Inclusion Properties for a Class of Meromorphic Functions Defined by a Linear Operator

Firas Ghanim¹ & Maslina Darus²

¹Department of Mathematics, College of Sciences, University of Sharjah, Sharjah, United Arab Emirates
²School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600 Selangor D. Ehsan, Malaysia
E-mail: fgahmed@sharjah.ac.ae

Abstract. This study targets a specific class of meromorphic univalent functions \( f(z) \) defined by the linear operator \( L(a,b)f(z) \). This paper aims to demonstrate some properties for the class \( \Sigma_{a,b}^{k,\lambda}(h) \) to satisfy a certain subordination.

Keywords: meromorphic functions; hypergeometric functions; subordination; linear operator; Hadamard product (convolution).

AMS Subject Classification: 30C45.

1 Introduction

Let \( \Sigma \) denote the class of meromorphic functions \( f(z) \) normalized by

\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,
\]

which are analytic in the punctured unit disk

\( \Delta^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \Delta \setminus \{0\} \).

For functions \( f_k(z) \) \((k = 1,2)\) given by

\[
f_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,k} z^n \quad (k = 1,2),
\]

we denote the Hadamard product (or convolution) of \( f_1(z) \) and \( f_2(z) \) by

\[
(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.
\]
Let the function $\phi(a,b;z)$ be defined by
\[
\phi(a,b;z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_{n+1}} z^n,
\]
for $b \neq 0,-1,-2,...$, and $a \in \mathbb{C} \setminus \{0\}$.

Here, and in the remainder of this paper, $(\lambda)_{(\lambda,\kappa \in \mathbb{C})}$ denotes the general Pochhammer symbol defined, in terms of the gamma function, by
\[
(\lambda)_{\kappa} = \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)} = \begin{cases} \frac{1}{\lambda(\lambda+1)...(\lambda+n-1)} & (\kappa = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \frac{\kappa!}{\lambda(\lambda+1)...(\lambda+n-1)} & (\kappa = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}.
\]

Corresponding to the function $\phi(a,b;z)$, using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L(a,b)$ on $\Sigma$ by
\[
L(a,b) f(z) = \phi(a,b;z) * f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_{n+1}} a_n z^n.
\]

The generalized and Gaussian hypergeometric functions together with the meromorphic functions were studied recently by several authors [1-9].

We define the following operator for the function $f \in L(a,b)f(z)$ by
\[
D^0(L(a,b)f(z)) = L(a,b)f(z)
\]
and for $k = 1,2,3,...,
\[
D^k(L(a,b)f(z)) = z \left( D^{k-1}L(a,b)f(z) \right)' + \frac{2}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(a)_{n+1}}{(b)_{n+1}} a_n z^n.
\]

The above differential operator $D^k$ was studied by Ghanim and Darus [10-12].

In addition, we derive from the Eq. (6) and Eq. (7)
\[
z(L(a,b)f(z))' = aL(a+1,b)f(z) - (a+1)L(a,b)f(z).
\]
and
\[
z(D^kL(a,b)f(z))' = aD^kL(a+1,b)f(z) - (a+1)D^kL(a,b)f(z).
\]
respectively.

Let $\Omega$ be the class of all analytic, convex and univalent functions in the open unit disk and let $h(z) \in \Omega$ satisfy $h(0) = 1$, with

$$\Re\{h(z)\} > 0, |z|<1.$$  \hspace{1cm} (10)

For two functions $f, g \in \Omega$, we say that $f$ is subordinate to $g$ or $g$ is superordinate to $f$ in $\Delta$ and write $f \prec g, z \in \Delta$, if there exists a Schwarz function $\omega$, analytic in $\Delta$ with $\omega(0) = 0$ and $|\omega(z)| \leq 1$ when $z \in \Delta$ such that $f(z) = g(\omega(z)), z \in \Delta$. Furthermore, if function $g$ is univalent in $\Delta$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta), \quad (z \in \Delta).$$

**Definition.** If a function $f \in \Sigma$ satisfies the following subordination condition

$$(1 + \lambda)z\left(D^k L(a,b) f(z)\right) + \lambda z^2 \left(D^k L(a,b) f(z)\right)' \prec h(z)$$ \hspace{1cm} (11)

then $f$ is in the class $\Sigma^{1,\lambda}_{a,b}(h)$, where $\lambda$ is a complex number and $h(z) \in \Omega$.

Let $A$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$ \hspace{1cm} (12)

which are analytic in $\Delta$.

A function $f(z) \in A$ is in the class of starlike functions $S^*(\alpha)$ of order $\alpha$ in $\Delta$, if

$$\Re\left[\frac{zf''(z)}{f'(z)}\right] > \alpha \quad (z \in \Delta),$$

for some $\alpha, 0 < \alpha < 1$.

A function $f(z) \in A$ is in the class of prestarlike function $R(\alpha)$ of order $\alpha$ in $\Delta$, if

$$\frac{z}{(1-z)^{\alpha}} f(z) \in S^*(\alpha) \quad (\alpha < 1)$$
Inclusion Properties on a Class of Meromorphic Functions

(see for example [13-15]). $f(z)$ is convex univalent in $\Delta$ and $R\left(\frac{1}{2}\right) = S^*(\frac{1}{2})$ if and only if $f(z) \in R(0)$.

2 Preliminary Results

Lemma 1. [16] Let $g(z)$ and $h(z)$ are two analytic functions in $\Delta$. $h(z)$ is convex univalent with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\mu} zg'(z) < h(z)$$

(13)

where $\Re\mu \geq 0$ and $\mu \neq 0$, then

$$g(z) < \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt < h(z)$$

and $\tilde{h}(z)$ is the best dominant of Eq. (13).

Lemma 2. [13] If $\Re a \geq 0$ and $a \neq 0$, then,

$$\Sigma^{k,a}_{\alpha,\phi}(h) \subset \Sigma^{k,a}_{\alpha,\phi}(h),$$

where

$$\tilde{h}(z) = az^{-a} \int_0^z t^{a-1} h(t) dt < h(z).$$

Lemma 3. [13] If $f(z) \in \Sigma^{k,a}_{\alpha,\phi}(h)$, $g(z) \in \Sigma$ and $\Re(zg(z)) > \frac{1}{2}$ $(z \in \Delta)$, then,

$$(f * g)(z) \in \Sigma^{k,a}_{\alpha,\phi}(h).$$

3 Main Results

Theorem 1. Let $f(z) \in \Sigma^{k,a}_{\alpha,\phi}(h)$. Then $F(z)$ is the function defined by

$$F(z) = \frac{\mu - 1}{\mu} \int_0^z t^{\mu-1} f(t) dt \quad (\Re \mu > 1)$$

(14)

and in the class $\Sigma^{k,a}_{\alpha,\phi}(\tilde{h})$, where
\[ \tilde{h}(z) = (\mu - 1)z^{1-\mu} \int_0^z t^{\mu-2}h(t)dt < h(z). \]

**Proof.** For \( f(z) \in \Sigma \) and \( \Re\mu > 1 \), we can obtain from (14) that \( F(z) \in \Sigma \) and
\[
(\mu - 1)f(z) = \mu F(z) + zF'(z), \quad F(z) \in \Sigma.
\]
Define \( H(z) \) by
\[
H(z) = (1 + \lambda)z\left(D^\lambda L(a,b)F(z)\right) + \lambda z^2\left(D^\lambda L(a,b)F(z)\right).
\]
From Eq. (15) and Eq. (16) it follows that:
\[
(1 + \lambda)z\left(D^\lambda L(a,b)f(z)\right) + \lambda z^2\left(D^\lambda L(a,b)f(z)\right)
= (1 + \lambda)\left(D^\lambda L(a,b)\left(\frac{\mu F(z) + zF'(z)}{\mu - 1}\right)\right) + \lambda z^2\left(D^\lambda L(a,b)\left(\frac{\mu F(z) + zF'(z)}{\mu - 1}\right)\right)
= \frac{\mu}{\mu - 1}H(z) + \frac{1}{\mu - 1}(zH'(z) - H(z)) = H(z) + \frac{zH'(z)}{\mu - 1}. \tag{17}
\]
Let \( f(z) \in \Sigma_{\alpha,\beta}^k(h) \). Then, by Eq. (17)
\[
H(z) + \frac{zH'(z)}{\mu - 1} < h(z) \quad (\Re\mu > 1),
\]
and hence we obtain from Lemma 1:
\[
H(z) < \tilde{h}(z) = (\mu - 1)z^{1-\mu} \int_0^z t^{\mu-2}h(t)dt < h(z).
\]
Thus, Lemma 2 contributes to
\[
F(z) \in \Sigma_{\alpha,\beta}^{k,\lambda}(\tilde{h}) \subset \Sigma_{\alpha,\beta}^{k,\lambda}(h).
\]

**Theorem 2.** Let \( F(z) \) be defined as in Eq. (14) and \( f(z) \in \Sigma \). If
\[
(1 + \alpha)z\left(D^\lambda L(a,b)F(z)\right) + \alpha z\left(D^\lambda L(a,b)f(z)\right) < h(z) \quad (\alpha > 0), \tag{18}
\]
then \( F(z) \in \Sigma_{\alpha,\beta}^k(\tilde{h}) = \Sigma_{\alpha,\beta}^{k,0}(\tilde{h}) \), where \( \Re\mu > 1 \) and
\[
\tilde{h}(z) = \left(\frac{\mu - 1}{\alpha}\right)z^{1-\mu} \int_0^z t^{\mu-1}h(t)dt < h(z).
\]
**Proof.** Let us define the analytic function \( H(z) \) in \( \Delta \) as follows:

\[
H(z) = z\left(D^k L(a,b)F(z)\right)
\]

(19)

with \( H(0) = 1 \), and

\[
zH'(z) = H(z) + z^2 \left(D^k L(a,b)F(z)\right)'.
\]

(20)

By using Eq. (15), Eq. (18), Eq. (19) and Eq. (20), we conclude that:

\[
(1-\alpha)z\left(D^k L(a,b)F(z)\right) + \alpha z\left(D^k L(a,b)f(z)\right)
\]

\[
= (1-\alpha)z\left(D^k L(a,b)F(z)\right) + \frac{\alpha}{\mu - 1}(\mu zD^k L(a,b)F(z)) + z^2 \left(D^k L(a,b)F(z)\right)'
\]

\[
= H(z) + \frac{\alpha}{\mu - 1}zH'(z) < h(z)
\]

for \( \Re\mu > 1 \) and \( \alpha > 0 \).

Therefore, an application of Lemma 1 asserts Theorem 2.

**Theorem 3.** Let \( f(z) \in \Sigma_{a,b}(h) \). If \( F(z) \) is the function given by

\[
F(z) = \frac{\mu - 1}{\mu} \int_0^{\mu} t^{\mu-1} f(t) dt \quad (\mu > 1)
\]

(21)

then,

\[
\sigma f(\sigma z) \in \Sigma_{a,b}(h)
\]

where

\[
\sigma = \sigma(\mu) = \sqrt{\frac{\mu^2 - 2(\mu - 1)}{(\mu - 1)}} \in (0,1).
\]

(22)

When

\[
h(z) = \delta + (1-\delta)\frac{1+z}{1-z} \quad (\delta \neq 1)
\]

(23)

consequently, bound \( \sigma \) is sharp.
Proof. For \( F(z) \in \Sigma_{\Delta_{\mu}}(h) \), we could verify that: \( F(z) = F(z) * \frac{z^{-1}}{1-z} \) and \( zF'(z) = F(z) * \left( \frac{1}{(1-z)^2} - \frac{1}{z(1-z)} \right) \).

Then, using Eq. (21), we obtain:

\[
f(z) = \frac{\mu F(z) + zF'(z)}{\mu - 1} = (F \ast g)(z) \quad (z \in \Delta^+, \mu > 1),
\]

where

\[
g(z) = \frac{1}{\mu - 1} \left( \frac{1}{z(1-z)^2} - (\mu - 1) \frac{1}{z(1-z)} \right) \in \Sigma.
\]

Now, we prove that:

\[
\Re(zg(z)) > \frac{1}{2}, \quad (|z| < \sigma),
\]

where \( \sigma = \sigma(\mu) \) is given by Eq. (22). Setting

\[
\frac{1}{1-z} = \text{Re}^{i\theta} \quad (R > 0, |z| = r < 1)
\]

we have:

\[
\cos \theta = \frac{1 + R^2(1-r^2)}{2R} \quad \text{and} \quad R \geq \frac{1}{1+r}.
\]

By Eq. (25) and Eq. (27) with \( \mu > 1 \), we have:

\[
2\Re(zg(z)) = \frac{2}{\mu - 1} \left[ (\mu - 1) R \cos \theta + R^2 (2 \cos^2 \theta - 1) \right]
\]

\[
= \frac{1}{\mu - 1} \left[ (\mu - 1) (1 + R^2 (1-r^2)) + \left[ 1 + R^2 (1-r^2) \right]^2 - R^2 \right]
\]

\[
= \frac{R^2}{\mu - 1} \left[ R^2 (1-r^2)^2 + \mu (1-r^2) - 1 \right] + 1 \geq \frac{R^2}{\mu - 1} \left[ (1-r^2)^2 + \mu (1-r^2) - 1 \right] + 1
\]

\[
= \frac{R^2}{\mu - 1} \left[ (1-\mu)r^2 + \mu - 2r \right] + 1.
\]
This would eventually give Eq. (26) and hence
\[ \Re(zg(\sigma z)) > \frac{1}{2} \quad (z \in \Delta). \] (28)

Let \( F(z) \in \Sigma_{\alpha, \beta}^\Delta(h) \). Using Eq. (24) and Eq. (28) with Lemma 3, we have:
\[ \sigma f(\sigma z) = F(z) \ast \sigma g(\sigma z) \in \Sigma_{\alpha, \beta}^\Delta(h). \]

For \( h(z) \) defined by Eq. (23), function \( F(z) \in \Sigma \) is given by:
\[
(1 + \lambda)z(D^\lambda L(a, b) F(z)) + \lambda z^2 (D^\lambda L(a, b) F(z))' = \delta + (1 - \delta)\frac{1 + z}{1 - z}.
\] (29) \( (\delta \neq 1) \). By using Eq. (29), Eq. (16) and Eq. (17), we obtain the following:
\[
(1 + \lambda)z(D^\lambda L(a, b) f(z)) + \lambda z^2 (D^\lambda L(a, b) f(z))' \]
\[
= \delta + (1 - \delta)\frac{1 + z}{1 - z} + \frac{z}{\mu - 1} \left( \delta + (1 - \delta)\frac{1 + z}{1 - z} \right)' \]
\[
= \delta + \frac{(1 - \delta)(\mu + 2z - 1 + (1 - \mu)z^2)}{(\mu - 1)(1 - z)^2} = \delta \quad (\sigma = -z).\]

Hence, for each \( \mu (\mu > 1) \) the bound \( \sigma = \sigma(\mu) \) cannot be increased.

**Acknowledgements**

The authors would like to thank the referees for giving suggestions for improving this work. The work was supported by FRGS/1/2016/STG06/UKM/01/1.

**References**


