now,
$x \vee y = x \vee (x' \wedge y) = x \vee (x' \wedge y \wedge x') = [x \wedge (x \vee x')] \vee [x' \wedge y \wedge (x' \wedge (x \vee x'))] =$ $[x \wedge (x \vee x')] \vee [x' \wedge y \wedge x' \wedge (x \vee x')] = [x \wedge (x \vee x')] \vee [x' \wedge y \wedge (x \vee x')] = (x \vee y)$ $\wedge (x \vee x') = x \vee (y \wedge x')$. Similarly $x \wedge y = x \wedge (y \vee x')$.

Lemma 3.2. Let A be a C-algebra and $x, y \in A$. Then $x \vee y \vee x' = x \vee y \vee y'$.
Proof. Let A be a C-algebra and $x, y \in A$. By Lemma 2.3[b],[f] and Lemma 3.1, we have $x \vee y \vee x' = x \vee ((y' \wedge x') \vee y) = [x \vee (y' \wedge x')] \vee y = (x \vee y') \vee$ $y = x \vee y' \vee y = x \vee y \vee y'$.

Lemma 3.3. Let A be a C-algebra with $T, x, y \in A$ and $x \wedge y = F$. Then $x \vee y = y \vee x$.

Proof. Suppose that $x \wedge y = F$. Then $F = x \wedge y = x \wedge (x' \vee y) = (x \wedge x') \vee$ $(x \wedge y) = (x \wedge x') \vee F = x \wedge x'$. now $x \vee y = F \vee (x \vee y) = (x \wedge y) \vee (x \vee y)$ $= (x \vee x \vee y) \wedge (x' \vee y \vee x \vee y)$ (By Def 2.1) $= (x \vee y) \wedge (x' \vee y \vee x)$ (By Lemma 2.3[g]) $= (x \wedge x') \vee (y \vee x) = F \vee (y \vee x) = y \vee x$.

In [6], it is proved that if A is a C-algebra with T then $B(A) = \{a \in A | a \vee a' = T\}$ is a Boolean algebra under the same operations $\wedge, \vee, '$ in the C-algebra A. Now we prove the following.

Theorem 3.4. Let A be a C-algebra with T and $a, b \in A$ such that $a \vee b \in B(A)$. Then $a \in B(A)$.

Proof. Let A be a C-algebra with T and $a, b \in A$ such that $a \vee b \in B(A)$. Then
$T = (a \vee b) \vee (a \vee b)' = (a \vee b) \vee (a' \wedge b')$
$= (a \vee b \vee a') \wedge (a \vee b \vee b') = (a \vee b \vee a') \wedge (a \vee b \vee a')$ (By Theorem 3.2)
$= a \vee b \vee a'$.

Therefore,

$$T = a \vee b \vee a' \tag{1}$$

$$\begin{aligned} \text{Now, } a \vee a' &= (a \vee a') \wedge T \\ &= (a \vee a') \wedge (a \vee b \vee a') \quad \text{(by (1))} \\ &= (a \wedge (a \vee b \vee a')) \vee (a' \wedge (a \vee b \vee a')) \\ &= a \vee (a' \wedge (a \vee b \vee a')) \\ &= a \vee (a \vee b \vee a') \quad \text{(By Lemma 2.3[b])} \\ &= a \vee b \vee a' = T. \end{aligned}$$

Hence $a \in B(A)$.

The converse of the above theorem need not be true. For example, in the C-algebra $C, F \in B(C)$ but $F \vee U = U \notin B(C)$. We have the following consequence of the above theorem.

Corollary 3.5. Let A be a C-algebra with T, $a, b \in A$ and $a \wedge b \in B(A)$. Then $a \in B(A)$.

Proof. Let $a \wedge b \in B(A)$. Then we have, $(a \wedge b)' \in B(A) \Rightarrow a' \vee b' \in B(A)$ $\Rightarrow a' \in B(A) \Rightarrow a \in B(A)$.

In [1], it is proved that if A is a C-algebra, then $\theta_x = \{(p, q) \mid x \wedge p = x \wedge q\}$ is a congruence on A and $\theta_x \cap \theta_{x'} = \theta_{x \vee x'}$. In [6], if A is C-algebra with T then θ_x is a factor congruence if and only if $x \in B(A)$. They also proved that θ_x, θ_y are permutable congruences whenever both $x, y \in B(A)$. Now we prove some important properties of these congruences.

Theorem 3.6. Let A be a C-algebra with T and $a, b \in A$. Then we have the following (i) $\theta_{a \wedge b} = \theta_{b \wedge a}$; (ii) $\theta_a \circ \theta_b \subseteq \theta_{a \wedge b}$.

Proof. (i) $(x, y) \in \theta_{a \wedge b}$

$$\begin{aligned} &\Rightarrow a \wedge b \wedge x = a \wedge b \wedge y \\ &\Rightarrow b \wedge a \wedge b \wedge x = b \wedge a \wedge b \wedge x \\ &\Rightarrow b \wedge a \wedge x = b \wedge a \wedge x \\ &\Rightarrow (x, y) \in \theta_{b \wedge a} \end{aligned}$$

Therefore $\theta_{a \wedge b} \subseteq \theta_{b \wedge a}$. Similarly, $\theta_{b \wedge a} \subseteq \theta_{a \wedge b}$. Hence $\theta_{a \wedge b} = \theta_{b \wedge a}$.

(ii) Let $(x,y)\in\theta_a\circ\theta_b$. Then there exists $z\in A$ such that $(x,z)\in\theta_b$ and $(z,y)\in\theta_a$. Thus $b\wedge x=b\wedge z$ and $a\wedge z=a\wedge y$. Now, $a\wedge b\wedge x=a\wedge b\wedge z=a\wedge b\wedge a\wedge z=a\wedge b\wedge a\wedge y=a\wedge b\wedge y$. Therefore, $(x,y)\in\theta_{a\wedge b}$. Thus $\theta_a\circ\theta_b\subseteq\theta_{a\wedge b}$.

In the following we give an example of a C-algebra G without T in which the Con A is not permute.

Example 3.7. Consider the C-algebra $G=\{a_1,a_2,a_3,a_4,a_5\}$ where $a_1=(T,U)$, $a_2=(F,U)$, $a_3=(U,T)$, $a_4=(U,F)$, $a_5=(U,U)$ under pointwise operations in C .

x	x'
a_1	a_2
a_2	a_1
a_3	a_4
a_4	a_3
a_5	a_5

$\wedge$	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	a_5	a_5	a_5
a_2	a_2	a_2	a_2	a_2	a_2
a_3	a_5	a_5	a_3	a_4	a_5
a_4	a_4	a_4	a_4	a_4	a_4
a_5	a_5	a_5	a_5	a_5	a_5

$\vee$	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_5	a_5	a_5
a_3	a_3	a_3	a_3	a_3	a_3
a_4	a_5	a_5	a_3	a_4	a_5
a_5	a_5	a_5	a_5	a_5	a_5

This algebra $(G,\vee,\wedge,')$ is a C-algebra with out T.
Let $\Delta =$ diagonal of A. Then we have the following:

$$\theta_{a_1} = \{(x,y) \mid a_1 \wedge x = a_1 \wedge y\}$$
$$= \Delta \cup \{(a_3,a_4),(a_4,a_5),(a_5,a_3),(a_4,a_3)(a_5,a_4),(a_3,a_5)\}$$
$$\theta_{a_3} = \Delta \cup \{(a_1,a_2),(a_2,a_5),(a_5,a_1),(a_2,a_1)(a_5,a_2),(a_1,a_5)\}$$

Now, $\theta_{a_1} \circ \theta_{a_3} = \Delta \cup \theta_{a_1} \cup \theta_{a_3} \cup \{(a_4,a_1),(a_4,a_2)(a_3,a_1),(a_3,a_2)\}$

$\theta_{a_3} \circ \theta_{a_1} = \Delta \cup \theta_{a_1} \cup \theta_{a_3} \cup \{(a_2,a_4),(a_2,a_3)(a_1,a_3),(a_1,a_4)\}$

Therefore $\theta_{a_1} \circ \theta_{a_3} \neq \theta_{a_3} \circ \theta_{a_1}$.

Theorem 3.8. Let A a C-algebra with T and $a \in B(A)$. Then for any $b \in A, \theta_a, \theta_b$ permute and $\theta_a \circ \theta_b = \theta_{a \wedge b}$.

Proof. Let A be a C-algebra with T and $a \in B(A)$. By Theorem 3.6, $\theta_a \circ \theta_b \subseteq \theta_{a \wedge b}$. Now let $(p,q) \in \theta_{a \wedge b}$. Then $a \wedge b \wedge p = a \wedge b \wedge q \Rightarrow b \wedge a \wedge b \wedge p = b \wedge a \wedge b \wedge q \Rightarrow b \wedge a \wedge p = b \wedge a \wedge q.$ Consider, $r = (a \wedge p) \vee (a' \wedge q)$. Now $a \wedge r = a \wedge [(a \wedge p) \vee (a' \wedge q)] = (a \wedge p) \vee (a \wedge a' \wedge q) = (a \wedge p) \vee (F \wedge q)$ $= (a \wedge p) \vee F = a \wedge p.$ Therefore $(r,p) \in \theta_a \Rightarrow (p,r) \in \theta_a$. Now, $b \wedge r =$ $b \wedge [(a \wedge p) \vee (a' \wedge q)] = [b \wedge a \wedge p] \vee [b \wedge a' \wedge q] = (b \wedge a \wedge q) \vee (b \wedge a' \wedge q)$ $= b \wedge ((a \wedge q) \vee (a' \wedge q)) = b \wedge ((a \vee a') \wedge q) = b \wedge (T \wedge q)$ (since $a \in B(A)$) $= b \wedge q$. Therefore $(q,r) \in \theta_b \Rightarrow (r,q) \in \theta_b$. Thus $(p,q) \in \theta_b \circ \theta_a$. Hence $\theta_b \circ \theta_a = \theta_{a \wedge b}$. Thus $\theta_b \circ \theta_a$ is a congruence on A and hence θ_a, θ_b are permutable congruences and hence $\theta_a \circ \theta_b = \theta_b \circ \theta_a = \theta_{a \wedge b}$.

Corollary 3.9. Let A be a C-algebra with T and $a,b \in A$. Then i) $a \vee b \in B(A) \Rightarrow \theta_a \circ \theta_b = \theta_{a \wedge b}$; ii) $a \wedge b \in B(A) \Rightarrow \theta_a \circ \theta_b = \theta_{a \wedge b}$.

Proof. i) We know that if $a \vee b \in B(A)$ then $a \in B(A)$ and hence by the above theorem $\theta_a \circ \theta_b = \theta_b \circ \theta_a = \theta_{a \wedge b}$. Similarly, we can prove ii).

Let A be a C-algebra. If Con(A) is permutable, then A need not be a Boolean algebra. For example, in the C-algebra C, the only congruences are Δ, ∇ and they are permutable. But C is not a Boolean algebra. Now we give equivalent conditions for a C-algebra to become a Boolean algebra in terms of congruence relations.

Theorem 3.10. Let $(A, \vee, \wedge, ')$ be a C-algebra with T. Then the following are equivalent. (i) Let $(A, \vee, \wedge, ')$ be a Boolean algebra. (ii) $\theta_x \cap \theta_{x'} = \Delta$ for all $x \in A$. (iii) $\theta_{x \vee x'} = \Delta$ for all $x \in A$.

Proof. (1)$\Rightarrow$(2): Let A be a Boolean algebra and $x \in A$. Let $(p,q) \in \theta_x \cap \theta_{x'}$. Then $x \wedge p = x \wedge q$ and $x' \wedge p = x' \wedge q$. Now, $p = (x \vee x') \wedge p = (x \wedge p) \vee (x' \wedge q) = (x \wedge q) \vee (x' \wedge q) = (x \vee x') \wedge q = q$. Thus $\theta_x \cap \theta_{x'} \subseteq \Delta$. Therefore $\theta_x \cap \theta_{x'} = \Delta$. Since $\theta_x \cap \theta_{x'} = \theta_{x \vee x'}$, we get (ii)$\Rightarrow$(iii). (iii)$\Rightarrow$(i): Suppose $\theta_{x \vee x'} = \Delta$ for all $x \in A$. We prove that $\theta_{x'} \circ \theta_x = A \times A$. Let $(p,q) \in A \times A$. Write $t = (x \wedge p) \vee (x' \wedge q)$. Now, $x \wedge t = x \wedge ((x \wedge p) \vee (x' \wedge q)) = (x \wedge p) \vee (x \wedge x' \wedge q) = (x \wedge p) \vee (x \wedge x') = x \wedge (p \vee x') = x \wedge p$. Also, $x' \wedge t = x' \wedge ((x \wedge p) \vee (x' \wedge q)) = (x' \wedge x \wedge p) \vee (x' \wedge x' \wedge q)) = (x' \wedge x) \vee (x' \wedge q)) = (x' \wedge (x \vee q)) = x' \wedge q$. Therefore $(p,t) \in \theta_x$ and $(t,q) \in \theta_{x'}$. Thus $(p,q) \in \theta_{x'} \circ \theta_x$. Hence we get $\theta_{x'} \circ \theta_x = A \times A$. Also $\theta_x \cap \theta_{x'} = \theta_{x \vee x'} = \Delta$. That is θ_x and $\theta_{x'}$ are permutable factor congruences. Therefore, by Theorem 2.6, we have $x \in B(A)$. Thus $A = B(A)$ and hence A is a Boolean algebra.

4 The C-algebra S_x

We prove that, for each $x \in A$, $S_x = \{x \vee t \mid t \in A\}$ is itself a C-algebra under induced operations $\wedge, \vee$ and the unary operation is defines by $(x \vee t)^* = x \wedge t'$. We observe that S_x need not be a subalgebra of *A* because the unary operation in S_x is not the restriction of the unary operation on A. Also for each $x \in A$, the set $A_x = \{x \wedge t \mid t \in A\}$ is a C-algebra in which the unary operation is given by $(x \wedge t)^* = x \wedge t'$. We prove that the *B(A)* is isomorphic to the Boolean algebra $\mathfrak{B}_{S(A)}$ of all C-algebras S_a where $a \in B(A)$. Also, we prove that B(A) is isomorphic to the Boolean algebra $\mathfrak{B}_{R(A)}$ of all C-algebras $A_a, a \in B(A)$.

Theorem 4.1. Let $< A, \wedge, \vee, ' >$ be a C-algebra, $x \in A$ and $S_x = \{x \vee t \mid t \in A\}$. Then $< S_x, \wedge, \vee, * >$ is a C-algebra with x as the identity for $\vee$, where $\wedge$ and $\vee$ are the operations in A restricted to S_x and for any $x \vee t \in S_x$, here $(x \vee t)^*$ is $x \vee t'$.

Proof. Let $t, r, s \in A$. Then $(x \vee t) \vee (x \vee r) = x \vee (t \vee r) \in S_x$ and $(x \vee t) \wedge (x \vee r) = x \wedge (t \vee r) \in S_x$. Thus $\vee, \wedge$ are closed in S_x. Also * is closed in S_x. Consider $(x \vee t)^{**} = x \vee (x \vee t')' = x \vee (x' \wedge t) = x \vee t$. Now $[(x \vee t) \wedge (x \vee r)]^* = [x \vee (t \wedge r)]^* = x \vee (t' \vee r') = x \vee t' \vee x \vee r' = (x \vee t)^* \vee (x \vee r)^*$. Now, consider $\left[(x \vee t) \vee (x \vee r)\right] \wedge (x \vee s) = x \wedge [(t \wedge r) \wedge s] = x \vee \left[(t \wedge s) \vee (t' \wedge r \wedge s)\right] = x \vee (t \wedge s) \vee x \vee (t' \wedge r \wedge s) = \left[(x \vee t) \wedge (x \vee s)\right] \vee \left[(x \vee t') \wedge (x \vee r) \wedge (x \vee s)\right] = \left[(x \vee t) \wedge (x \vee s)\right] \vee \left[(x \vee t)^* \wedge (x \vee r) \wedge (x \vee s)\right]$. The remaining identities of a C-algebra also hold in S_x because they hold in A. Hence, S_x is itself a C-algebra. Also x is the identity for $\vee$ because $x \vee x \vee t = x \vee t = x \vee t \vee x$. Here $x \vee x'$ is the identity for $\wedge$.

Theorem 4.2. Let A be a C-algebra. Then the following holds.

(i) $S_x = S_y$ if and only if $x = y$;

(ii) $S_x \cap S_y \subseteq S_{x \vee y}$;

(iii) $S_x \cap S_{x'} = S_{x \vee x'}$;

(iv) $(S_x)_{x \vee y} = S_{x \vee y}$.

Proof. (i) Suppose $S_x = S_y$. Since $x = x \vee x \in S_x = S_y$ and $y = y \vee y \in S_y = S_x$. Therefore $x = y \vee t$ and $y = x \vee r$ for some $t, r \in A$. Now, $x = y \vee t = (y \vee t \vee y) \wedge (y \vee y \vee t) = (x \vee y) \wedge (y \vee x) = (y \vee x) \wedge (x \vee y) = (x \vee r \vee x) \wedge (x \vee x \vee r) = x \vee r = y.$ The converse is trivial. (ii) Suppose $t \in S_x \cap S_y$. Then $t = x \vee s = y \vee r$ for some $s, r \in A$. Now, $t = x \vee x \vee s = x \vee t = x \vee y \vee r \in S_{x \vee y}$. (iii) $S_x \cap S_{x'} \subseteq S_{x \vee x'}$ by (ii). Since $x \vee x' = x' \vee x$ we have $S_{x \vee x'} \subseteq S_x \cap S_{x'}$. Hence $S_x \cap S_{x'} = S_{x \vee x'}$. (iv) $(S_x)_{x \vee y} = \{x \vee y \vee t \mid t \in S_x\} = \{x \vee y \vee x \vee r \mid r \in A\} = \{x \vee y \vee r \mid r \in A\} = S_{x \vee y}$.

Theorem 4.3. Let A be a C-algebra with T and $x \in A$, then the mapping $\alpha_x : A \to S_x$ defined by $\alpha_x(t) = x \vee t$ for all $t \in A$ is a homomorphism of A to S_x with kernel $\theta_{x'}$ and hence $A/\theta_{x'} \cong S_x$.

Proof. Let $t, r \in A$. Then $\alpha_x(t \vee r) = x \vee t \vee r = x \vee t \vee x \vee r = \alpha_x(t) \vee \alpha_x(r)$ and $\alpha_x(t') = x \vee t' = (x \vee t)^* = (\alpha_x(t))^*$. Clearly, $\alpha_x(t \wedge r) = \alpha_x(t) \wedge \alpha_x(r)$. Also $\alpha_x(T) = x \vee T = x \vee x'$, which is the identity for $\wedge$ in $S_{x.}$ Therefore α_x is a homomorphism. Hence by the fundamental theorem of homomorphism $A/Ker\,\alpha_x \cong S_x$ and $Ker\alpha_x = \{(t,r) \in A \times A \mid \alpha_x(t) = \alpha_x(r)\} = \{(t,r) \in A \times A \mid x \vee t = x \vee r\} = \{(t,r) \in A \times A \mid x' \wedge t = x' \wedge r\} \theta_{x'}$ (by Lemma 2.3 [b]) = and hence $A/\theta_{x'} \cong S_x$.

Theorem 4.4. Let A be a C-algebra with T and $a \in B(A)$, then $A \cong S_a \times S_{a'}$.

Proof. Define $\alpha : A \to S_a \times S_{a'}$ by $\alpha(x) = (\alpha_a(x), \alpha_{a'}(x))$ for all $x \in A$. Then, by Theorem 4.3, α is well-defined and α is a homomorphism. Now, we prove that α is one-one. Let $x, y \in A$. Then $\alpha(x) = \alpha(y) \Rightarrow (\alpha_a(x), \alpha_{a'}(x))$ $= (\alpha_a(y), \alpha_{a'}(y)) \Rightarrow (a \vee x, a' \vee x) = (a \vee y, a' \vee y) \Rightarrow a \vee x = a \vee y$ and $a' \vee x = a' \vee y$. Now $x = F \vee x = (a \wedge a') \vee x = (a \vee x) \wedge (a' \vee x) = (a \vee y) \wedge (a' \vee y) = y$. Finally, we prove that α is onto. Let $(x, y) \in S_a \times S_{a'}$. Then $x = a \vee t$, and $y = a' \vee r$ for some $t, r \in A$. Therefore, $a \vee x = x$, $a \vee y = a \vee a' \vee y = T \vee y = T$ and $a' \vee x = T, a' \vee y = y$. Now,

$$
\begin{aligned}
\alpha(x \wedge y) &= (\alpha_a(x \wedge y), \alpha_{a'}(x \wedge y)) \\
&= (a \vee (x \wedge y), a' \vee (x \wedge y)) \\
&= ((a \vee x) \wedge (a \vee y), (a' \vee x) \wedge (a' \vee y)) \\
&= (x \wedge T, T \wedge y) \\
&= (x, y).
\end{aligned}
$$

Therefore, α is onto and hence α is an isomorphism. Therefore $A \cong S_a \times S_{a'}$.

Lemma 4.5. Let A be a C-algebra. Then for $a, b \in A$:

(i) $a \vee b = b \vee a$ if and only if $S_{a \vee b} = S_a \cap S_b$

(ii) $S_{a \wedge b} = Sup\{S_a, S_b\}$ in the poset ($\{S_x \mid x \in A\}, \subseteq$), then $a \wedge b = b \wedge a$. The converse is not true.

Proof. (i) Suppose that $a\vee b=b\vee a$. Then clearly $S_{a\vee b}\subseteq S_a\cap S_b$. By Theorem 4.2(ii) $S_a\cap S_b\subseteq S_{a\vee b}$. Hence $S_{a\vee b}=S_a\cap S_b$. Conversely assume that $S_{a\vee b}=S_a\cap S_b$. Clearly $a\vee b\in S_{a\vee b}=S_a\cap S_b$. Therefore $a\vee b\in S_b\Rightarrow$ $a\vee b=b\vee t$ for some $t\in A$. Now $b\vee a=b\vee a\vee b=b\vee b\vee t=b\vee t=$ $a\vee b$. (ii) Assume that $a,b\in A$ and $S_{a\wedge b}=Sup\{S_a,S_b\}$. Then $S_{a\wedge b}=S_{b\wedge a}$ and hence $a\wedge b\in S_{a\wedge b}=S_{b\wedge a}$. Therefore $a\wedge b=(b\wedge a)\vee t$ for some $t\in A$. Now $(b\wedge a)\vee(a\wedge b)=(b\wedge a)\vee((b\wedge a)\vee t)=(b\wedge a)\vee t=a\wedge b$. Similarly we can prove that $(a\wedge b)\vee(b\wedge a)=b\wedge a$. Hence $a\wedge b=b\wedge a$. The converse need not be true, for example for the C-algebra C, $S_U=\{U\},S_T=\{T\}$ and $U\wedge T=T\wedge U$. But $S_{U\wedge T}(=S_U)$ is not an upper bound of $\{S_U,S_T\}$.

Now we prove $\mathfrak{B}_{S(A)}=\{S_a\mid a\in B(A)\}$ is a Boolean algebra under set inclusion.

Theorem 4.6. Let $<A,\wedge,\vee,'>$ be a C-algebra with T. Then $\mathfrak{B}_{S(A)}=\{S_a\mid a\in B(A)\}$ is a Boolean algebra under set inclusion.

Proof. Clearly $(\mathfrak{B}_{S(A)},\subseteq)$ is a partially ordered set under inclusion. First we show for $a,b\in B(A)$, $S_{a\vee b}$ is the infimum of $\{S_a,S_b\}$ and $S_{a\wedge b}$ is the supremum of $\{S_a,S_b\}$ for all $a,b\in B(A)$. Let $a,b\in B(A)$. Then $a\wedge b=b\wedge a$ and $a\vee b=b\vee a$. Hence by the above Lemma 4.5, $S_{a\vee b}$ is the infimum of $\{S_a,S_b\}$. Let $t\in S_a$. Then $t=a\vee x$ for some $x\in A$. Now $t=a\vee x=(a\wedge(a\vee b))\vee x=(a\wedge (b\vee a)\vee x=(a\wedge b)\vee a\vee x\in S_{a\wedge b}$. Similarly $S_b\subseteq S_{b\wedge a}=S_{a\wedge b}$. Therefore $S_{a\wedge b}$ is an upper bound of S_a,S_b. Suppose S_c is an upper bound of S_a,S_b. $t\in S_{a\wedge b}$. Then $t=(a\wedge b)\vee x$ for some $x\in A$. Now $t=(a\wedge b)\vee x=(a\vee x)\ \wedge(a'\vee b\vee x)=(a\vee x)\wedge(b\vee a'\vee x)\in S_c$ (since $a\vee x\in S_a\subseteq S_c$, $b\vee a'\vee x\in S_b\subseteq S_c$ and S_c is closed under $\wedge$). Therefore $S_{a\wedge b}$ is the supremum of $\{S_a,S_b\}$. Denote the supremum of $\{S_a,S_b\}$ by $S_a\vee S_b$ and the infimum of $\{S_a,S_b\}$ by $S_a\wedge S_b$. Now $S_T\wedge S_a=S_{T\vee a}=S_T$ and $S_F\vee S_a=S_{F\wedge a}=S_F$. Therefore S_T is the least element and S_F is the greatest element of $(\mathfrak{B}_{S(A)}, \subseteq)$. Now for any $a,b,c \in B(A)$, $(S_a \vee S_b) \wedge S_c = S_{(a \wedge b) \vee c} = S_{(a \vee c) \wedge (b \vee c)} = S_{(a \vee c)} \vee S_{(b \vee c)} = (S_a \wedge S_c) \vee (S_b \wedge S_c)$. Also $S_a \wedge S_{a'} = S_{a \vee a'} = S_T$ and $S_a \vee S_{a'} = S_{a \wedge a'} = S_F$. Therefore $(\mathfrak{B}_{S(A)}, \subseteq)$ is a complimented distributive lattice and hence it is a Boolean algebra.

Theorem 4.7. Let A be a C-algebra with T Define $\varphi: B(A) \to \mathfrak{B}_{S(A)}$ by $\phi(a) = s_{a'}$ for all $a \in B(A)$. Then ϕ is an isomorphism.

Proof. Let $a,b \in B(A)$. Then $\varphi(a \wedge b) = S_{(a \wedge b)'} = S_{a'} \wedge S_{b'} = \varphi(a) \wedge \varphi(b)$. $\varphi(a \vee b) = S_{(a \vee b)'} = S_{a'} \vee S_{b'} = \varphi(a) \vee \varphi(b)$, $\varphi(a') = S_{a'} = (S_a)' = (\varphi(a))'$. Clearly ϕ is both one-one and onto. Hence $B(A) \cong \mathfrak{B}_{S(A)}$.

In [3] we defined a partial ordering on a C-algebra by $x \leq y$ if and only if $y \wedge x = x$ and we studied the properties of this partial ordering. We gave a number of equivalent conditions in terms of this partial ordering for a C-algebra to become a Boolean algebra. In [4] we proved that, for each $x \in A$, $A_x = \{s \in A \mid s \leq x\}$ is itself a C-algebra under induced operations $\wedge, \vee$ and the unary operation is defined by $s^* = x \wedge s'$ we also observed that A_x need not be an algebra of A because the unary operation in A_x is not the restriction of the unary operation. For each $x \in A$, we proved that A_x is isomorphic to the quotient algebra A/θ_x where $\theta_x = \{(p,q) \in A \times A \mid x \wedge p = x \wedge q\}$. We can easily see that the C-algebras S_x, A_x are different in general where $x \in A$.

Now, we prove that the set of all A_a's where $a \in B(A)$ is a Boolean algebra under set inclusion. The following theorem can be proved analogous to Theorem 4.6.

Theorem 4.8. Let A be a C-algebra with T. Then $\mathfrak{B}_{R(A)} := \{A_a \mid a \in B(A)\}$ is a Boolean Algebra under set inclusion in which the supremum of $\{A_a, A_b\} = A_{a \vee b}$ and the infimum of $\{A_a, A_b\} = A_{a \wedge b}$.

The proof of the following theorem is analogous to that of Theorem 4.7.

Theorem 4.9. Let A be a C-algebra with T Define $f : B(A) \to \mathfrak{B}_{R(A)}$ by $f(a) = A_a$ for all $a \in B(A)$. Then f is an isomorphism.

The following corollary can be proved directly from Theorems 4.7 and 4.9.

Corollary 4.10. Let A be a C-algebra with T. Then $\mathfrak{B}_{R(A)}, B(A)$ and $\mathfrak{B}_{S(A)}$ are isomorphic to each other.

References

[1] Guzman, F. & Squier, C., *The Algebra of Conditional Logic*, Algebra Universalis, **27**, pp. 88-110, 1990.

[2] Swamy, U.M., Rao, G.C., Sundarayya, P. & Kalesha Vali, S., *Semilattice Structures on a C-algebra,* Southeast Asian Bulletin of Mathematics, **33**, pp. 551-561, 2009.

[3] Rao, G.C., Sundarayya, P., C-*algebra as a Poset*, International Journal of Mathematical Sciences, Serials Pub., New Delhi, **4**(2), pp. 225-236, Dec. 2005.

[4] Rao, G.C., Sundarayya, P., *Decompositions of a C-algebra*, International Journal of Mathematics and Mathematical Sciences, Hindawi Pub. Cor. U.S.A., **2006**(3), Article ID 78981, pp. 1-8, 2006.

[5] Swamy, U.M., Rao,G.C. & Ravi Kumar, R.V.G., *Centre of a C-algebra,* Southeast Asian Bulletin of Mathematics, **27**, pp. 357-368, 2003.

[6] Stanely, B. & Sankappanavar, H.P., *A Course in Universal Algebra*, Springer Verlag, 1981.