Trees with Certain Locating-Chromatic Number

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Abstract. The locating-chromatic number of a graph $G$ can be defined as the cardinality of a minimum resolving partition of the vertex set $V(G)$ such that all vertices have distinct coordinates with respect to this partition and every two adjacent vertices in $G$ are not contained in the same partition class. In this case, the coordinate of a vertex $v$ in $G$ is expressed in terms of the distances of $v$ to all partition classes. This concept is a special case of the graph partition dimension notion. Previous authors have characterized all graphs of order $n$ with locating-chromatic number either $n$ or $n-1$. They also proved that there exists a tree of order $n$, $n \geq 5$, having locating-chromatic number $k$ if and only if $k \in \{3,4,\ldots,n-2,n\}$. In this paper, we characterize all trees of order $n$ with locating-chromatic number $n-t$, for any integers $n$ and $t$, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$.

Keywords: color code; leaves; locating-chromatic number; stem; tree.

1 Introduction

The topic of locating-chromatic number was introduced by Chartrand, et al. [1] in 2002. They determined the locating-chromatic numbers of paths, cycles, and double stars. Inspired by Chartrand, et al., other authors have determined the locating-chromatic numbers of some well known classes of graphs, i.e. amalgamation of stars and firecrackers by Asmiati, et al. [2,3], Kneser graphs by Behtoei and Omoomi [4], Halin graphs by Purwasih, et al. [5], Cartesian product of graphs and joint product graphs by Behtoei and Omoomi [6] and Behtoei [7], and homogeneous lobster graphs by Syofyan, et al. [8].

Let $G = (V,E)$ be a connected graph. We define the distance as the minimum length of path connecting vertices $u$ and $v$ in $G$, denoted by $d(u,v)$. A $k$-coloring of $G$ is a function $c: V(G) \rightarrow \{1,2,\ldots,k\}$ where $c(u) \neq c(v)$ for any two adjacent vertices $u$ and $v$ in $G$. Thus, the coloring $c$ induces a partition $\Pi$ of $V(G)$ into $k$ color classes (independent sets) $C_1, C_2, \ldots, C_k$ where $C_i$ is the set of all vertices colored by $i$ for $1 \leq i \leq k$. The color code $c_i(v)$ of a vertex $v$ in $G$ is defined as the $k$-vector $(d(v,C_1), d(v,C_2), \ldots, d(v,C_k))$ where $d(v,C_i) = \min\{d(v,x) \mid x \in C_i\}$ for $1 \leq i \leq k$. The $k$-coloring $c$ of $G$ such that all
vertices have different color codes is called a locating coloring of $G$. The least integer $k$ is such that there is a locating coloring in $G$ that is called the locating-chromatic number of $G$, denoted by $\chi_L(G)$.

Chartrand, et al. in [1] have determined all graphs of order $n$ with locating-chromatic number $n$, namely a complete multipartite graph of $n$ vertices. Furthermore in Chatrand, et al. [9], all graphs of order $n$ with locating-chromatic number $n - 1$ were characterized. Chartrand, et al. [1] also proved that there exists a tree of order $n$, $n \geq 5$, having locating-chromatic number $k$ if and only if $k \in \{3, 4, ..., n - 2, n\}$. Recently, Baskoro and Asmiati [10] have characterized all trees with locating-chromatic number $3$. In this investigation, we have characterized all trees of order $n$ with locating-chromatic number $n - t$, for any integers $n$ and $t$, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$.

The following results were proved in Chartrand, et al. [1].

**Lemma 1.** Let $G$ be a simple, connected and non-directed graph. Let function $c$ be a locating coloring of $G$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for every $w \in V(G) \setminus \{u, v\}$, then $c(u) \neq c(v)$.

**Corollary 1.** If $G$ is a connected graph containing a vertex that is adjacent to $k$ leaves of $G$, then $\chi_L(G) \geq k + 1$.

### 2 Main Results

In the following theorem, we provide a method to construct a tree $T$ of order $n$ from any tree of smaller order $t + 1$ where $n > 5$ and $2 \leq t < \frac{n}{2}$, such that $\chi_L(T) = n - t$. A vertex $v$ of degree $\geq 2$ in a tree $T$ is called a stem if it is adjacent to a leaf.

**Theorem 1.** For any integer $n$ and $t$, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$, let $T_{t+1}$ be any tree of order $t + 1$. Let $T_n$ be a tree of order $n$ obtained by joining $n - t - 1$ new vertices to a vertex $x \in V(T_{t+1})$, where $x$ is not a stem. Then, $\chi_L(T_n) = n - t$.

**Proof.** Let $V(T_n) = \{x, y_i, z_j | 1 \leq i \leq n - t - 1, 1 \leq j \leq t\}$, where $x$ is adjacent to $n - t - 1$ leaves, $y_i$ are the leaves adjacent to $x$, and $z_j$ are other vertices in $T_n$. Define a $(n - t)$-coloring $c: V(T_n) \to \{1, 2, ..., n - t\}$ as follows:

1. $c(x) = n - t$,
2. \( c(y_i) = i \) for \( 1 \leq i \leq n - t - 1 \),
3. \( c(z_j) = j \) for \( 1 \leq j \leq t \).

Next, we show that the color codes of all vertices under the coloring \( c \) are distinct. We only consider pairs of vertices with the same color. The possibilities are for the pairs of vertices \( y_i \) and \( z_j \) for some \( i, j \). If \( z_j \) is adjacent to \( x \), then \( c_\Pi(y_i) \neq c_\Pi(z_j) \) because \( c_\Pi(y_i) \) contains exactly one entry 1, while \( c_\Pi(z_j) \) contains at least two entries 1. If \( z_j \) is not adjacent to \( x \) then \( c_\Pi(y_i) \neq c_\Pi(z_j) \) because \( c_\Pi(y_i) \) contains entry 1 in the \((n-t)\)th ordinate, while \( c_\Pi(z_j) \) does not contain entry 1 in the \((n-t)\)th ordinate. Since every vertex of \( T_n \) has distinct color codes, \( c \) is a locating coloring of \( T_n \). So, \( \chi_L(T_n) \leq n - t \).

Now, since \( T_n \) contains a vertex \( x \) that is adjacent to \( n - t - 1 \) leaves, then by Corollary 1, \( \chi_L(T_n) \geq n - t \). Hence, \( \chi_L(T_n) = n - t \). \( \square \)

In the following theorem, we give a necessary condition of a tree of order \( n \) whose locating-chromatic number is \( n - t \), where \( 2 \leq t < \frac{n}{2} \).

**Theorem 2.** For any integer \( n \) and \( t \), where \( n > t + 3 \) and \( 2 \leq t < \frac{n}{2} \), let \( T_n \) be a tree of order \( n \). If \( \chi_L(T_n) = n - t \), then \( T_n \) has exactly one stem with \( n - t - 1 \) leaves.

**Proof.** Since \( \chi_L(T_n) = n - t \), every stem of \( T_n \) is adjacent to at most \( n - t - 1 \) leaves. Suppose that there is no stem of \( T_n \) having \( n - t - 1 \) leaves. Then every stem of \( T_n \) is adjacent to at most \( n - t - 2 \) leaves. Furthermore, we have a locating coloring for \( T_n \) by using \( n - t - 1 \) colors as follows.

Let there be \( b \) stems in \( T_n \). First, we denote all stems of \( T_n \) by \( s_i \), for \( 1 \leq i \leq b \), the leaves of \( T_n \) adjacent to \( s_i \) by \( l_{ij} \), for \( 1 \leq i \leq b \) and \( 1 \leq j \leq n - t - 2 \), and the remaining vertices by \( v_k \), for \( 0 \leq k \leq n - 4 \). Let \( N(s_i) \) be the set of neighbors of \( s_i \), for \( 1 \leq i \leq b \). For a coloring \( c \) of \( V(T_n) \), define \( c(N(s_i)) = \{ c(x) \mid x \in N(s_i) \} \). Now, define a \((n-t-1)\)-coloring \( c \) of \( T_n \) with the following steps:

1. For all stems \( s_i \), define \( c(s_i) = 1 \) or 2 such that if there are at least two stems adjacent to the same \( v_k \) for some \( k \), then two of these stems receive different colors.
2. For all vertices \( v_k \) adjacent to a stem, assign \( c(v_k) = \alpha \), for some \( \alpha \in \{3, 4, 5, \ldots, n - t - 1\} \) such that \( c(v_k) \neq c(v_l) \) for \( k \neq l \).
3. For all vertices \( v_k \) not adjacent to a stem, define \( c(v_k) = a \), for some \( a \in \{3, 4, 5, ..., n - t - 1\} \) such that \( c(v_k) \neq c(v_l) \) if \( d(v_k, C_i) = d(v_l, C_i) \) for \( i = 1, 2 \).

4. For all leaves \( l_{ij} \), define \( c(l_{ij}) = a \), for some \( a \in \{1, 2, ..., n - t - 1\} \) such that all vertices (including leaves) adjacent to stems \( s_i \) and \( s_p \) satisfy \( c(N(s_i)) \neq c(N(s_p)) \) for any \( i \neq p \).

Figure 1: Trees \( H_1, H_2, H_3, H_4, H_5, H_6 \).

Observe that, with the exception of the six trees depicted in Figure 1, the coloring \( c \) can always be done for any tree \( T_n, n \geq 6 \). Meanwhile, for all trees in Figure 1 we cannot use the coloring \( c \) because the number of colors is smaller than the number of vertices \( v_k \) adjacent to a stem. However, we can define another coloring for \( T_n \) in Figure 1 by \( n - t - 1 \) colors such that if \( n = 8, 9, 10, 11, 13 \) then \( t = 3, 4, 5, 6 \), respectively.

Next, we show that \( c \) is a locating coloring of \( T_n \). Let \( x \) and \( y \) be two vertices of \( T_n \) such that \( c(x) = c(y) \). We distinguish five cases:

**Case 1.** \( x = s_i \) and \( y = s_j \), for \( i \neq j \).

Since \( c(N(s_i)) \neq c(N(s_j)) \) for \( i \neq j \) (from Step (4)), we obtain that \( c_\Pi(x) \neq c_\Pi(y) \).

**Case 2.** \( x = v_k \) and \( y = v_l \), for \( k \neq l \).
If $v_k$ is adjacent to a stem $s_i$ and $v_l$ is not adjacent to any stem, then $c_{\Pi}(v_k) \neq c_{\Pi}(v_l)$ because $c_{\Pi}(v_k)$ contains entry 1 in the first or second ordinate, while $c_{\Pi}(v_l)$ does not contain entry 1 in the first and second ordinate (from Step (1), (2) and (3)).

**Case 3.** $x = l_{ij}$ and $y = l_{pq}$, for $i \neq p$.
Since $c(N(s_i)) \neq c(N(s_p))$ for $i \neq p$ (from step (4)), we obtain that $c_{\Pi}(x) \neq c_{\Pi}(y)$.

**Case 4.** $x = s_i$ and $y = l_{pq}$.
Since $c(N(s_i)) \neq c(N(s_p))$ for $i \neq p$ (from step (4)), we obtain that $c_{\Pi}(x) \neq c_{\Pi}(y)$.

**Case 5.** $x = v_k$ and $y = l_{ij}$.
Then there are two possibilities for this case:

i) If $v_k$ is adjacent to a stem, then $c_{\Pi}(v_k) \neq c_{\Pi}(l_{ij})$ because $c_{\Pi}(v_k)$ contains at least two entries 1, while $c_{\Pi}(l_{ij})$ contains exactly one entry 1 (from Step (1),(2),(4)).

ii) If $v_k$ is not adjacent to any stem, then $c_{\Pi}(v_k) \neq c_{\Pi}(l_{ij})$ because $c_{\Pi}(v_k)$ does not contain entry 1 in the first and second ordinate, while $c_{\Pi}(l_{ij})$ contains entry 1 in the first or second ordinate (from Step (1),(3),(4)).

By the above cases, we prove that $c$ is a locating coloring of $T_n$. Then $\chi_L(T_n) \leq n - t - 1$, which contradicts $\chi_L(T_n) = n - t$. Hence, there is a stem of $T$ having $n - t - 1$ leaves.

Next, we will show that there is only one stem of $T_n$ having $n - t - 1$ leaves. We suppose that there are two stems of $T_n$ adjacent to $n - t - 1$ leaves. Then, $|V(T_n)| \geq 2(n - t)$. Since $t < \frac{n}{2}$, $|V(T_n)| \geq 2(n - t) > n$, a contradiction with $|V(T_n)| = n$. □

Applying Theorem 1 and Theorem 2, we obtain the following theorem.

**Theorem 3.** For any integer $n$ and $t$, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$, let $T_n$ be a tree of order $n$. Then $\chi_L(T_n) = n - t$ if and only if $T_n$ has exactly one stem with $n - t - 1$ leaves.

Based on Theorem 3, we can determine all trees $T_n$ on $n$ vertices with $\chi_L(T_n) = n - t$ for any integers $n$ and $t$, where $n > t + 3$ and $2 \leq t < \frac{n}{2}$. In particular, if $t = 2, 3, \text{ or } 4$, all trees $T_n$ with $\chi_L(T_n) = n - t$ are the caterpillars shown in Figures 2, 3, and 4. But for $t \geq 5$, there are trees $T_n$ on $n$ vertices other than caterpillars with $\chi_L(T_n) = n - t$, for example the cases of $t = 5$ and 6, all trees
with $\chi_L(T_n) = n - 5$ and $\chi_L(T_n) = n - 6$ are depicted in Figure 5 and Figure 6, respectively. Therefore, as a special case of Theorem 3, we have the following corollary.

First, we give the definition of a caterpillar. Let $P_m = x_1x_2...x_m$ be a path with $m$ vertices. A caterpillar $C(m; n_1, n_2, ..., n_m)$, is obtained by joining $n_i$ new vertices to every vertex $x_i$ in a path $P_m$, $n_i \geq 0$, $1 \leq i \leq m$.

**Corollary 2.** For any integer $n$ and $t$, where $n > t + 3$ and $t = 2, 3, 4$, let $T_n$ be a tree of order $n$. Then $\chi_L(T_n) = n - t$ if and only if $T_n$ is a caterpillar $C(m; n_1, n_2, ..., n_m)$ where $0 \leq n_i \leq n - t - 1$, $2 \leq m \leq t$, $n_1, n_m \neq 0$, and exactly one of $n_i$ is equal to $n - t - 1$.

All caterpillars in the Corollary 2 are shown in Figure 2, 3, and 4.
Figure 4 (continued) All trees of order $n > 7$ with locating chromatic number $n - 4$.

Figure 5 All trees of order $n > 8$ with locating chromatic number $n - 5$. 
Figure 6  All trees of order $n > 9$ with locating chromatic number $n - 6$.

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