Expanding Super Edge-Magic Graphs

E. T. Baskoro¹ & Y. M. Cholily¹,²

¹Department of Mathematics, Institut Teknologi Bandung
Jl. Ganesa 10 Bandung 40132, Indonesia
Emails: {ebaskoro,yus}@dns.math.itb.ac.id

²Department of Mathematics, Universitas Muhammadiyah Malang
Jl. Tlogomas 246 Malang 65144, Indonesia
Email: yus@ummm.ac.id

Abstract. For a graph \( G \), with the vertex set \( V(G) \) and the edge set \( E(G) \) an edge-magic total labeling is a bijection \( f \) from \( V(G) \cup E(G) \) to the set of integers \( \{1,2,\cdots,|V(G)|+|E(G)|\} \) with the property that \( f(u)+f(v)+f(uv)=k \) for each \( uv \in E(G) \) and for a fixed integer \( k \). An edge-magic total labeling \( f \) is called super edge-magic total labeling if \( f(V(G))=\{1,2,\cdots,|V(G)|\} \) and \( f(E(G))=\{|V(G)|+1,|V(G)|+2,\cdots,|V(G)|+|E(G)|\} \). In this paper we construct the expanded super edge-magic total graphs from cycles \( C_n \), generalized Petersen graphs and generalized prisms.

Keywords: Edge-magic; super edge-magic; magic-sum.

1 Introduction

All graphs considered here are finite, undirected and simple. As usual, the vertex set and edge set will be denoted \( V(G) \) and \( E(G) \), respectively. The symbol \( |A| \) will be the cardinality of the set \( A \). Other terminologies or notations not defined here can be found in [2,7,15].

Edge-magic total labelings were introduced by Kotzig and Rosa [8] as follow. An edge-magic total labeling on \( G \) is a bijection \( f \) from \( V(G) \cup E(G) \) onto \( \{1,2,\cdots,|V(G)|+|E(G)|\} \) with the property that, given any edge \( uv \),

\[
f(u)+f(v)+f(uv)=k
\]

for some constant \( k \). It will be convenient to call \( f(u)+f(v)+f(uv) \) the edge sum of \( uv \) and \( k \) the magic sum of \( f \). A graph is called edge-magic total if it admits any edge-magic total labeling.

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Kotzig and Rosa [9] showed that no complete graph $K_n$ with $n > 6$ is edge-magic total and neither is $K_4$, and edge-magic total labelings for $K_3, K_5$ and $K_6$ for all feasible values of $k$, are described in [14].

In [8] it is proved that every cycle $C_n$, every caterpillar (a graph derived from a path by hanging any number of pendant vertices from vertices of the path) and every complete bipartite graph $K_{m,n}$ (for any $m$ and $n$) are edge-magic total.

Wallis et.al. [14] showed that all paths $P_n$ and all $n$-suns (a cycle $C_n$ with an additional edge terminating in a vertex of degree 1 attached to each vertex of the cycle) are edge-magic total. It was shown in [16] that the Cartesian product $C_n \times P_m$ admits an edge-magic total labeling for odd $n$.

It is conjectured that all trees are edge-magic total [8] and all wheels $W_n$ are edge-magic total whenever $n \equiv 3 \pmod{4}$ [4]. Enomoto et.al. [4] showed that the conjectures are true for all trees with less than 16 vertices and wheels $W_n$ for $n \leq 30$. Philips et.al. [12] solved the conjecture partially by showing that a wheel $W_n$, $n \equiv 0$ or $1 \pmod{4}$, is edge-magic total. Slamin et.al [13] showed that for $n \equiv 6 \pmod{8}$ every wheel $W_n$ has an edge-magic total labeling.

An edge-magic total labeling $f$ is called super edge-magic total if $f(V(G)) = \{1, 2, \ldots, |V(G)| \}$ and $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)| \}$. Enomoto et.al [4] proved that the complete bipartite graphs $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$. They also proved the complete graphs $K_n$ is super edge-magic if and only if $n = 1, 2$ or $3$.

In this paper we will construct the super edge-magic total graphs by hanging any number of pendant vertices from vertices of the cycles, generalized prisms and generalized Petersen graphs.

2 Results

For $n \geq 3$ and $p \geq 1$ we denote by $C_n + A_p$ a graph which is obtained by adding $p$ vertices and $p$ edges to one vertex of cycles $C_n$ (say $v_1$). The vertex set and the edge set of $C_n + A_p$ are $V(C_n + A_p) = \{v_i : 1 \leq i \leq n\} \cup \{u_j : 1 \leq j \leq p\}$ and $E(C_n + A_p) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i u_j : 1 \leq j \leq p\}$. 
Let \((n,p) - \text{sun}\) be a graph derived from a cycle \(C_n, n \geq 3\), by hanging \(p\) pendant vertices from all vertices of the cycle. Let us denote the vertex set of \((n,p) - \text{sun}\) by \(V((n,p) - \text{sun}) = \{v_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}\) and the edge set by \(E((n,p) - \text{sun}) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_{u_{i,j}} : 1 \leq i \leq n, 1 \leq j \leq p\}\). Observe that \(|V((n,p) - \text{sun})| = |E((n,p) - \text{sun})| = n(p+1)\). The cycle \(C_n, n \geq 3\), is super edge-magic total if and only if \(n\) is odd (see [4]). Now, we shall investigate super edge-magic total labelings for graphs of \(C_n + A_p\) and \((n,p) - \text{sun}\) which are expanded from a cycle \(C_n\).

Define a vertex labeling \(f_1\) and an edge labeling \(f_2\) of \(C_n + A_p\) as follows,

\[
f_1(v_i) = \begin{cases} \frac{n+i}{2} & \text{if } i \text{ odd,} \\ \frac{i}{2} & \text{if } i \text{ even,} \end{cases}
\]

\[
f_1(u_j) = n + j \quad \text{for } 1 \leq j \leq p,
\]

\[
f_2(v_i v_{i+1}) = 2(n+p)+1-i \quad \text{for } 1 \leq i \leq n-1,
\]

\[
f_2(v_n v_1) = n+2p+1,
\]

\[
f_2(v_{u_{i,j}}) = n+2p+1-j \quad \text{for } 1 \leq j \leq p.
\]

**Theorem 1.** If \(n\) is odd, \(n \geq 3\) and \(p \geq 1\), then graph \(C_n + A_p\) is super edge-magic total.

**Proof.** It is easy to verify that the values of \(f_1\) are \(1, 2, \ldots, n+p\) and the values of \(f_2\) are \(n + p + 1, n + p + 2, \ldots, 2n + 2p\) and furthermore the common edge sum is \(k = 2p + \frac{5n+3}{2}\).

**Theorem 2.** If \(n\) is odd, \(n \geq 3\) and \(p \geq 1\), then graph \((n,p) - \text{sun}\) is super edge-magic total.

**Proof.** Label the vertices and the edges of \((n, p) - \text{sun}\) in the following way.

\[
f_3(v_i) = f_1(v_i) \quad \text{for } 1 \leq i \leq n,
\]

\[
f_3(u_{i,j}) = nj + 1 \quad \text{for } 1 \leq j \leq p,
\]

\[
f_3(u_{i,j}) = n(j+1)+2-i \quad \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq p,
\]

\[
f_4(v_i v_{i+1}) = 2(n+p)+1-i \quad \text{for } 1 \leq i \leq n,
\]
We can see that the vertices of \((n, p) - \text{sun}\) are labeled by values \(1, 2, \ldots, n(p+1)\) and the edges are labeled by \(n(p+1)+1, n(p+1)+2, \ldots, 2n(p+1)\). Furthermore, all edges have the same magic number \(k = 2n(p+1) + \frac{n+3}{2}\).

A generalized Petersen graph \(P(n, m)\), \(n \geq 3\) and \(1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor\), consists of an outer \(n\)-cycle \(v_1, v_2, \ldots, v_n\) a set of \(n\) spokes \(v_i z_i\), \(1 \leq i \leq n\), and inner edges \(z_i z_{i+m}\), \(1 \leq i \leq n\), with indices taken modulo \(n\).

For \(n \geq 5\), \(m = 2\) and \(p \geq 1\), we denote by \(P(n, 2) + A_p\) for a graph which is obtained by adding \(p\) vertices and \(p\) edges to one vertex of \(P(n, 2)\), say \(v_1\). Hence, \(V(P(n, 2) + A_p) = V(P(n, 2)) \cup \{u_j : 1 \leq j \leq p\}\) and \(E(P(n, 2) + A_p) = E(P(n, 2)) \cup \{v_j u_j : 1 \leq j \leq p\}\).

Let \(P(n, 2, p)\) be a graph derived from \(P(n, 2)\), \(n \geq 5\), by hanging \(p\) pendant vertices from all vertices \(v_i\), \(1 \leq i \leq n\) of \(P(n, 2)\). Then the vertex set of \(P(n, 2, p)\) is \(V(P(n, 2, p)) = V(P(n, 2)) \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}\) and the edge set is \(E(P(n, 2, p)) = E(P(n, 2)) \cup \{v_{j} u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}\).

In [11] it is proved that generalized Petersen graphs \(P(n, 2)\) are edge-magic total. Fukuchi [6] showed that \(P(n, 2)\) are super edge-magic total.

**Theorem 3.** If \(n\) is odd, \(n \geq 5\) and \(p \geq 1\), then the graph \(P(n, 2) + A_p\) has a super edge-magic total labeling.

**Proof.** Consider a bijection, \(f_5 : V(P(n, 2) + A_p) \rightarrow \{1, 2, \ldots, 2n + p\}\) where,

\[
f_5(v_i) = \begin{cases}  n + \frac{i}{2} & \text{if } i \text{ is even, } 2 \leq i \leq n-1, \\  3n + \frac{i-1}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq n, \end{cases}
\]
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$$f_5(z_i) = \begin{cases} \frac{n-i+4}{4} & \text{if } i \equiv 1 \pmod{4}, \\ \frac{2n-i+4}{4} & \text{if } i \equiv 2 \pmod{4}, \\ \frac{3n-i+4}{4} & \text{if } i \equiv 3 \pmod{4}, \\ \frac{4n-i+4}{4} & \text{if } i \equiv 0 \pmod{4}, \end{cases}$$

$$f_5(u_j) = 2n + j \text{ for } 1 \leq j \leq p.$$ 

We can observe that under the labeling $f_5$, \{ $f_5(v_i) + f_5(v_{i+1}) : 1 \leq i \leq n$\} = \{ $\frac{5n+1}{2} + i : 1 \leq i \leq n$\} and \{ $f_5(z_i) + f_5(z_{i+2}) : 1 \leq i \leq n$\} = \{ $\frac{n+1}{2} + i : 1 \leq i \leq n$\} with indices taken modulo $n$. Moreover, \{ $f_5(v_i) + f_5(v_j) : 1 \leq i \leq n$\} = \{ $\frac{3n+1}{2} + i : 1 \leq i \leq n$\} and \{ $f_5(v_j) + f_5(u_j) : 1 \leq j \leq p$\} = \{ $\frac{7n+1}{2} + j : 1 \leq j \leq p$\}. The elements of the set \{ $f_5(v_i) + f_5(v_{i+1}) : 1 \leq i \leq n$\} $\cup$ \{ $f_5(z_i) + f_5(z_{i+2}) : 1 \leq i \leq n$\} $\cup$ \{ $f_5(v_j) + f_5(z_j) : 1 \leq j \leq p$\} form an arithmetic sequence $\frac{n+1}{2} + 1$, $\frac{n+1}{2} + 2$, $\frac{7n+1}{2}$, $\frac{2n+1}{2} + 1$, $\frac{7n+1}{2} + p$. We are able to arrange the values $2n + p + 1, 2n + p + 2, \cdots, 5n + 2p$ to the edges of $P(n,2) + A_p$ in such way that the resulting labeling is total and every edge $xy \in E(P(n,2) + A_p)$, $f_5(x) + f_5(y) + f_5(xy) = \frac{11n+3}{2} + 2p$. Thus we arrive at the desired result.

**Theorem 4.** If $n$ is odd, $n \geq 5$ and $p \geq 1$, then the graph $P(n,2, p)$ has a super edge-magic total labeling.

**Proof.** Define a bijection, $f_6 : V(P(n,2, p)) \rightarrow \{1, 2, \cdots, n(p+2)\}$ as follows,

- $f_6(v_i) = f_5(v_i)$ and $f_6(z_i) = f_5(z_i)$ for $1 \leq i \leq n$,
- $f_6(u_i,j) = n(j+1) + 1$ for $1 \leq j \leq p$,
- $f_6(u_{i,j}) = n(j+2) + 2 - i$ for $2 \leq i \leq n$ and $1 \leq j \leq p$.

We can see that under the vertex labeling $f_6$ the values $f_6(x) + f_6(y)$ of all edges $xy \in E(P(n,2, p))$ constitute an arithmetic sequence $\frac{2n+1}{2} + 1, \frac{2n+1}{2} + 2, \cdots, \frac{7n+1}{2}, \frac{7n+1}{2} + 1, \cdots, \frac{7n+1}{2} + np$. If we complete the edge labeling with the consecutive values in the set \{ $n(p+2) + 1, n(p+2) + 2, n(p+2) + 3, \cdots, 5n + 2np$\} then we can obtain total labeling where $f_6(x) + f_6(y) + f_6(xy) = \frac{11n+3}{2} + 2np$ for every edge $xy \in E(P(n,2, p))$. 


In the sequel we shall consider a graph of a generalized prism which can be defined as the Cartesian product $C_n \times P_m$ of a cycle on $n$ vertices with a path on $m$ vertices.

Let $V(C_n \times P_m) = \{v_{i,k} : 1 \leq i \leq n \text{ and } 1 \leq k \leq m\}$ be the vertex set and $E(C_n \times P_m) = \{v_{i,k}v_{i+1,k} : 1 \leq i \leq n \text{ and } 1 \leq k \leq m\} \cup \{v_{i,k}v_{i,k+1} : 1 \leq i \leq n \text{ and } 1 \leq k \leq m-1\}$ be the edge set, where $i$ is taken modulo $n$. For $n \geq 3$, $m \geq 2$ and $p \geq 1$, we will consider a graph $(C_n \times P_m) + A_p$ (respectively a graph $(C_n \times P_m) + \sum_{i=1}^{n} A^i_p$) which is obtained by adding $p$ vertices and $p$ edges to one vertex of $C_n \times P_m$, say $v_{i,m}$ (respectively to all vertices $v_{i,m}$, $1 \leq i \leq n$ of $C_n \times P_m$). Thus $V((C_n \times P_m) + A_p) = V(C_n \times P_m) \cup \{u_j : 1 \leq j \leq p\}$,

$$V((C_n \times P_m) + \sum_{i=1}^{n} A^i_p) = V(C_n \times P_m) \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\},$$

$E((C_n \times P_m) + A_p) = E(C_n \times P_m) \cup \{v_{i,m}u_j : 1 \leq j \leq p\}$, and

$$E((C_n \times P_m) + \sum_{i=1}^{n} A^i_p) = E(C_n \times P_m) \cup \{v_{i,m}u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}.$$

Figueroa-Centeno et.al. [5] showed that the generalized prism $C_n \times P_m$ is super edge-magic if $n$ is odd and $m \geq 2$.

The next two theorems show super edge-magic total labelings of graphs $(C_n \times P_m) + A_p$ and $(C_n \times P_m) + \sum_{i=1}^{n} A^i_p$.

**Theorem 5.** If $n$ is odd, $n \geq 3$, $m \geq 2$ and $p \geq 1$, then the graph $(C_n \times P_m) + A_p$ has a super edge-magic total labeling.

**Proof.** If $m$ is even, $m \geq 2$, $1 \leq k \leq m$, $1 \leq i \leq n$, then we construct a vertex labeling $f_1$ in the following way,
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If \( m \) is odd, \( m \geq 3, 1 \leq k \leq m, 1 \leq i \leq n \), then we define a vertex labeling \( f_8 \) as follows,

\[
\begin{cases}
\frac{n+i}{2} + n(k-1) & \text{if } i \text{ is odd and } k \text{ is odd}, \\
\frac{1}{2} + n(k-1) & \text{if } i \text{ is even and } k \text{ is odd}, \\
nk & \text{if } i = 1 \text{ and } k \text{ is even}, \\
n(k-1) + \frac{i-1}{2} & \text{if } i \text{ is odd and } k \text{ is even}, \\
n(k-1) + \frac{n+i-1}{2} & \text{if } i \text{ is even and } k \text{ is even},
\end{cases}
\]

\( f_8(u_j) = mn + j \) for \( 1 \leq j \leq p \).

It is easy to verify that for each edge \( xy \in E((C_n \times P_m) + A_p) \) the values \( f_7(x) + f_7(y) \) and \( f_6(x) + f_6(y) \) form an arithmetic sequence \( \frac{n+1}{2} + 1, \frac{n+1}{2} + 2, \ldots, 2mn - \frac{n-1}{2}, 2mn - \frac{n-3}{2}, \ldots, 2mn - \frac{n-1}{p} + p \).

Let \( f_9 \) be a bijection from \( E((C_n \times P_m) + A_p) \) onto \( \{1, 2, \ldots, 2mn - n + p\} \). We can combine the vertex labeling \( f_7 \) (or \( f_8 \)) and the edge labeling \( f_9 + mn + p \) such that the resulting labeling is total and the edge sum for each edge \( xy \in E((C_n \times P_m) + A_p) \) is equal to \( 3mn + \frac{3-n}{2} + 2p \).

**Theorem 6.** If \( n \) is odd, \( n \geq 3, m \geq 2, \) and \( p \geq 1 \), then the graph \((C_n \times P_m) + \sum_{i=1}^{n} A'_i\) has a super edge-magic total labeling.

**Proof.** Define vertex labeling \( f_{10} \) and \( f_{11} \) such that:

\[
\begin{align*}
\text{if } m \text{ is even, } 1 \leq k \leq m, 1 \leq i \leq n, & \quad f_{10}(v_{i,k}) = f_7(v_{i,k}), \\
\text{if } m \text{ is odd, } 1 \leq k \leq m, 1 \leq i \leq n, & \quad f_{11}(v_{i,k}) = f_8(v_{i,k}), \\
\text{for } 1 \leq j \leq p, & \quad f_{10}(u_{i,j}) = f_{11}(u_{i,j}) = n(m + j - 1) + 1.
\end{align*}
\]
\[ f_{10}(u_{i,j}) = f_{11}(u_{i,j}) = n(m + j) - i + 2 \text{ for } 2 \leq i \leq n \text{ and } 1 \leq j \leq p. \]

We can see that vertices of \((C_n \times P_m) + \sum_{i=1}^{n} A^t_p\) are labeled by values 1, 2, 3, \ldots, \(n(m + p)\) and \(f_i(x) + f_i(y)\) for all edges \(xy \in (C_n \times P_m) + \sum_{i=1}^{n} A^t_p\) and \(t \in \{10, 11\}\) constitute an arithmetic sequence \(\frac{n+1}{2} + 1, \frac{n+1}{2} + 2, \ldots, 2mn - \frac{n-1}{2} + np\).

We can complete the edge labeling of \((C_n \times P_m) + \sum_{i=1}^{n} A^t_p\) with values in the set \(\{n(m + p) + 1, n(m + p) + 2, \ldots, n(3m + 2p - 1)\}\) consecutively such that the common edge sum is \(k = 3mn + 2pn - \frac{n-3}{2}\). Thus the total labeling of \((C_n \times P_m) + \sum_{i=1}^{n} A^t_p\) is super edge-magic and the theorem is proved.

References