On The Total Irregularity Strength of Regular Graphs

Rismawati Ramdani1,2, A.N.M. Salman1 & Hilda Assiyatun1

1 Combinatorial Mathematics Research Group, Department of Mathematics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesha No. 10, Bandung 40132, Indonesia
2 Department of Mathematics, Faculty of Sciences and Technologies, Universitas Islam Negeri Sunan Gunung Djati, Jalan A.H. Nasution No. 105, Bandung, 40614, Indonesia
Email: rismawatiramdani@gmail.com

Abstract. Let \( G = (V,E) \) be a graph. A total labeling \( f: V \cup E \rightarrow \{1, 2, \ldots, k\} \) is called a totally irregular total \( k \)-labeling of \( G \) if every two distinct vertices \( x \) and \( y \) in \( V \) satisfy \( w_f(x) \neq w_f(y) \) and every two distinct edges \( x_1x_2 \) and \( y_1y_2 \) in \( E \) satisfy \( w_f(x_1x_2) \neq w_f(y_1y_2) \), where \( w_f(x) = f(x) + \sum_{x \in E(G)} f(xx) \) and \( w_f(x_1x_2) = f(x_1) + f(x_1x_2) + f(x_2) \). The minimum \( k \) for which a graph \( G \) has a totally irregular total \( k \)-labeling is called the total irregularity strength of \( G \), denoted by \( ts(G) \). In this paper, we consider an upper bound on the total irregularity strength of \( m \) copies of a regular graph. Besides that, we give a dual labeling of a totally irregular total \( k \)-labeling of a regular graph and we consider the total irregularity strength of \( m \) copies of a path on two vertices, \( m \) copies of a cycle, and \( m \) copies of a prism \( C_n \square P_2 \).

Keywords: cycle; dual labeling; path; prism; regular graph; the total irregularity strength; totally irregular total \( k \)-labeling.

1 Introduction

In 2007, Baća, et al. [1] introduced vertex irregular total \( k \)-labelings and edge irregular total \( k \)-labelings. A total labeling \( f: V \cup E \rightarrow \{1, 2, \ldots, k\} \) is called a vertex irregular total \( k \)-labeling of \( G \) if every two distinct vertices \( x \) and \( y \) in \( V \) satisfy \( w_f(x) \neq w_f(y) \), where \( w_f(x) = f(x) + \sum_{x \in E(G)} f(xx) \). The minimum \( k \) for which a graph \( G \) has a vertex irregular total \( k \)-labeling, denoted by \( tvs(G) \), is called the vertex irregularity strength of \( G \).

Baća, et al. [1] proved that for any graph \( G = (V,E) \),

\[
\frac{|V(G)| + \delta(G)}{\Delta(G)+1} \leq tvs(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1. \tag{1}
\]

Another result about \( tvs(G) \) was given by Nurdin, et al. in [2] as follows:

\[
tvs(G) \geq \max \left\{ \left\lfloor \frac{\delta+n_0}{\delta+1} \right\rfloor, \left\lfloor \frac{\delta+n_0+n_1}{\delta+2} \right\rfloor, \ldots, \left\lfloor \frac{\delta+n_0+n_1+n_2}{\Delta+1} \right\rfloor \right\}. \tag{2}
\]
where \( n_i \) denotes the number of vertices of degree \( i, i = \delta, \delta + 1, \ldots, \Delta \).

In [3], Majerski and Przybylo gave the best result for dense graphs so far. In [4], Anholcer, Kalkowski and Przybylo gave the best known result for general graphs. Some other results about vertex irregular total \( k \)-labeling were given by Nurdin, et al. in [5] and [6], and Wijaya, et al. in [7] and [8].

A total labeling \( f: V \cup E \to \{1, 2, \ldots, k\} \) is called an edge irregular total \( k \)-labeling of \( G \) if every two distinct edges \( x_1x_2 \) and \( y_1y_2 \) in \( E \) satisfy \( w_f(x_1x_2) \neq w_f(y_1y_2) \), where \( w_f(x_1x_2) = f(x_1) + f(x_2) \). The minimum \( k \) for which a graph \( G \) has an edge irregular total \( k \)-labeling, denoted by \( tes(G) \), is called the edge irregularity strength of \( G \).

In [1], Bača, et al. derived a lower and an upper bounds on the total edge irregularity strength of any graph \( G = (V, E) \) as follows:

\[
\left\lfloor \frac{|E(G)|+2}{3} \right\rfloor \leq tes(G) \leq |E(G)|. \tag{3}
\]

Ivančo and Jendroľ in [9] proved that

\[
tes(T) = \max \left\{ \left\lfloor \frac{|E(T)|+2}{3} \right\rfloor, \left\lfloor \frac{\Delta(T)+1}{2} \right\rfloor \right\}, \tag{4}
\]

where \( T \) is a tree.

In [10], Nurdin, et al. determined the total edge irregularity strength of the corona product of a path with some graphs, which are a path, a cycle, a star, a gear, a friendship graph, and a wheel.

Some other results about edge irregular total \( k \)-labelings were given by Bača and Siddiqui in [11], Jendroľ, Miškuf, and Sotášk in [12] and [13], and Miškuf and Jendroľ in [14].

Combining vertex irregular total \( k \)-labelings and edge irregular total \( k \)-labelings, Marzuki, Salman, and Miller, in [15], introduced a new irregular total \( k \)-labeling of a graph \( G \). It is called ‘totally irregular total \( k \)-labelings’, which is required to be at the same time both vertex and edge irregular. The minimum \( k \) for which a graph \( G \) has a totally irregular total \( k \)-labeling, denoted by \( ts(G) \), is called the total irregularity strength of \( G \).

In the same paper, Marzuki, et al. gave a lower bound on \( ts(G) \) and exact values of the total irregularity strength of cycles and paths as follows:
For every graph $G$,
\[ ts(G) \geq \max\{tes(G), tvs(G)\}, \]  
(5)
\[ ts(C_n) = \left\lfloor \frac{n+2}{3} \right\rfloor \text{ for } n \geq 3, \]  
(6)
\[ ts(P_n) = \left\lfloor \frac{n+2}{3} \right\rfloor \text{ for } n = 2 \text{ or } n = 5; \]  
\[ \left\lfloor \frac{n+1}{3} \right\rfloor \text{ otherwise.} \]  
(7)
In [16], Ramdani and Salman determined the total irregularity strength of some Cartesian product graphs. One of some results in the paper is given as follows:
\[ ts(C_n \square P_2) = n + 1 \text{ for } n \geq 3. \]  
(8)

2 Main Results

A totally irregular total $k$-labeling $f$ of $G$ is called an optimal labeling of $G$ if $ts(G) = k$. In the following theorem, we derive an upper bound on the total irregularity strength of $m$ copies of a regular graph.

**Theorem 2.1** Let $G$ be an $r$-regular connected graph with $r \geq 1$. Then,
\[ ts(mG) \leq m(ts(G)) - \left\lfloor \frac{m-1}{2} \right\rfloor. \]

**Proof.** Let $G = (V, E)$ be an $r$-regular graph with order $n$, $ts(G) = t$, and $f$ be an optimal labeling of $G$. Then, $|E| = \frac{nr}{2}$. Let $mG$ be $m$ copies of $G$ where the copies of $G$ are denoted by $G_i$ for $1 \leq i \leq m$. Let $V(G_i) = \{v_1, v_2, \ldots, v_n\}$, $E(G) = \{e_1, e_2, \ldots, e_{\frac{nr}{2}}\}$, $V(G_i) = \{v_1^i, v_2^i, \ldots, v_n^i\}$, and $E(G_i) = \{e_1^i, e_2^i, \ldots, e_{\frac{nr}{2}}^i\}$, where $G_i$ is isomorphic with $G$ with the isomorphism
\[ s: V(G) \cup E(G) \rightarrow V(G_i) \cup E(G_i), \]
where
\[ s(v_a) = v_a^i \text{ and } s(e_x) = e_x^i \]
for every $1 \leq i \leq m, 1 \leq a \leq n$, and $1 \leq x \leq \frac{nr}{2}$.

Define a total labeling $g$ of $mG$ as follows:

- For $i$ odd,
  1) $g(v_a^i) = f(v_a) + (i-1)t - \left(\frac{i-1}{2}\right)$;
  2) $g(e_x^i) = f(e_x) + (i-1)t - \left(\frac{i-1}{2}\right)$;

- For $i$ even,
  1) $g(v_a^i) = f(v_a) + (i-1)t - \left(\frac{i}{2}\right)$;
  2) $g(e_x^i) = f(e_x) + (i-1)t - \left(\frac{i}{2}\right)$;
For $i$ even,
1) $g(v_a^i) = f(v_a) + (i - 1)t - \left(\frac{i}{2}\right)$;
2) $g(e_x^i) = f(e_x) + (i - 1)t - \left(\frac{i}{2}\right) + 1$;

for every $1 \leq i \leq m$, $1 \leq a \leq n$, and $1 \leq x \leq \frac{nr}{2}$.

Next, it will be shown that in the labeling $g$, there are no two edges of the same weight and there are no two vertices of the same weight.

1. It will be shown that there are no two edges in $G_i$ with the same weight for every $i$, $1 \leq i \leq m$.

Let $e_x^i = v_a^i v_b^i$ be an edge in $G_i$ for $1 \leq i \leq m$. We consider two cases.

- Case 1: For $i$ odd,
  
  \[
  w_g(e_x^i) = g(v_a^i) + g(e_x^i) + g(v_b^i) = f(v_a) + (i - 1)t - \left(\frac{i-1}{2}\right) + f(e_x) + (i - 1)t - \left(\frac{i-1}{2}\right) + f(v_b) + (i - 1)t - \left(\frac{i-1}{2}\right) + 3 \left( (i - 1)t - \left(\frac{i-1}{2}\right) \right) = w_f(e_x) + 3 \left( (i - 1)t - \left(\frac{i-1}{2}\right) \right).
  \]

- Case 2: For $i$ even,
  
  \[
  w_g(e_x^i) = g(v_a^i) + g(e_x^i) + g(v_b^i) = f(v_a) + (i - 1)t - \left(\frac{i}{2}\right) + f(e_x) + (i - 1)t - \left(\frac{i}{2}\right) + 1 + f(v_b) + (i - 1)t - \left(\frac{i}{2}\right) = f(v_a) + f(e_x) + f(v_b) + 3 \left( (i - 1)t - \left(\frac{i}{2}\right) \right) + 1 = w_f(e_x) + 3 \left( (i - 1)t - \left(\frac{i}{2}\right) \right) + 1.
  \]

Since $w_f(e_x) \neq w_f(e_y)$ for every $x \neq y$, $3 \left( (i - 1)t - \left(\frac{i-1}{2}\right) \right)$ and $3 \left( (i - 1)t - \left(\frac{i}{2}\right) \right) + 1$ are constants, we get $w_g(e_x^i) \neq w_g(e_y^i)$ for every $x \neq y$, $1 \leq i \leq m$, and $x, y \in \{1,2,\ldots,\frac{nr}{2}\}$. 

2. Define $j = i + 1$ for $1 \leq i \leq m$. It will be shown that $w_g(e_x^i) < w_g(e_y^j)$ for all edges $e_x^i \in G_i$ and $e_y^j \in G_j$ for $x, y \in \{1, 2, \ldots, \frac{nr}{2}\}$.

Let $e_x^i = v_au_i$ and $e_y^j = v_cv_d$. We consider two cases.

- **Case 1**: For $i$ odd,

  \[
  w_g(e_x^i) = g(v_a) + g(e_x^i) + g(v_b) \\
  = f(v_a) + f(e_x) + f(v_b) + 3\left((i-1)t - \left(\frac{i-1}{2}\right)\right) \\
  \leq 3t + 3\left((i-1)t - \left(\frac{i-1}{2}\right)\right) \\
  = 3it - 3\left(\frac{i-1}{2}\right) \tag{9}
  \]

  On the other hand,

  \[
  w_g(e_y^j) = f(v_c) + f(e_x) + f(v_d) + 3\left((j-1)t - \left(\frac{j}{2}\right)\right) + 1 \\
  \geq 3 + 3\left((j-1)t - \left(\frac{j}{2}\right)\right) + 1 \\
  = 3 + 3\left(it - \left(\frac{i+1}{2}\right)\right) + 1 \\
  > 3it - 3\left(\frac{i-1}{2}\right) + 1 \tag{10}
  \]

  From (9) and (10), it follows $w_g(e_y^j) > w_g(e_x^i)$.

- **Case 2**: For $i$ even,

  \[
  w_g(e_x^i) = g(v_a) + g(e_x^i) + g(v_b) \\
  = f(v_a) + f(e_x) + f(v_b) + 3\left((i-1)t - \left(\frac{i}{2}\right)\right) + 1 \\
  \leq 3t + 3\left((i-1)t - \left(\frac{i}{2}\right)\right) + 1 \\
  = 3it - 3\left(\frac{i}{2}\right) + 1 \tag{11}
  \]

  On the other hand,
\[ w_g(e^i_y) = f(v_c) + f(e_y) + f(v_d) + 3 \left( (j - 1)t - \left( \frac{i-1}{2} \right) \right) \]
\[ \geq 3 + 3 \left( (j - 1)t - \left( \frac{i-1}{2} \right) \right) \]
\[ = 3 + 3 \left( it - \left( \frac{j}{2} \right) \right) \]
\[ > 3it - 3 \left( \frac{1}{2} \right) + 1. \] (12)

From (11) and (12), we have \( w_g(e^i_y) > w_g(e^i_x) \).

Hence, \( w_g(e^p_u) \neq w_g(e^q_w) \) for all edges \( e^p_u \in G_p \) and \( e^q_w \in G_q \) with \( p \neq q \), \( p, q \in \{1, 2, \ldots, m\} \), and \( u, w \in \{1, 2, \ldots, n\} \).

1. It will be shown that there are no two vertices in \( G_i \) with the same weight for every \( i, 1 \leq i \leq m \).

Let \( v^i_a \) be a vertex in \( G_i \) for \( 1 \leq i \leq m \). Let the edges incident with \( v^i_a \) be \( e^i_{a_1}, e^i_{a_2}, \ldots, e^i_{a_r} \). We consider two cases.

- Case 1: For \( i \) odd,
  \[ w_f(v^i_a) = f(v_a) + (i - 1)t - \left( \frac{i-1}{2} \right) \]
  \[ + \sum_{s=1}^{r} \left( f(e_{a_s}) + (i - 1)t - \left( \frac{i-1}{2} \right) \right) \]
  \[ = f(v_a) + \sum_{s=1}^{r} f(e_{a_s}) + (r + 1) \left( (i - 1)t - \left( \frac{i-1}{2} \right) \right) \]
  \[ = w_f(v_a) + (r + 1) \left( (i - 1)t - \left( \frac{i-1}{2} \right) \right) \].

- Case 2: For \( i \) even,
  \[ w_f(v^i_a) = f(v_a) + (i - 1)t - \left( \frac{i}{2} \right) \]
  \[ + \sum_{s=1}^{r} \left( f(e_{a_s}) + (i - 1)t - \left( \frac{i}{2} \right) + 1 \right) \]
  \[ = f(v_a) + \sum_{s=1}^{r} f(e_{a_s}) + (r + 1) \left( (i - 1)t - \left( \frac{i}{2} \right) \right) + r \]
  \[ = w_f(v_a) + (r + 1) \left( (i - 1)t - \left( \frac{i}{2} \right) \right) + r. \]

Since \( w_f(v_a) \neq w_f(v_b) \) for every \( a \neq b \), \( (r + 1) \left( (i - 1)t - \left( \frac{i-1}{2} \right) \right) \) and \( (r + 1) \left( (i - 1)t - \left( \frac{i}{2} \right) \right) + r \) are constants, we get \( w_g(v^i_a) \neq w_g(v^i_b) \) for every \( a \neq b, 1 \leq i \leq m \), and \( a, b \in \{1, 2, \ldots, n\} \).
Define \( j = i + 1 \) for \( 1 \leq i \leq m \). It will be shown that \( w_g(v^i_a) < w_g(v^i_b) \) for all vertices \( v^i_a \in G_i \) and \( v^i_b \in G_j \) for \( a, b \in \{1, 2, \ldots, n\} \).

Let the edges incident with \( v^i_a \) be \( e^i_{a_1}, e^i_{a_2}, \ldots, e^i_{a_r} \) and the edges incident with \( v^i_b \) be \( e^i_{b_1}, e^i_{b_2}, \ldots, e^i_{b_r} \). We consider two cases.

- **Case 1** : For \( i \) odd, 
  \[
  w_g(v^i_a) = f(v_a) + \sum_{s=1}^{r} f(e^i_{a_s}) + (r + 1) \left( (i - 1) t - \left( \frac{i-1}{2} \right) \right) \\
  \leq (r + 1) t + (r + 1) \left( (i - 1) t - \left( \frac{i-1}{2} \right) \right) \\
  = (r + 1) t - (r + 1) \left( \frac{i}{2} \right) + r \quad (13)
  \]

  On the other hand, 
  \[
  w_g(v^i_b) = f(v_b) + \sum_{s=1}^{r} f(e^i_{b_s}) + (r + 1) \left( (j - 1) t - \left( \frac{j}{2} \right) \right) + r \\
  \geq (r + 1) + (r + 1) \left( (j - 1) t - \left( \frac{j}{2} \right) \right) + r \\
  = (r + 1) t - (r + 1) \left( \frac{j-1}{2} \right) + r \\
  > (r + 1) t - (r + 1) \left( \frac{j-1}{2} \right) + r \quad (14)
  \]

  From (13) and (14), we obtain \( w_g(v^i_a) > w_g(v^i_b) \).

- **Case 2** : For \( i \) even, 
  \[
  w_g(v^i_a) = f(v_a) + \sum_{s=1}^{r} f(e^i_{a_s}) + (r + 1) \left( (i - 1) t - \left( \frac{i}{2} \right) \right) + r \\
  \leq (r + 1) t + (r + 1) \left( (i - 1) t - \left( \frac{i}{2} \right) \right) + r \\
  = (r + 1) t - (r + 1) \left( \frac{i}{2} \right) + r \quad (15)
  \]

  Moreover, 
  \[
  w_g(v^i_b) = f(v_b) + \sum_{s=1}^{r} f(e^i_{b_s}) + (r + 1) \left( (j - 1) t - \left( \frac{j-1}{2} \right) \right) \\
  \geq (r + 1) + (r + 1) \left( t - \left( \frac{j}{2} \right) \right) \\
  > (r + 1) t - (r + 1) \left( \frac{j}{2} \right) + r \quad (16)
  \]

  From (15) and (16), we obtain \( w_g(v^i_b) > w_g(v^i_a) \).
Hence, \( w_g(v^p_u) \neq w_g(v^q_w) \) for all vertices \( v^p_u \in G_p \) and \( v^q_w \in G_q \) with \( p \neq q \), \( p, q \in \{1, 2, \ldots, m\} \), and \( u, w \in \{1, 2, \ldots, n\} \).

It can easily be seen that the maximum label of \( g \) is not greater than \( t + (m - 1)t - \left\lfloor \frac{m-1}{2} \right\rfloor = m t - \left\lfloor \frac{m-1}{2} \right\rfloor \).

Since there are no two edges of the same weight and there are no two vertices of the same weight in \( mG \), \( g \) is a totally irregular total \( m \) labels of \( mG \). We can conclude that

\[
\text{ts}(mG) \leq m(\text{ts}(G)) - \left\lfloor \frac{m-1}{2} \right\rfloor.
\]

The upper bound in Theorem 2.1 can be decreased for some graphs.

**Theorem 2.2** Let \( G \) be an \( r \)-regular connected graph with \( r \geq 1 \). Let \( f \) be an optimal labeling of \( G \) such that \( w_f(e) < 3 \text{ts}(G) \) for every \( e \in E(G) \) and \( w_f(v) < (r + 1) \text{ts}(G) \) for every \( v \in V(G) \). Then,

\[
\text{ts}(mG) \leq m(\text{ts}(G)) - 1 + 1.
\]

**Proof.** Let \( G = (V, E) \) be an \( r \)-regular graph with order \( n \), \( \text{ts}(G) = t \), and \( f \) be an optimal labeling of \( G \). Then, \( |E| = \frac{nr}{2} \). Let \( mG \) be \( m \) copies of \( G \) where the copies of \( G \) are denoted by \( G_i \) for \( 1 \leq i \leq m \). Let \( (G) = \{v_1, v_2, \ldots, v_n\} \), \( E(G) = \{e_1, e_2, \ldots, e_{\frac{nr}{2}}\} \), \( V(G_i) = \{v^i_1, v^i_2, \ldots, v^i_n\} \), and \( E(G_i) = \{e^i_1, e^i_2, \ldots, e^i_{\frac{nr}{2}}\} \), where \( G_i \) is isomorphic with \( G \) with the isomorphism

\[
s: V(G) \cup E(G) \rightarrow V(G_i) \cup E(G_i),
\]

where \( s(v_a) = v^i_a \) and \( s(e_x) = e^i_x \) for every \( 1 \leq i \leq m \), \( 1 \leq a \leq n \), and \( 1 \leq x \leq \frac{nr}{2} \).

Define a total labeling \( g \) of \( mG \) as follows:

1) \( g(v^i_a) = f(v_a) + (i - 1)(t - 1) \);
2) \( g(e^i_x) = f(e_x) + (i - 1)(t - 1) \);

for every \( 1 \leq i \leq m \), \( 1 \leq a \leq n \), and \( 1 \leq x \leq \frac{nr}{2} \).

Next, it will be shown that in the labeling \( g \), there are no two edges of the same weight and there are no two vertices of the same weight.
1. It will be shown that there are no two edges in $G_i$ with the same weight for every $i$, $1 \leq i \leq m$.

Let $e^i_x = v^i_a v^i_b$ be an edge in $G_i$ for $1 \leq i \leq m$. Then,

$$w_g(e^i_x) = g(v^i_a) + g(e^i_x) + g(v^i_b) = f(v_a) + f(e_x) + f(v_b) + (i-1)(t-1)$$

$$= w_f(e_x) + 3(i-1)(t-1).$$

Since $w_f(e_x) \neq w_f(e_y)$ for every $x \neq y$ and $3(i-1)(t-1)$ is a constant, $w_g(e^i_x) \neq w_g(e^i_y)$ for every $x \neq y$, $1 \leq i \leq m$, and $x, y \in \{1, 2, \ldots, \frac{nr}{2}\}$.

2. Define $j = i + 1$ for $1 \leq i \leq m$. Let $e^i_x = v^i_a v^i_b$ and $e^j_y = v^j_c v^j_d$. Then,

$$w_g(e^i_x) = f(v_a) + f(e_x) + f(v_b) + 3(i-1)(t-1)$$

$$< 3t + 3(i-1)(t-1)$$

$$= 3i(t-1) + 3.$$  \hspace{1cm} (17)

On the other hand,

$$w_g(e^j_y) = f(v_c) + f(e_y) + f(v_d) + 3(j-1)(t-1)$$

$$\geq 3 + 3(j-1)(t-1)$$

$$= 3 + 3i(t-1).$$  \hspace{1cm} (18)

From (17) and (18), $w_g(e^j_y) > w_g(e^i_x)$.

Hence, $w_g(e^p_u) \neq w_g(e^q_w)$ for all edges $e^p_u \in G_p$ and $e^q_w \in G_q$ with $p \neq q$, $p, q \in \{1, 2, \ldots, m\}$, and $u, w \in \{1, 2, \ldots, \frac{nr}{2}\}$.

1. It will be shown that there are no two vertices in $G_i$ with the same weight for every $i$, $1 \leq i \leq m$.

Let $v^i_a$ be a vertex in $G_i$ for $1 \leq i \leq m$. Let the edges incident with $v^i_a$ be $e^i_{a_1}, e^i_{a_2}, \ldots, e^i_{a_r}$. Then,
\[
w_g(v^i_a) = f(v_a) + (i-1)(t-1) + \sum_{s=1}^{r} (f(a_s) + (i-1)(t-1)) \\
= f(v_a) + \sum_{s=1}^{r} f(a_s) + (r+1)(i-1)(t-1) \\
= w_f(v_a) + (r+1)(i-1)(t-1).
\]

Since \(w_f(v_a) \neq w_f(v_b)\) for every \(a \neq b\) and \((r+1)(i-1)(t-1)\) is a constant, \(w_g(v^i_a) \neq w_g(v^i_b)\) for every \(a \neq b\), \(1 \leq i \leq m\), and \(a, b \in \{1, 2, \ldots, n\}\).

2. It will be shown that \(w_g(v^i_u) \neq w_g(v^i_w)\) for all vertices \(v^i_u \in G_p\) and \(v^i_w \in G_q\) with \(p \neq q\), \(p, q \in \{1, 2, \ldots, m\}\), and \(u, w \in \{1, 2, \ldots, n\}\).

Define \(j = i + 1\) for \(1 \leq i \leq m\). Let the edges incident with \(v^i_a\) be \(e_{a_1}, e_{a_2}, \ldots, e_{a_r}\) and the edges incident with \(v^j_b\) be \(e_{b_1}, e_{b_2}, \ldots, e_{b_r}\). Then,

\[
w_g(v^i_a) = w_f(v_a) + (r+1)(i-1)(t-1) \\
< (r+1)t + (r+1)(i-1)(t-1) \tag{19} \\
= (r+1)(t-1)i + (r+1).
\]

Also,

\[
w_g(v^j_b) = w_f(v_a) + (r+1)(j-1)(t-1) \\
\geq (r+1) + (r+1)(t-1)i \tag{20}
\]

From (19) and (20), \(w_g(v^i_a) > w_g(v^j_b)\). Hence, the claim follows.

It can easily be seen that the maximum label of \(g\) is \(t + (m-1)(t-1) = m(t-1) + 1\).

Since there are no two edges of the same weight and there are no two vertices of the same weight in \(mG\), \(g\) is a totally irregular total \((m(t-1) + 1)\)-labeling of \(mG\). We can conclude that

\[ts(mG) \leq m(ts(G) - 1) + 1.\]

In the third theorem, we determine a dual labeling of a totally irregular total \(k\)-labeling of arbitrary regular graph.

**Definition 2.1** Let \(G\) be an \(r\)-regular graph. Let \(f\) be an optimal labeling of \(G\). The dual labeling of \(f\), denoted by \(\hat{f}\), is defined by

\[\hat{f}(v) = ts(G) + 1 - f(v), \forall v \in V(G)\]
\[ \hat{f}(e) = ts(G) + 1 - f(e), \forall e \in E(G). \]

**Theorem 2.3.** Let \( G \) be an \( r \)-regular graph. Let \( f \) be an optimal labeling of \( G \). Then, \( \hat{f} \) is also an optimal labeling of \( G \).

**Proof.** It will be shown that in the labeling \( \hat{f} \), there are no two edges of the same weight and there are no two vertices of the same weight.

Let \( c = u_i v_i \) and \( d = u_j v_j \) be different edges in \( E(G) \). Then,

\[
\begin{align*}
\hat{w}_f(c) &= \hat{f}(u_i) + \hat{f}(c) + \hat{f}(v_i) \\
&= ts(G) + 1 - f(u_i) + ts(G) + 1 - f(c) + ts(G) + 1 - f(v_i) \\
&= 3(ts(G) + 1) - (f(u_i) + f(c) + f(v_i)) \\
&= 3(ts(G) + 1) - w_f(c).
\end{align*}
\]

Also,

\[
\begin{align*}
\hat{w}_f(d) &= \hat{f}(u_j) + \hat{f}(d) + \hat{f}(v_j) \\
&= ts(G) + 1 - f(u_j) + ts(G) + 1 - f(d) + ts(G) + 1 - f(v_j) \\
&= 3(ts(G) + 1) - (f(u_j) + f(d) + f(v_j)) \\
&= 3(ts(G) + 1) - w_f(d).
\end{align*}
\]

Since \( \hat{w}_f(c) \neq \hat{w}_f(d) \) for every \( c \neq d \) and \( 3(ts(G) + 1) \) is a constant, \( \hat{w}_f(c) \neq \hat{w}_f(d) \).

Let \( v_a \) and \( v_b \) be different vertices in \( G \). Let the edges incident with \( v_a \) be \( e_{a_1}, e_{a_2}, \ldots, e_{a_r} \) and the edges incident with \( v_b \) be \( e_{b_1}, e_{b_2}, \ldots, e_{b_r} \). Then,

\[
\begin{align*}
\hat{w}_f(v_a) &= \hat{f}(v_a) + \sum_{s=1}^{r} \hat{f}(e_{a_s}) \\
&= ts(G) + 1 - f(v_a) + r(ts(G) + 1) - \sum_{s=1}^{r} f(e_{a_s}) \\
&= (r + 1)(ts(G) + 1) - (f(v_a) + \sum_{s=1}^{r} f(e_{a_s})) \\
&= (r + 1)(ts(G) + 1) - w_f(v_a).
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\hat{w}_f(v_b) &= \hat{f}(v_b) + \sum_{s=1}^{r} \hat{f}(e_{b_s}) \\
&= ts(G) + 1 - f(v_b) + r(ts(G) + 1) - \sum_{s=1}^{r} f(e_{b_s}) \\
&= (r + 1)(ts(G) + 1) - (f(v_b) + \sum_{s=1}^{r} f(e_{b_s})) \\
&= (r + 1)(ts(G) + 1) - w_f(v_b).
\end{align*}
\]
Since \( w_f(v_a) \neq w_f(v_b) \) for every \( v_a \neq v_b \) and \((r + 1)(ts(G) + 1)\) is a constant, \( w_f(v_a) \neq w_f(v_b) \).

Therefore, in the labeling \( \tilde{f} \), there are no two edges of the same weight and there are no two vertices of the same weight. Moreover, the maximum label of \( \tilde{f} \) is less than or equal to \( ts(G) \). We can conclude that \( \tilde{f} \) is an optimal labeling of \( G \).

\[ \square \]

Three last theorems in this paper consider the total irregularity strength of a 1-regular graph, a 2-regular graph, and a 3-regular graph.

**Theorem 2.4** Let \( P_2 \) be a path with 2 vertices. Then, \( ts(mP_2) = m + 1 \) for \( m \geq 1 \).

**Proof.** The graph \( mP_2 \) has \( 2m \) vertices and \( m \) edges and is 1-regular graph. From (1) and (3), we get \( tvs(mP_2) \geq \left\lceil \frac{2m+1}{2} \right\rceil = m + 1 \) and \( tes(mP_2) \geq \left\lceil \frac{m+2}{3} \right\rceil \). Therefore, from (5), we get \( ts(mP_2) \geq m + 1 \). Besides that, from (7) we get \( ts(P_2) = 2 \).

Let the vertex set of \( P_2 \) be \( \{v_1, v_2\} \). Given a totally irregular total 2-labeling \( f \) of \( P_2 \) as follows:

\[ f(v_i) = i \text{ for } 1 \leq i \leq 2; \quad f(v_1v_2) = 1. \]

It can be seen that \( f \) is an optimal labeling of \( P_2 \) such that \( w_f(v_1v_2) < 3(ts(P_2)) \) and \( w_f(v) < 2(ts(P_2)) \) for every \( v \in V(P_2) \). Therefore, from Theorem 2.2, we get

\[ ts(mP_2) \leq m(ts(P_2) - 1) + 1 \]
\[ = m(2 - 1) + 1 \]
\[ = m + 1. \]

We conclude that \( ts(mP_2) = m + 1 \).

\[ \square \]

**Theorem 2.5.** Let \( C_n \) be a cycle of order \( n \). For \( n \geq 3 \) and \( n \equiv 0 \text{ mod } 3 \), \( ts(mC_n) = \left\lceil \frac{mn+2}{3} \right\rceil \).

**Proof.** The \( mC_n \) has \( mn \) vertices and \( mn \) edges and is 2-regular. From (1), (3), and (5), we get

\[ ts(mC_n) \geq \left\lceil \frac{mn+2}{3} \right\rceil. \]  

(21)
Next, we will prove that $ts(mC_n) \leq \left\lceil \frac{mn+2}{3} \right\rceil$.

Let the disconnected graph $C_n$ consists of the vertex and edge set as follows:

- $V(C_n) = \{v_i \mid 1 \leq i \leq n\}$;
- $E(C_n) = \{e_i = v_i v_{i+1} \mid 1 \leq i \leq n\}$;

where the subscript $n+1$ is replaced by 1.

From (6), we get $ts(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$.

Given a totally irregular total $\left\lceil \frac{n+2}{3} \right\rceil$-labeling $f$ of $C_n$ for $n \equiv 0 \mod 3$ as follows:

$$f(v_i) = \begin{cases} \left\lfloor \frac{i}{3} \right\rfloor + \left\lfloor \frac{i}{3} \right\rfloor + 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ \left\lfloor \frac{n-i+1}{3} \right\rfloor + \left\lfloor \frac{n-i+1}{3} \right\rfloor + 1 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n; \end{cases}$$

$$f(e_i) = \begin{cases} \left\lfloor \frac{i}{3} \right\rfloor + \left\lfloor \frac{i+1}{3} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ \left\lfloor \frac{n-i}{3} \right\rfloor + \left\lfloor \frac{n-i+1}{3} \right\rfloor + 1 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n. \end{cases}$$

The labeling gives the weight of vertices and the weight of edges of $C_n$ as follows:

$$w_f(v_i) = \begin{cases} 3 & \text{for } i = 1; \\ 2i & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1; \\ 2(n-i) + 5 & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n; \end{cases}$$

$$w_f(e_i) = \begin{cases} 2i + 1 & \text{for } 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor; \\ 2(n-i) + 4 & \text{for } \left\lfloor \frac{m}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

Hence, there are no two vertices of the same weight and there are no two edges of the same weight. Moreover, it can be seen $w_f(e) < 3(ts(C_n))$ for every $e \in E(C_n)$ and $w_f(v) < 3(ts(C_n))$ for every $v \in V(P_2)$. Therefore, from Theorem 2.2, we get
\[ ts(mC_n) \leq m(ts(C_n) - 1) + 1 \]
\[ = m\left(\left\lceil \frac{n+2}{3} \right\rceil - 1 \right) + 1 \]
\[ = m\left(\frac{n}{3} + 1 - 1 \right) + 1 \]
\[ = m\left(\frac{n}{3}\right) + 1 \]
\[ = \left\lceil \frac{mn+2}{3} \right\rceil. \] (22)

From (21) and (22), we conclude that \( ts(mC_n) = \left\lceil \frac{mn+2}{3} \right\rceil \) for \( n \equiv 0 \mod 3 \).

**Theorem 2.6** For \( n \geq 3 \), \( ts(m(C_n \square P_2)) = mn + 1 \).

**Proof.** The graph \( m(C_n \square P_2) \) has \( 2mn \) vertices and \( 3mn \) edges and is 3-regular. From (1) and (3), we get \( tvs(m(C_n \square P_2)) \geq \left\lceil \frac{2mn+3}{4} \right\rceil \) and \( tes(m(C_n \square P_2)) \geq mn + 1 \). Therefore, from (5), \( ts(m(C_n \square P_2)) \geq nm + 1 \). Moreover, from (8), \( ts(C_n \square P_2) = n + 1 \).

In [14], Ramdani and Salman gave an optimal labeling \( f \) of \( C_n \square P_2 \) such that \( \omega_f(e) < 3(ts(C_n \square P_2)) \) for every \( e \in E(C_n \square P_2) \) and \( \omega_f(v) < 4(ts(C_n \square P_2)) \) for every \( v \in V(C_n \square P_2) \). Therefore, from Theorem 2.2, we get

\[ ts(m(C_n \square P_2)) \leq m(ts(C_n \square P_2) - 1) + 1 \]
\[ = m(n + 1 - 1) + 1 \]
\[ = mn + 1. \]

We conclude that \( ts(m(C_n \square P_2)) = mn + 1 \). \( \blacksquare \)

**References**


