J. Math. Fund. Sci., Vol. 51, No. 2, 2019, 191-203

The Second Hankel Determinant Problem for a Class of Bi-Univalent Functions

Mohammad Hasan Khani[1], Ahmad Zireh[2] & Ebrahim Analouei Adegani[2]

[1]Department of Mathematics, Shahinshahr Branch, Islamic Azad University, Shahinshahr, Iran
[2]Faculty of Mathematical Sciences, Shahrood University of Technology, PO Box 316-36155, Shahrood, Iran
E-mail: khanimohammad@yahoo.com

Abstract. Hankel matrices are related to a wide range of disparate determinant computations and algorithms and some very attractive computational properties are allocated to them. Also, the Hankel determinants are crucial factors in the research of singularities and power series with integral coefficients. It is specified that the Fekete-Szegö functional and the second Hankel determinant are equivalent to $H_1(2)$ and $H_2(2)$, respectively. In this study, the upper bounds were obtained for the second Hankel determinant of the subclass of bi-univalent functions, which is defined by subordination. It is worth noticing that the bounds rendered in the present paper generalize and modify some previous results.

Keywords: *bi-univalent functions; Hankel determinant; subordinate.*

Mathematical Subject Classification 2010: 30C45; 30C50

1 Introduction

Suppose we have a class $\mathcal{A}$ consisting of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. All univalent functions in the subclass of $\mathcal{A}$ are denoted by $\mathcal{S}$. Obviously, the inverse f^{-1} of $f \in \mathcal{S}$ is expressed by

$$f^{-1}(f(z)) = z \;\; (z \in \mathbb{U}) \;\text{ and }\; f(f^{-1}(w)) = w \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{2}$$

If both $f$and f^{-1} are univalent in $\mathbb{U}$, then function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$. Let σ describe the class of bi-univalent functions in $\mathbb{U}$. Class σ was first investigated by Lewin [1]. He obtained the bound for the second

Received March 1st, 2017, 1st Revision October 24th, 2018, 2nd Revision November 22nd, 2018, Accepted for publication March 4th, 2019.
 ISSN: 2337-5760, DOI: 10.5614/j.math.fund.sci.2019.51.2.8

coefficient. Recently, several researches have focused on studying the class Σ, which consists of the bi-univalent functions, and acquired non-sharp estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, e.g. [2-8]. The coefficient estimate issue for certain subfamilies of class σ of Taylor-Maclaurin coefficients $|a_n|$ for $n \geq 4$ is presumably still a concern. Either way, some researchers have investigated the Faber polynomial expansions to obtain the upper bounds for various subclasses of class σ [9-14].

The Fekete-Szegö functional $a_3 - \delta a_2^2$ for $f \in \mathcal{A}$, where δ is a real number, is famous due to its importance in the history of the geometric function theory. In [15], the Fekete-Szegö problem is reported for odd univalent functions. In 1976, the q-th Hankel determinant was stated for integers $n \geq 1$ and $q \geq 1$ [16], as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

Hankel determinants are advantageous due to their pivotal application in the study of singularities and power series with integral coefficients [17]. It is well-known that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \text{ and } H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

where the Hankel determinant $H_2(1) = a_3 - a_2^2$ is called the Fekete-Szegö functional and $H_2(2) = a_2 a_4 - a_3^2$ is defined as the second Hankel determinant functional. Recently, several researchers have investigated similar problems in this direction, [18-27] to name a few.

Definition 1.1. [28] Let h and H be analytic in $\mathbb{U}$. We state that h is subordinate to H, written as $h(z) \prec H(z)$ provided there is an analytic function ϖ, described on $\mathbb{U}$ with the conditions $\varpi(0) = 0$ and, $|\varpi(z)| < 1$ satisfying $h(z) = H(\varpi(z))$. In particular, if H is univalent then $h(z) \prec H(z)$ is equivalent to $h(\mathbb{U}) \subseteq H(\mathbb{U})$ and $h(0) = H(0)$.

Different subclasses of starlike and convex functions were introduced by Ma and Minda [29], where each factor $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinated to the total function. To this aim, they determined an analytic function ϕ with the characteristics of a positive real part of $\mathbb{U}$, $\phi(\mathbb{U})$ is symmetric respecting the real axis $\phi'(0) > 0$ and starlike considering $\phi(0) = 1$. The series expansion of this function can be demonstrated in the form of

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0). \tag{3}$$

Definition 1.2. [2] We say that $f \in \sigma$ is in the subclass $\mathcal{H}_\sigma(\phi)$ if the following condition is considered:

$$f'(z) \prec \phi(z) \quad \text{and} \quad g'(w) \prec \phi(w),$$

where function g is given by Eq. (2).

Lemma 1.3. [28] Suppose that the analytic functions $t(z)$ and $l(z)$ are in $\mathbb{U}$ with conditions, $t(0) = l(0) = 0$, $|t(z)| < 1$, $|l(z)| < 1$ and with respect to:

$$t(z) = \sum_{n=1}^{\infty} r_n z^n \text{ and } l(z) = \sum_{n=1}^{\infty} q_n z^n \qquad (z \in \mathbb{U}). \tag{4}$$

Then for $n = 1,2,3, \ldots$ we have $|r_n| \leq 1$ and $|q_n| \leq 1$.

Lemma 1.4. [30] Let $\mathcal{P}$ comprise all analytic functions ρ in $\mathbb{U}$ such that $\rho(z) = 1 + \sum_{n=1}^{\infty} \rho_n z^n$ and $Re\rho(z) > 0$. Suppose $\rho \in \mathcal{P}$, then $|\rho_k| \leq 2$ for any $k \in \mathbb{N}$.

Lemma 1.5. [31] Suppose $\rho \in \mathcal{P}$, $\rho_1 > 0$, then for some h, s with $|h| \leq 1$ and $|s| \leq 1$ we have

$$2\rho_2 = \rho_1^2 + h(4 - \rho_1^2)$$
$$4\rho_3 = \rho_1^3 + 2(4 - \rho_1^2)\rho_1 h - \rho_1(4 - \rho_1^2)h^2 + 2(4 - \rho_1^2)(1 - |h|^2)s.$$

Based on the results presented in previous researches, in the current study, the coefficient for the functional $|H_2(2)| = |a_2 a_4 - a_3^2|$ was estimated for the function $f \in \mathcal{H}_\sigma(\phi)$ .It is worthwhile mentioning that the given bounds in this paper generalize and enhance some results obtained in [18].

2 Main Results

The subordination classes consist of some important subclasses of univalent functions and the obtained outcomes for these specific subclasses are called corollaries. Therefore, the following lemma will be used to establish our main result of obtaining the upper bounds for $|H_2(2)|$ for subclass $\mathcal{H}_\sigma(\phi)$, which is defined by subordination.

Lemma 2.1. Suppose the function $\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n \in A$ is analytic somehow $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in\mathbb{U}$. Then we have

$$\omega_2 = h(1-\omega_1^2)$$
$$\omega_3 = (1-\omega_1^2)(1-|h|^2)s - \omega_1(1-\omega_1^2)h^2,$$

for some h, s with $|h|\le 1$ and $|s|\le 1$.

Theorem 2.2. If, $B_2=\alpha B_1$, $\frac{1}{192}\le\alpha\le 1$ then for $f\in\mathcal{H}_\sigma(\phi)$, as shown by (1.1), we have

$$|a_2a_4-a_3^2|\le B_1\begin{cases}\dfrac{B_1}{9}, & T\le 0, S\le -T\\[2ex] \left|\dfrac{-B_1^3}{16}+\dfrac{B_3}{8}\right|, & (T\ge 0, S\ge -\dfrac{T}{2}) \text{ or } (T\le 0, S\ge -T)\\[2ex] \dfrac{4SU-T^2}{4S}, & T>0, S\le -\dfrac{T}{2},\end{cases} \tag{5}$$

where $S=\left|\dfrac{-B_1^3}{16}+\dfrac{B_3}{8}\right|-\dfrac{B_1^2}{24}-\dfrac{|B_2|}{4}-\dfrac{B_1}{72}$, $T=\dfrac{B_1^2}{24}+\dfrac{|B_2|}{4}-\dfrac{7B_1}{72}$ and $U=\dfrac{B_1}{9}$.

Remark 2.3. For $0\le\beta<1$, we take

$$f \in \mathcal{H}_\sigma\left(\frac{1+(1-2\beta)z}{1-z}\right) = 1+2(1-\beta)z+2(1-\beta)z^2+2(1-\beta)z^3+\cdots).$$

In this case, with respect to Theorem 2.2, $T \geq 0$, $S+T/2 \leq 0$ and we have the next corollary, which is a refinement of the results presented in [18, Theorem 1].

Corollary 2.4. Suppose $f \in \mathcal{H}_\sigma\left(\frac{1+(1-2\beta)z}{1-z}\right) = \mathcal{N}_\Sigma(\beta)$ is given by Eq. (1). Then

$$|a_2a_4 - a_3^2| \leq (1-\beta)^2\left\{\frac{4}{9} - \frac{[17-6\beta]^2}{36^2\left[|(1-\beta)^2 + \frac{1}{2}|-\frac{25}{18}+\frac{1}{3}\beta\right]}\right\} \qquad \beta \in [0,1).$$

Remark 2.5. For $\frac{1}{192} \leq \alpha \leq 1$ let

$$f \in \left(\mathcal{H}_\sigma\left(\frac{1+z}{1-z}\right)^\alpha = 1+2\alpha z+2\alpha^2z^2+\frac{8\alpha^3+4\alpha}{6}z^3+\cdots\right).$$

In this case, $T \geq 0, S+T/2 \leq 0$, then from Theorem 2.2 we get the next corollary as a refinement of the results presented in [18, Theorem 2].

Corollary 2.6. Let $f \in \mathcal{H}_\sigma\left(\left(\frac{1+z}{1-z}\right)^\alpha\right) = \mathcal{N}_\Sigma^\alpha$ be given by Eq. (1). Then

$$|a_2a_4 - a_3^2| \leq \alpha^2\left\{\frac{4}{9} - \frac{\left[\frac{2\alpha}{3}-\frac{7}{36}\right]^2}{2|-\frac{1}{3}\alpha^2+\frac{1}{12}|-\frac{2}{3}\alpha-\frac{1}{36}}\right\} \qquad \alpha \in [\frac{1}{192},1].$$

3 Proof of Results

Proof of Lemma 2.1. Define $q(z) = \frac{1+\omega(z)}{1-\omega(z)}$ where $q(z) = 1+\sum_{n=1}^{\infty} c_n z^n$ is such that $Re q(z) > 0$ for $|z| < 1$. To compare the coefficients corresponding to the powers of z resulted from $c_1 = 2\omega_1, c_2 = 2(\omega_2 + \omega_1^2)$ and $c_3 = 2\omega_3 + 4\omega_1\omega_2 + 2\omega_1^3$. By Lemma 1.5 we get that

$$4(\omega_2 + \omega_1^2) = 4\omega_1^2 + h(4 - 4\omega_1^2)$$

$$4(2\omega_3 + 4\omega_1\omega_2 + 2\omega_1^3) = 8\omega_1^3 + 4\omega_1(4 - 4\omega_1^2)h$$

$$-2\omega_1(4 - 4\omega_1^2)h^2 + 2(4 - 4\omega_1^2)(1 - |h|^2)s.$$

So we obtain our result.

Proof of Theorem 2.2. Suppose $f \in \mathcal{H}_\sigma(\phi)$. In this case there are two Schwartz functions, $t, l : \mathbb{U} \to \mathbb{U}$, with conditions $t(0) = l(0) = 0$, presented by Eq. (4), such that

$$f'(z) = \phi(t(z)), \tag{6}$$

and

$$g'(w) = \phi(l(w)), \tag{7}$$

where by Eq. (3), we get

$$\phi(t(z)) = 1 + B_1r_1z + (B_1r_2 + B_2r_1^2)z^2 + (B_1r_3 + 2r_1r_2B_2 + B_3r_1^3)z^3 + \cdots, \tag{8}$$

and

$$\phi(l(w)) = 1 + B_1q_1w + (B_1q_2 + B_2q_1^2)w^2 + (B_1q_3 + 2q_1q_2B_2 + B_3q_1^3)w^3 + \cdots. \tag{9}$$

It follows from Eqs. (6), (8) and (7), (9) that

$$2a_2 = B_1r_1 \tag{10}$$

$$3a_3 = B_1r_2 + B_2r_1^2 \tag{11}$$

$$4a_4 = B_1r_3 + 2r_1r_2B_2 + B_3r_1^3, \tag{12}$$

and

$$-2a_2 = B_1q_1 \tag{13}$$

$$6a_2^2 - 3a_3 = B_1q_2 + B_2q_1^2 \tag{14}$$

$$-4a_4 + 20a_2a_3 - 20a_2^3 = B_1q_3 + 2q_1q_2B_2 + B_3q_1^3. \tag{15}$$

From Eqs. (10) and (13), we have

$$r_1 = -q_1 \tag{16}$$

and

$$a_2 = \frac{B_1 r_1}{2} \tag{17}$$

Now, from Eqs. (11) and (14) we obtain

$$a_3 = \frac{B_1^2 r_1^2}{4} + \frac{B_1(r_2 - q_2)}{6}. \tag{18}$$

Also, from Eqs. (12) and (15) we find that

$$a_4 = \frac{5B_1^2 r_1(r_2 - q_2)}{24} + \frac{B_1(r_3 - q_3)}{8} + \frac{B_2 r_1(r_2 + q_2)}{4} + \frac{B_3 r_1^3}{4}. \tag{19}$$

Therefore

$$|a_2 a_4 - a_3^2| = \left| \left(\frac{-B_1^4}{16} + \frac{B_3 B_1}{8} \right) r_1^4 + \frac{B_1^3 r_1^2 (r_2 - q_2)}{48} \right.$$
$$\left. + \frac{B_2 B_1 r_1^2 (r_2 + q_2)}{8} + \frac{B_1^2 r_1 (r_3 - q_3)}{16} - \frac{B_1^2 (r_2 - q_2)^2}{36} \right|. \tag{20}$$

From Lemma 2.1 and (16) we obtain

$$\left. \begin{aligned} r_2 &= h(1 - r_1^2) \\ q_2 &= j(1 - q_1^2) \end{aligned} \right\} \Rightarrow r_2 - q_2 = (1 - r_1^2)(h - j), \tag{21}$$

and

$$r_3 = (1 - r_1^2)(1 - |h|^2)s - r_1(1 - r_1^2)h^2$$

$$q_3 = (1 - q_1^2)(1 - |j|^2)w - q_1(1 - q_1^2)j^2,$$

where

$$r_3 - q_3 = (1 - r_1^2)((1 - |h|^2)s - (1 - |j|^2)w) - r_1(1 - r_1^2)(h^2 + j^2),$$

for some h, j, s, w where $|h| \leq 1, |j| \leq 1$, $|s| \leq 1$ and $|w| \leq 1$. Then, employing Eq. (21) and the above equation in Eq. (20) yields

$$|a_2a_4 - a_3^2| = B_1 \Bigg| \left(\frac{-B_1^3}{16} + \frac{B_3}{8} \right) r_1^4 + \left(\frac{B_1^2(h-j)}{48} + \frac{B_2(h+j)}{8} \right) \times r_1^2(1-r_1^2) - \frac{B_1 r_1^2(1-r_1^2)}{16}(h^2+j^2) - \frac{B_1(1-r_1^2)^2}{36}(h-j)^2 + \frac{B_1 r_1(1-r_1^2)}{16}((1-|h|^2)s - (1-|j|^2)w) \Bigg|.$$

As $|r_1| \leq 1$, we may assume without restriction that $r_1 = r \in [0,1]$, so

$$|a_2a_4 - a_3^2| \leq B_1 (\left| \frac{-B_1^3}{16} + \frac{B_3}{8} \right| r^4 + \left[\frac{B_1^2}{48} + \frac{|B_2|}{8} \right] r^2(1-r^2)(|h|+|j|) + \frac{B_1 r^2(1-r^2)}{16}(|h|^2+|j|^2) + \frac{B_1(1-r^2)^2}{36}(|h|+|j|)^2 + \frac{B_1 r(1-r^2)}{16}[(1-|h|^2)|s| + (1-|j|^2)|w|]$$

$$\leq B_1 (\left| \frac{-B_1^3}{16} + \frac{B_3}{8} \right| r^4 + \left[\frac{B_1^2}{48} + \frac{|B_2|}{8} \right] r^2(1-r^2)(|h|+|j|) + \frac{B_1 r^2(1-r^2)}{16}(|h|^2+|j|^2) + \frac{B_1(1-r^2)^2}{36}(|h|+|j|)^2 + \frac{B_1 r(1-r^2)}{16}[(1-|h|^2) + (1-|j|^2)]$$

$$= B_1 (\left| \frac{-B_1^3}{16} + \frac{B_3}{8} \right| r^4 + \frac{B_1 r(1-r^2)}{8} + \left[\frac{B_1^2}{48} + \frac{|B_2|}{8} \right] r^2(1-r^2)(|h|+|j|) + \left[\frac{B_1 r^2(1-r^2)}{16} - \frac{B_1 r(1-r^2)}{16} \right](|h|^2+|j|^2) + \frac{B_1(1-r^2)^2}{36}(|h|+|j|)^2).$$

Now, for $\lambda = |h| \leq 1$ and $\gamma = |j| \leq 1$, we get

$$|a_2a_4 - a_3^2| \leq B_1[T_1 + (\lambda+\gamma)T_2 + (\lambda^2+\gamma^2)T_3 + (\lambda+\gamma)^2T_4] = B_1F(\lambda,\gamma),$$

where

$$T_1 = T_1(p) = \left|\frac{-B_1^3}{16} + \frac{B_3}{8}\right| r^4 + \frac{B_1 r(1-r^2)}{8} \geq 0$$

$$T_2 = T_2(p) = \left[\frac{B_1^2}{48} + \frac{|B_2|}{8}\right] r^2(1-r^2) \geq 0$$

$$T_3 = T_3(p) = \frac{B_1 r(r-1)(1-r^2)}{16} \leq 0$$

$$T_4 = T_4(p) = \frac{B_1(1-r^2)^2}{36} \geq 0.$$

Now the function $F(\lambda,\gamma)$ has to be maximized on the closed square $[0,1] \times [0,1]$ for $r \in [0,1]$. To this aim, the maximum of $F(\lambda,\gamma)$ is investigated with respect to $r \in (0,1)$ and $r = 1$ considering the sign of $F_{\lambda\lambda}.F_{\gamma\gamma} - (F_{\lambda\gamma})^2$.

Take $r \in (0,1)$. As $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $r \in (0,1)$, we conclude that $F_{\lambda\lambda}.F_{\gamma\gamma} - (F_{\lambda\gamma})^2 < 0$. Therefore, there is not a local maximum for function F in the interior of the square.

For $0 \leq \gamma \leq 1$ and $\lambda = 0$ (in the same way $0 \leq \lambda \leq 1$ and $\gamma = 0$) it is concluded that

$$F(0,\gamma) = H(\gamma) = (T_3 + T_4)\gamma^2 + T_2\gamma + T_1.$$

(i) If $T_3 + T_4 > 0$, obviously, $H'(\gamma) = 2(T_3 + T_4)\gamma + T_2 > 0$ for $0 < \gamma < 1$ and each fixed $r \in [0,1)$ and so $H(\gamma)$ is a non-decreasing function. Thus, we get the maximum of $H(\gamma)$ on $\gamma = 1$ for fixed $r \in [0,1)$, and

$$\max H(\gamma) = H(1) = T_3 + T_4 + T_2 + T_1.$$

(ii) If $T_3 + T_4 < 0$ it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\gamma + T_2 < T_2$. By, $B_2 = \alpha B_1$, $\frac{1}{192} \leq \alpha \leq 1$ therefore $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \gamma < 1$, and $r \in [0,1)$. So $H'(\gamma) > 0$ and therefore we obtain the maximum of $H(\gamma)$ on $\gamma = 1$ for fixed $r \in [0,1)$, and

$$\max H(\gamma) = H(1) = T_3 + T_4 + T_2 + T_1.$$

Moreover, for $r = 1$ it follows that

$$F(\lambda,\gamma) = \left|\frac{-B_1^3}{16} + \frac{B_3}{8}\right|. \tag{22}$$

Taking into account the value of Eq. (22) for the case $\lambda = 0$, $0 \leq \gamma \leq 1$ and any fixed $r \in [0,1]$

$$\max \quad H(\gamma) = H(1) = T_3 + T_4 + T_2 + T_1.$$

For $\lambda = 1$ and $0 \leq \gamma \leq 1$ (similarly $\gamma = 1$ and $0 \leq \lambda \leq 1$) we get

$$F(1,\gamma) = G(\gamma) = (T_3 + T_4)\gamma^2 + (T_2 + 2T_4)\gamma + T_1 + T_2 + T_3 + T_4.$$

Similarly, from the above **(i)** and **(ii)** for $T_3 + T_4$ yields

$$\max \quad G(\gamma) = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

As $G(1) \geq H(1)$ for $r \in [0,1]$, it follows that max $F(\lambda,\gamma) = F(1,1)$. Thus the maximum of F takes place at $\lambda = 1$ and $\gamma = 1$ on the boundary $[0,1] \times [0,1]$.

We define the real function W on $[0,1]$ by

$$W(r) = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_3$$

Now putting T_1, T_2, T_3, and T_4 in the function W, we have

$$W(r) = B_1\left\{\left[\left|\frac{-B_1^3}{16} + \frac{B_3}{8}\right| - 2\left(\frac{B_1^2}{48} + \frac{|B_2|}{8}\right) - \frac{B_1}{72}\right]r^4 + \left[2\left(\frac{B_1^2}{48} + \frac{|B_2|}{8}\right) - \frac{7B_1}{72}\right]r^2 + \frac{B_1}{9}\right\}.$$

Let $r^2 = t$ and

$$S = \left|\frac{-B_1^3}{16} + \frac{B_3}{8}\right| - \frac{B_1^2}{24} - \frac{|B_2|}{4} - \frac{B_1}{72}$$

$$T = \frac{B_1^2}{24} + \frac{|B_2|}{4} - \frac{7B_1}{72} \tag{23}$$

$$U = \frac{B_1}{9}.$$

Since

$$\underset{0\le t\le 1}{(St^2+Tt+U)}=\begin{cases} U & T\le 0, S\le -T \\ S+T+U & (T\ge 0, S\ge -\dfrac{T}{2}) \text{ or } (T\le 0, S\ge -T) \\ \dfrac{4SU-T^2}{4S} & T>0, S\le -\dfrac{T}{2}, \end{cases}$$

it gives,

$$|a_2a_4-a_3^2|\le B_1\begin{cases} U & T\le 0, S\le -T \\ S+T+U & (T\ge 0, S\ge -\dfrac{T}{2}) \text{ or } (T\le 0, S\ge -T) \\ \dfrac{4SU-T^2}{4S} & T>0, S\le -\dfrac{T}{2}, \end{cases}$$

where S, T and U are shown by Eq. (23). This completes the proof.

4 Conclusion

In the final sections we found upper bounds for $|H_2(2)|$ of subclass σ, which is defined by Definition 1.2, and then we discussed some new results, which can be deduced from the main theorem. Thus, regarding the proofs of Theorem 2.2, this technique can be applied for all classes that have been defined similarly to Definition 1.2 in several papers, enhancing their outcomes.

Acknowledgements

The authors are grateful to the reviewers of this article for giving valuable remarks, comments and advice in order to revise and improve the results of this paper. Also, the authors would like to thank the Shahinshahr Branch, Islamic Azad University for their financial support.

References

[1] Lewin, M., On a Coefficient Problem for Bi-univalent Functions, Proc. Amer. Math. Soc., **18**, pp. 63-68, 1967.

[2] Ali, R.M., Lee, S.K., Ravichandran, V. & Subramaniam, S., *Coefficient Estimates for Bi-univalent Ma-Minda Starlike and Convex Functions*, Appl. Math. Lett., **25**, pp. 344-351, 2012.

[3] Kedzierawski, A.W., *Some Remarks on Bi-univalent Functions*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **39**, pp. 77-81, 1985.

[4] Frasin, B.A. & Aouf, M.K., *New Subclasses of Bi-univalent Functions*, Appl. Math. Lett., **24**, pp. 1569-1573, 2011.

[5] Srivastava, H.M., Mishra A.K. & Gochhayat, P., *Certain Subclasses of Analytic and Bi-univalent Functions*, Appl. Math. Lett., **23**, pp. 1188-1192, 2010.

[6] Xu, Q.H., Gui, Y.C. & Srivastava, H.M., *Coefficient Estimates for A Certain Subclass of Analytic and Bi-univalent Functions*, Appl. Math. Lett., **25**, pp. 990-994, 2012.

[7] Xu, Q.H., Xiao, H.G. & Srivastava, H.M., *A Certain General Subclass of Analytic and Bi-univalent Functions and Associated Coefficient Estimate Problems*, Appl. Math. Comput., **218**, pp. 11461-11465, 2012.

[8] Zireh, A. & Adegani, E.A., *Coefficient Estimates for a Subclass of Analytic and Bi-univalent Functions*, Bull. Iranian Math. Soc., **42**, pp. 881-889, 2016.

[9] Adegani, E.A., Hamidi, S.G., Jahangiri, J.M. & Zireh, A., *Coefficient Estimates of M-fold Symmetric Bi-subordinate Functions by Faber Polynomial*, Hacet. J. Math. Stat., **48**, pp. 365-371, 2019.

[10] Bulut, S., Magesh, N. & Balaji, V.K., *Faber Polynomial Coefficient Estimates for Certain Subclasses of Meromorphic Bi-univalent Functions*, C.R. Acad. Sci. Paris, Ser. I, **353**, pp. 113-116, 2015.

[11] Hamidi, S.G. & Jahangiri, J.M., *Faber Polynomial Coefficients of Bi-subordinate Functions*, C.R. Math. Acad. Sci. Paris, **354**, pp. 365-370, 2016.

[12] Jahangiri, J.M., Hamidi, S.G. & Halim, S.A., *Coefficients of Bi-univalent Functions with Positive Real Part Derivatives*, Bull. Malays. Math. Sci. Soc., **37**, pp. 633-640, 2014.

[13] Zireh, A., Adegani, E.A. & Bulut, S., *Faber Polynomial Coefficient Estimates for A Comprehensive Subclass of Analytic Bi-univalent Functions Defined by Subordination*, Bull. Belg. Math. Soc. Simon Stevin, **23**, pp. 487-504, 2016.

[14] Zireh, A., Adegani, E.A., & Bidkham, M., *Faber Polynomial Coefficient Estimates for Subclass of Bi-univalent Functions Defined by Quasi-subordinate*, Mathematica Slovaca, **68**, pp. 369-378, 2018.

[15] Fekete, M. & Szegö, G., *Eine Bemerkung über Ungerade Schlichte Funktionen*, J. London Math. Soc., **8**, pp. 85-89, 1933. (Text in Germany)

[16] Noonan, J.W. & Thomas, D.K., *On the Second Hankel Determinant of Areally Mean p-valent Functions*, Trans. Am. Math. Soc., **223**, pp. 337-346, 1976.

[17] Cantor, D.G., *Power Series with Integral Coefficients*, Bull. Amer. Math. Soc., 69, pp. 362-366, 1963.

[18] Caglar, M., Deniz, E. & Srivastava, H.M., *Second Hankel Determinant for Certain Subclasses of Bi-univalent Functions*, Turk. J. Math., **41**, pp. 694-706, 2017.

[19] Deniz, E., Caglar, M. & Orhan, H., *Second Hankel Determinant for Bi-starlike and Bi-convex Functions of Order β*, Appl. Math. Comput., **271**, pp. 301-307, 2015.

[20] Kanas, S., *An Unified Approach to the Fekete-Szegö Problem*, Appl. Math. Comput., **218**, pp. 8453-8461, 2012.

[21] Kanas, S., Adegani, E.A. & Zireh, A., *An Unified Approach to Second Hankel Determinant of Bi-Subordinate Functions*, Mediterr. J. Math., **14**, pp.233, 2017.

[22] Kanas, S. & Darwish, H.E., *Fekete-Szegö Problem for Starlike and Convex Functions of Complex Order*, Appl. Math. Lett., **23**, pp. 777-782, 2010.

[23] Murugusundaramoorthy G. & Magesh, N., *Coefficient Inequalities for Certain Classes of Analytic Functions Associated with Hankel Determinant*, Bull. Math. Anal. Appl., **1**, pp. 85-89, 2009.

[24] Orhan, H., Magesh, N. & Balaji, V.K., *Second Hankel Determinant for Certain Class of Bi-univalent Functions Defined by Chebyshev Polynomials*, Asian-European J. Math., **12**, pp. 1-16, 2019.

[25] Orhan, H., Magesh, N. & Yamini, *J., Bounds for the Second Hankel Determinant of Certain Bi-univalent Functions*, Turk. J. Math., **40**, pp. 679-687, 2016.

[26] Zaprawa, P., *On the Fekete-Szegö Problem for Classes of Bi-univalent Functions*, Bull. Belg. Math. Soc. Simon Stevin, **21**, pp. 169-178, 2014.

[27] Zaprawa, P., *Estimates of Initial Coefficients for Bi-univalent Functions*, Abstr. Appl. Anal., Art. ID 357480, pp. 1-6, 2014.

[28] Duren, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983. (Text in Germany)

[29] Ma, W.C. & Minda, D., *A Unified Treatment of Some Special Classes of Univalent Functions*, in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992. Conf. Proc. Lecture Notes Anal. I, Int. Press, Cambridge, MA, pp. 157-169, 1994.

[30] Pommerenke, C., *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[31] Grenander, U. & Szegö, G., *Toeplitz Forms and Their Applications*, California Monographs in Mathematical Sciences Univ, California Press, Berkeley, 1958.