P-properties in Near-rings

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Abstract. In this paper, assuming that N is a near-ring and P is an ideal of N, the P-center of N, the P-center of an element in N, the P-identities of N are defined. Their properties and relations are investigated. It is shown that the set of all P-identities in N is a multiplicative subsemigroup of N. Also, P-right and P-left permutable and P-medial near-rings are defined and some properties and connections are given. P-regular and P-strongly regular near-rings are studied. P-completely prime ideals are introduced and some characterizations of P-completely prime near-rings are provided. Also, some properties of P-idempotents, P-centers, P-identities in P-completely prime near-rings are investigated. The results that were obtained in this study are illustrated with many examples.

Keywords: *near-ring; P-center; P-completely prime ideal; P-regular; P-strongly regular.*

1 Introduction

Roos [1] was the first to define the concept of regularity for rings. Later certain regularities in associative rings were obtained by other authors. Most of these regularities were defined for near-rings and several authors, such as Groenewald and Potgieter [2], have improved large part of the general theory about those. Mason [3] has examined the concepts of regular and strongly regular for right near-rings. Also, he argued that it is necessary to distinguish between strong left and right regularity. In more recent years, he proved that for a zero-symmetric near-ring with identity, the notions of left regularity, right regularity and left strong regularity are equivalent. Reddy and Murty [4] have proven that these three notions are equivalent for arbitrary near-rings. Also, Hongan [5] has proven that these three notions and right strongly regular are equivalent. Several authors have researched the relationships between the concepts of primality and strongly regular. For example, Argaç and Groenewald [6] used left 0-prime and left prime ideals to characterize strongly regular near-rings. Moreover, it was attempted to adapt the concept of strongly to the notions of ring and near-ring. Handelman and Lawrence [7] introduced strongly prime rings. Groenewald [8] proposed the idea of strongly prime near-rings. Booth, *et al.* [9] defined a

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strongly equiprime near-ring as an alternative definition of a strongly prime near-ring.

Some concepts, such as center, idempotent element, identity, right and left permutable, medial, commutative, abelian, internal multiplier in near-rings, have been known for a long time. In [10-12], the authors developed the basic properties of medial, left permutable, right permutable and commutative near-rings. Furthermore, Mason [3] and Drazin [13] studied the concepts of center and idempotent element and also examined some relationships between these concepts. Also, several authors studied relationships with regular forms, strongly regular forms and prime ideals of these concepts. Birkenmeier [14] examined relationships between sets of idempotent elements and completely semiprime ideals. Mason [15] introduced strong forms of regularity for near-rings and examined some relations between the concepts of idempotent element and strongly regular. Dheena [16] presented a generalization of strongly regular near-rings. Drazin [13] studied regularity in near-rings where all idempotent elements are central.

Andrunakievich [17] defined P-regular rings and Choi [18] extended the P-regularity of rings to the P-regularity of near-rings. In 2012, Dheena and Jenila [19] introduced the notion of P-strongly regular near-rings and obtained equivalent conditions for near-rings to be P-strongly regular. They also were the first to define the concept of P-prime [19].

In this paper, generalizations of some important concepts in near-rings are given, such as the center of a near-ring, center of an element, (left-right) identities, (left-right) permutability, mediality and completely primeness by using a given ideal P. Also, several results on P-regularity, P-strong regularity and P-idempotents related to these generalizations were obtained.

2 Preliminaries

Let N be a right near-ring. The set $\{x \in N: xn = nx, \forall n \in N\}$ is called the center of N and is denoted by $C(N)$. Elements of $C(N)$ are called central elements. The set $\{n \in N: an = na\}$ is called the center of element $a \in N$ and is denoted by $C_a(N)$. An element $e \in N$ is called an idempotent if $e = e^2$. An idempotent e is called central if $en = ne$ for all $n \in N$. Element e of N is called right identity if $n = ne$ for all $n \in N$. It is called left identity if $n = ne$ for all $n \in N$. It is called an identity if it is both right and left identity. If N has a unity 1, then N is called a unital near-ring. A near-ring N is said to be right permutable if $xyz = xzy$, left permutable if $xyz = yxz$ for all $x, y, z \in N$. N is called a medial near-ring if $xyzt = xzyt$ for all $x, y, z, t \in N$. A near-ring N is said to be commutative if $xy = yx$ for all $x, y \in N$. The near-ring $(N, +)$ is said to be an abelian near-ring if $(N, +)$ is an abelian group. If there exists an element n such that $xy = xny$ for all $x, y \in N$, then n is called an internal multiplier of N. An element $d \in N$ is called a distributive element if for all $, y \in N$ $d(x + y) = dx + dy$. The set of distributive elements is denoted by N_d. Let $I \trianglelefteq N$. If $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in N$, then I is called a completely prime ideal. If $a^2 \in I$ implies $a \in I$ for all $a \in N$, then I is called a completely prime ideal. If the zero-ideal of N is completely prime, then N is called a completely prime near-ring. If the zero-ideal of N is completely semiprime, then N is called a completely semiprime near-ring. For $A, B \trianglelefteq N$, the multiplication of ideals A and B is defined as $AB = \{ab\colon a \text{ belongs to } A \text{ and } b \text{ belons to } B\}$. A near-ring N is said to be regular if for each $a \in N$, there exists an element $x \in N$ such that $a = axa$. It is said to be strongly regular if for each $a \in N$ there exists an element $x \in N$ such that $a = xa^2$[20].

Throughout this paper, N will denote a right near-ring and P will denote an ideal of N.

Definition 2.1 [19]: An element $e \in N$ is called a P-idempotent if $e - e^2 \in P$.

Definition 2.2 [19]: A near-ring N is said to be P-regular if for each $a \in N$, there exists an element $x \in N$ such that $a = axa + p$ for some $p \in P$.

Such $a \in N$ is called a P-regular element and $x \in N$ is called a P-regular component of element a.

Definition 2.3 [19]: A near-ring N is said to be P-strongly regular if for each $a \in N$ there exists an element $x \in N$ such that $a = xa^2 + p$ for some $p \in P$.

If $P = 0$, then a P-strongly regular near-ring is a strongly regular near-ring. If N is strongly regular, then N is P-strongly regular for all ideals P of N. But in general, a P-strongly regular near-ring does not have to be a strongly regular near-ring.

Definition 2.4 [19]: An ideal A of N is said to be P-prime if for any ideals $B, C \in N$ $BC + P \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$. An ideal A of N is said to be P-semiprime if $B^2 + P \subseteq A$ implies $B \subseteq A$ for any ideal B of N.

If A is a prime ideal then clearly A is a A-prime ideal for any ideal P. The concept P-primeness in near-rings is a generalization of 0-primeness in near-rings. The following gives an example of a P-prime ideal that is not a prime ideal.

Example 2.5 Let $N = \{0, a, b, c\}$ be Klein's four-group. Multiplication in N is defined with the following table:

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then $(N, +, .)$ is a commutative near-ring with identity [20]. The ideals of N are $\{0\}$, $\{0, a\}$, $\{0, b\}$ and N. Let $P = \{0, b\}$. Clearly $\{0\}$ is P-prime but not prime since $\{0, a\}\{0, b\} \subseteq \{0\}$ but $\{0, a\} \nsubseteq \{0\}$ and $\{0, b\} \nsubseteq \{0\}$.

3 *P*- Regularities in Near-rings

Definition 3.1 Let N be a near-ring. Also, let $P \trianglelefteq N$. Then the set $\{x \in N: xn - nx \in P, \forall n \in N\}$ is called the P-center of N and denoted by $C^P(N)$. Elements of $C^P(N)$ are called P-central elements.

If $P = 0$, then the elements of the P-center of N are also elements of $C(N)$. If $x \in C(N)$, then $x \in C^P(N)$ for all ideals P of N. But in general an element of $C^P(N)$ does not have to be an element of $C(N)$.

Definition 3.2 Let N be a near-ring, $P \trianglelefteq N$ and $a \in N$. Then, the set $\{n \in N: an - na \in P\}$ is called the P-center of element a and is denoted by $C_a^P(N)$.

If $P = 0$, then the elements of P-center of $a \in N$ are also elements of $C_a(N)$. If $x \in C_a(N)$, then $x \in C_a^P(N)$ for all ideals P of N. But in general an element of $C_a^P(N)$ does not not have to be an element of $C_a(N)$.

Example 3.3 Let $N = \{0, a, b, c\}$ be Klein's four-group. Multiplication in N is defined with the following table:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Then $(N, +, .)$ is an abelian near-ring with identity [20]. The ideals of N are $\{0\}$, $\{0, a\}$ and N. Let $P_1 = \{0\}$. Then

$C^{P_1}(N) = C(N) = \{0, c\}$, $C_0^{P_1}(N) = C_0(N) = N$, $C_a^{P_1}(N) = C_a(N) = \{0, a, c\}$, $C_b^{P_1}(N) = C_b(N) = \{0, b, c\}$, $C_c^{P_1}(N) = C_c(N) = N$.

Also, let $P_2 = \{0, a\}$. Then

$$C^{P_2}(N) = C_0^{P_2}(N) = C_a^{P_2}(N) = C_b^{P_2}(N) = C_c^{P_2}(N) = N.$$

It can be seen that $C^{P_1}(N) \subseteq C_x^{P_1}(N)$ and $C^{P_2}(N) \subseteq C_x^{P_2}(N)$ for all $x \in N$.

It is well known that in a near-ring N, $C(N) \subseteq C_a(N)$ for all $a \in N$. We have the following:

Proposition 3.4 Let N be a near-ring and $P \trianglelefteq N$. Then $C^P(N) \subseteq C_a^P(N)$ for all $a \in N$.

Proof. Let $x \in C^P(N)$, then $nx - xn \in P$ for all $n \in N$ by Definition 3.1. $x \in C_a^P(N)$ since $ax - xa \in P$ for $n = a$.

Definition 3.5 Let N be a near-ring, $P \trianglelefteq N$ and $e \in N$. An element e of N is called a P-right identity if $n - ne \in P$ for all $n \in N$. It is called a P-left identity if $n - en \in P$ for all $n \in N$. It is called a P-identity if it is both P-right and P-left identity. The set consisting of P-identity elements of N is denoted by U_P.

If $P = 0$, then P-identity of N is also identity of N. If e is an identity element of N, then e is P-identity for all ideals P of N. But in general a P-identity element of N does not have to be an identity element of N.

Example 3.6 Multiplication $*$ on the group $(Z_6, +)$ is defined with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Then $(Z_6, +, *)$ is a near-ring [21]. Let $P_1 = \{0,3\} \lhd Z_6$. Then 4 is not left identity, since $1 \neq 4 * 1$, but is P_1-left identity since $n - 4n \in P_1$ for all $n \in N$. It is right identity and P_1-right identity. Thus, it is not identity but P_1-identity. Also $1 \in Z_6$, is P_1-identity. Hence $\{1,4\}$ is the set of P_1-identities. On the other hand, if $P_2 = \{0,2,4\} \lhd Z_6$, then the set of P_2-right identities is N and the set of P_2-left identities is $\emptyset$. Thus the set of P_2-identities is $\emptyset$.

Theorem 3.7 Let $(N, +, .)$ be a near-ring and $P \trianglelefteq N$. If $U_P \neq \emptyset$ then U_P is a subsemigroup of $(N, .)$.

Proof. Let $e_1, e_2 \in U_P$. There exist $p, p^I, p^{II}, p^{III} \in P$ such that

$$\begin{aligned} e_1e_2n - n &= e_1e_2n - e_2n + e_2n - n \\ &= e_1(n+p) - (n+p) + p^I \\ &= e_1(n+p) - e_1n + e_1n - (n+p) + p^I \\ &= p^{II} + e_1n - p - n + p^I \\ &= p^{II} + e_1n - n + p^{III} \in P. \end{aligned}$$

In a similar way, $ne_1e_2 - n \in P$ can also be demonstrated. Then we have $e_1, e_2 \in U_P$ and hence U_P is a subsemigroup of $(N,.)$.

Corollary 3.8 Let N be a near-ring, $P, J \trianglelefteq N$, and U_P and U_J be the sets of P-identities and J-identities, respectively.
(i) If $P \subseteq J$, then $U_P \subseteq U_J$.
(ii) If $P = J$, then $U_P = U_J$.

Proof. **(i)** By Definition 3.5, if $x \in U_P$ then $xn - n \in P$ and $nx - n \in P$ for all $n \in N$. Since $P \subseteq J$, $xn - n \in J$ and $nx - n \in J$ for all $n \in N$, hence $x \in U_J$. Therefore, $U_P \subseteq U_J$.

(ii) Can easily be seen, hence omitted.

Definition 3.9 Let N be a near-ring and $P \trianglelefteq N$. For $x \in N$, the set $x_r^{-1} = \{y \in N : xy \in U_P\}$ is called the set of P-right inverses of x, the set $x_l^{-1} = \{y \in N : yx \in U_P\}$ is called the set of P-left inverses of x. The set $x_P^{-1} = x_r^{-1} \cap x_l^{-1}$ is called the set of P-inverses of x.

Example 3.10 Let $N = (Z_6, +, *)$ be a near-ring and $P = \{0,3\}$ given in Example 3.6. We have $1_P^{-1} = 4_P^{-1} = \{1,4\}$ and $2_P^{-1} = 5_P^{-1} = \{2,5\}$, since $0_r^{-1} = \emptyset = 0_l^{-1}$, $1_r^{-1} = \{1,4\} = 1_l^{-1}$, $2_r^{-1} = \{2,5\} = 2_l^{-1}$, $3_r^{-1} = \emptyset = 3_l^{-1}$, $4_r^{-1} = \{1,4\} = 4_l^{-1}$, $5_r^{-1} = \{2,5\} = 5_l^{-1}$.

Proposition 3.11 If e is a P-identity element of N, then it is P-idempotent in N.

Proof. This can easily be seen from Definition 3.5.

Let the near-ring $N = (Z_6, +, *)$ and $P = \{0,3\}$ given in Example 3.6. Then $e = 4$ is a P-identity. Also, $x = 4$ is a P-idempotent, since $4 - 4 * 4 = 0 \in P$. In a similar way, it can easily be seen that $1 \in Z_6$ is also a P-idempotent.

Proposition 3.12 Let N be a near-ring and $P \trianglelefteq N$.
(i) If $NP \nsubseteq P$, then $P \cap C^P(N) = \emptyset$.

(ii) If $NP \subseteq P$, then $P \subseteq C^P(N)$.
(iii) If $C(N) \neq \emptyset$, then $C(N) \subseteq C^P(N)$.
(iv) If $C(N) \neq \emptyset$ and $NP \subseteq P$, then $P \cup C(N) \subseteq C^P(N)$.
(v) If N is a unital near-ring with unity 1, then $1 \in C^P(N)$.
(vi) If e is a P-identity of N, then $e \in C^P(N)$.

Proof. Let N be a near-ring and $P \trianglelefteq N$.

(i) Let $NP \nsubseteq P$. Let $x \in C^P(N)$. Then for all $n \in N$, $xn - nx \in P$ from Definition 3.1. If $x \in P$, then there exists an $x = p \in P$ such that $pn - np \in P$ for all $n \in N$. Hence $np \in P$ for all $n \in N$. This is a contradiction, since $NP \nsubseteq P$. Then $x \notin P$, so $P \cap C^P(N) = \emptyset$.

(ii) Let $NP \subseteq P$. Then $P \subseteq C^P(N)$, since $np - pn \in P$ for each $p \in P$ and all $n \in N$.

(iii) Let $C(N) \neq \emptyset$. If $x \in C(N)$, then $xn - nx = 0$ for all $n \in N$, so $xn - nx \in P$. Hence, $C(N) \subseteq C^P(N)$.

(iv) The proof is clear using parts (ii) and (iii).

(v) Let N be a unital near-ring with unity 1 and $P \trianglelefteq N$. Then for all $\in N$ $1n - n1 \in P$, since $1n = n1$. Hence, $1 \in C^P(N)$.

(vi) Let element e be a P-identity. Then for all $n \in N$, $en - ne \in P$, since $n = en + p = ne + p^I$ for some $p, p^I \in P$. Hence $e \in C^P(N)$.

Example 3.13 Let $N = \{0,1,2,3\}$ be Klein's four-group. Multiplication in N is defined with the following table:

.	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	0	0	2
3	1	1	1	3

Then $(N, +, .)$ is a near-ring [22]. The ideals of N are $\{0\}$, $\{0,1\}$, $\{0,2\}$ and N. Also $C(N) = \emptyset$. For $P_1 = \{0,1\}$ and $NP_1 \subseteq P_1$,

$$C^{P_1}(N) = \{0,1\}, C_0^{P_1}(N) = C_1^{P_1}(N) = N, C_2^{P_1}(N) = \{0,1,2\}, C_3^{P_1}(N) = \{0,1,3\}.$$

It can be seen that $P_1 \subseteq C^{P_1}(N)$. For $P_2 = \{0,2\}$ and $NP_2 \nsubseteq P_2$,

$$C^{P_2}(N) = \emptyset, C_0^{P_2}(N) = C_2^{P_2}(N) = \{0,2\}, C_1^{P_2}(N) = C_3^{P_2}(N) = \{1,3\}.$$

It can be seen that $P_2 \cap C^{P_2}(N) = \emptyset$.

Also in Example 3.3, let $P = \{0, a\}$. Then $NP \subseteq P$ and $C^P(N) = N$. Besides $C(N) = \{0, c\}$. Thus, it can be easily seen that if $C(N) \neq \emptyset$ and $NP \subseteq P$ then $P \cup C(N) \subseteq C^P(N)$.

Proposition 3.14 Let N be a near-ring, $P, J \trianglelefteq N$. If $P \subseteq J$ then $C^P(N) \subseteq C^J(N)$.

Proof. Let $P \subseteq J$ and $x \in C^P(N)$. Then for all $n \in N$, $xn - nx \in P$ by Definition 3.1. $xn - nx \in J$ since $P \subseteq J$. Hence, $x \in C^J(N)$. This completes the proof.

Example 3.15 Addition and multiplication in N are respectively defined as follows:

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

$.$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	0	0	0	0
3	0	3	2	1	4	5	6	7
4	0	4	2	6	4	0	6	2
5	0	5	0	5	0	5	0	5
6	0	6	2	4	4	0	6	2
7	0	7	0	7	0	5	0	5

Then $(N, +, .)$ is a near-ring with identity [12,20]. Let $P = \{0,2\} \triangleleft N$ and $J = \{0,2,5,7\} \triangleleft N$. Then, we obtain $C^P(N) = N$ and $C^J(N) = N$. It can easily be seen that if $P \subseteq J$, then $C^P(N) \subseteq C^J(N)$.

Definition 3.16 Let N be a near-ring and $P \trianglelefteq N$. N is said to be P-right permutable if $xyz - xzy \in P$, P-left permutable if $xyz - yxz \in P$ for all $x, y, z \in N$. It is said to be a P-medial near-ring, if $xyzt - xzyt \in P$ for all $x, y, z, t \in N$.

Proposition 3.17 If N is right permutable (resp. left permutable, medial), then N is P-right permutable (P-left permutable, P-medial) for all ideals P of N. But in general the converse does not hold.

Proof. Let N be right permutable. For all $x, y, z \in N$, $xyz = xzy$, so $xyz - xzy = 0$. Then $xyz - xzy \in P$ for all ideals P of N. Hence, N is P-right permutable. Similarly, it can easily be shown that if N is left permutable (resp. medial), then it is P-left permutable P-medial).

In the following example, it can easily be seen that the converse does not hold.

Example 3.18 Let $N = \{0, a, b, c\}$ be Klein's four-group. Multiplication in N is defined with the following table:

.	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

Then $(N, +, .)$ is a near-ring [20]. This near-ring is not right permutable, since $bca \neq bac$, but is P-right permutable, since $bca - bac \in P$ for $P = \{0, a\} \lhd N$. Then, it can easily be seen that a P-right permutable near-ring does not have to be right permutable. Also, it is left permutable, medial, and hence P-left permutable and P-medial.

Definition 3.19 A near ring $(N, +, .)$ is said to be P-commutative if $xy - yx \in P$ for all $x, y \in N$. It is said to be P-abelian if $x + y - x - y \in P$ for all $x, y \in N$.

If $P = 0$, then a P-commutative (resp. P-abelian) near-ring is also a commutative (resp. abelian) near-ring. If N is commutative (resp. abelian), then it is P-commutative (resp. P-abelian) for all ideals P of N. But in general a P-commutative (resp. P-abelian) near-ring does not have to be a commutative (resp. abelian) near-ring.

In Example 3.18, N is not commutative since $ca \neq ac$, but it is P-commutative. For instance, we see that $ca - ac \in P$ for $P = \{0, a\} \lhd N$. Also, it is abelian, and hence P-abelian for all $P \unlhd N$.

In the Example 3.15, N isn't abelian since $4 + 5 \neq 5 + 4$, but it is a P-abelian near-ring. For instance, $4 + 5 - 4 - 5 \in P$ for $P = \{0,2\} \lhd N$.

Theorem 3.20 Let N be a near-ring, $P \unlhd N$ and $C^P(N) = N$. Then N is a P-regular near-ring if and only if N is a P-strongly regular near-ring.

Proof. $\Longrightarrow$: For all $a \in N$, there exists an element $x \in N$ such that $a = axa + p$ for some $p \in P$ since N is P-regular and for all $a \in N$, there exists a $p^I \in P$ such that $ax = xa + p^I$ since $C^P(N) = N$. Then for $p, p^I, p^{II}, p^{III} \in P$,

$$a = axa + p = (xa + p^I)a + p = xa^2 + p^{II} + p = xa^2 + p^{III}.$$

Hence N is a P-strongly regular near-ring by Definition 2.3.

$\Longleftarrow$: For all $a \in N$, there exists an element $x \in N$ such that $a = xa^2 + p$ for some $p \in P$ since N is P-strongly regular. Then for $p, p^I, p^{II}, p^{III} \in P$,

$$a = xa^2 + p = xaa + p = (ax + p^I)a + p = axa + p^{II} + p = axa + p^{III}.$$

Hence N is a P-regular near-ring by Definition 2.2.

We give a characterization for P-idempotent (resp. P-regular) elements of a near-ring with the following two theorems.

Theorem 3.21 $x \in N$ is a P-idempotent element if and only if $x + p$ is a P-idempotent element for all $p \in P$.

Proof. $\Longrightarrow$: Let $x \in N$ be a P-idempotent element. Then $x^2 - x \in P$ by Definition 2.1. Hence, for all $p \in P$ and $p^I, p^{II}, p^{III}, p^{IV} \in P$,

$$\begin{aligned}(x+p)^2 - (x+p) &= (x+p)(x+p) - (x+p)\\ &= x(x+p) + p(x+p) - p - x\\ &= x(x+p) + p^I - p - x\\ &= x(x+p) - x^2 + x^2 + p^{II} - x\\ &= p^{III} + x^2 + p^{II} - x\\ &= p^{IV} + x^2 - x \in P.\end{aligned}$$

Then, $x + p$ is a P-idempotent element for all $p \in P$.

$\Longleftarrow$: We assume that for all $p \in P$, $x + p$ is a P-idempotent element of N, then it is true for $p = 0$. Hence, $x \in N$ is a P-idempotent element of N by Definition 2.1.

Theorem 3.22 $x \in N$ is a P-regular element if and only if $x + p$ is a P-regular element for all $p \in P$.

Proof. $\Longrightarrow$: Let $x \in N$ be a P-regular element. Then there exists an element $y \in N$ such that $x = xyx + p^I$ for some $p^I \in P$. Hence, for all $p \in P$ and $p^I, p^{II}, p^{III}, p^{IV} \in P$,

$$\begin{aligned}(x+p)y(x+p) - (x+p) &= xy(x+p) + py(x+p) - (x+p)\\ &= xy(x+p) + p^{II} - (x+p)\\ &= xy(x+p) - xyx + xyx + p^{II} - (x+p)\\ &= p^{III} + xyx + p^{II} - p - x\\ &= xyx - x + p^{IV} \in P.\end{aligned}$$

Then, $x + p$ is a P-regular element for all $p \in P$.

$\Longleftarrow$: We assume that for all $p \in P$, $x + p$ is a P-regular element of N, then it is true for $p = 0$. Hence, $x \in N$ is a P-regular element of N.

4 *P*-Completely Prime Ideals

Definition 4.1 Let N be a near-ring and $P, I \trianglelefteq N$ such that $I \neq N$. If for $a, b \in N$, $ab + P \subseteq I$ implies $a \in I$ or $b \in I$, then I is called a P-completely prime ideal. If for $a \in N$, $a^2 + P \subseteq I$ implies $a \in I$, then I is called a P-completely semiprime ideal.

If $P = 0$, then a P-completely prime (resp. P-completely semiprime) ideal is a completely prime (resp. completely semiprime) ideal. If I is a completely prime (resp. completely semiprime) ideal, then it is a P-completely prime (resp. P-completely semiprime) ideal for all ideals P of N. But in general a P-completely prime (resp. P-completely semiprime) ideal does not have to be a completely prime (resp. completely semiprime) ideal.

Example 4.2 In the Example 3.15, let $I = \{0,1\} \triangleleft N$. Then I is not a completely prime ideal and a completely semiprime ideal since $2 \notin I \Rightarrow 2.2 = \{0\} \in I$, but it is a P-completely prime ideal and a P-completely semiprime ideal. For example, we can see that $2 \notin I$ and $2.2 + P \nsubseteq I$ for $P = \{0,2\} \triangleleft N$.

Lemma 4.3 If I is a P-completely prime ideal, then it is a P-completely semiprime ideal.

Proof. In Definiton 4.1, if we take $a = b$, then the result is obvious.

Definition 4.4 Let N be a near-ring and $P, I \trianglelefteq N$. If P is a P-completely prime ideal, then N is called a P-completely prime near-ring. If P is a P-completely semiprime ideal, then N is called a P-completely semiprime near-ring.

Corollary 4.5 If N is a P-completely prime near-ring, then it is a P-completely semiprime near-ring.

Proposition 4.6 Let N be a P-left permutable P-completely prime near-ring. Then, each P-idempotent element of N is P-central.

Proof. Let e be a P-idempotent element of N and let $e \notin P$. For all $n \in N$ there exist $p, p^I, p^{II}, p^{III} \in P$ such that

$$\begin{aligned}(ne - en)e + P &= nee - ene + P \\ &= n(e + p) - ene + P \\ &= n(e + p) - (n(e + p) + p^I) + P \\ &= \mathrm{p^{II}} + \mathrm{ne} - \mathrm{p^I} - \mathrm{ne} - \mathrm{p^{II}} + \mathrm{P} \\ &= p^{III} + P \subseteq P.\end{aligned}$$

Since P is a P-completely prime ideal and $e \notin P$, then $ne - en \in P$.
If $e \in P$, then it can be easily seen that $ne - en \in P$. Hence the proof is completed.

Proposition 4.7 Let N be a P-completely prime near-ring and $e \notin P$ be a P-idempotent element of N. Then

(i) e is P-right identity.
(ii) If e is P-regular and $x \in N$ is a P-regular component of element e, then $x \in N$ is P-idempotent.

Proof. **(i)** Let $e \notin P$. For all $n \in N$ there exist $p, p^I \in P$ such that

$$(ne - n)e + P = nee - ne + P = n(e + p) - ne + P = p^I + P \subseteq P.$$

Since N is a P-completely prime near-ring and $e \notin P$, we obtain $ne - n \notin P$ for all $n \in N$. Hence e is P-right identity.

(ii) Since e is a P-regular, there exists an $x \in N$ such that $e - exe \in P$. We know that $x - xe \in P$ from case (i) and $e^2 - e \in P$, since e is P-idempotent. There exist $p, p^I, p^{II}, p^{III}, p^{IV} \in P$ such that

$$\begin{aligned} x &= xe + p \\ &= x(exe + p^{I}) - xexe + xexe + p \\ &= xexe + p^{II} \\ &= (x + p^{III})(x + p^{III}) + p^{II} \\ &= x(x + p^{III}) - x^2 + x^2 + p^{IV} \\ &= x^2 + p^{IV}. \end{aligned}$$

Thus $x \in N$ is P-idempotent.

Proposition 4.8 Let N be a P-left permutable P-completely prime near-ring and $e \notin P$ be a P-idempotent element of N.

(i) e is P-left identity.
(ii) $e \in C^P(N)$.

Proof. **(i)** Let $e \notin P$. There exist $p, p^I, p^{II}, p^{III} \in P$ such that

$$\begin{aligned} (en - n)e + P &= (ene - ne) + P \\ &= (nee + p - ne) + P \\ &= (n(e + p^{I}) + p - ne) + P \end{aligned}$$

$$= (\mathrm{n}(\mathrm{e}+\mathrm{p}^{\mathrm{I}}) - \mathrm{ne} + \mathrm{ne} + \mathrm{p} - \mathrm{ne}) + \mathrm{P}$$
$$= \mathrm{p}^{\mathrm{II}} + \mathrm{p}^{\mathrm{III}} + \mathrm{P} \subseteq \mathrm{P}.$$

Since N is a P-left permutable P-completely prime near-ring and $e \notin P$, we obtain $en - n \in P$. Thus, e is P-left identity.

(ii) It can easily be seen that $en - ne \in P$ from case (i) of Proposition 4.7 and case (i) of this Proposition. Hence, $e \in C^P(N)$.

Definition 4.9 If $xny - xy \in P$ for all $x, y \in N$, then $n \in N$ is called a P-internal multiplier of N.

If $P = 0$, then a P-internal multiplier of N is also an internal multiplier of N. If an element n is an internal multiplier of N, then it is a P-internal multiplier of N for all ideals P of N. But in general a P-internal multiplier of N does not have to be an internal multiplier of N.

Corollary 4.10 Let N be a P-completely prime near-ring and $e \notin P$ be P-idempotent. Then e is a P-internal multiplier in N.

Proof. **(i)** Under the assumption of e is a P-right identity element in N, i.e. for all $x \in N$ $xe - x \in P$ by Proposition 4.7, then for all $x, y \in N$ there exist $p, p^I, p^{II} \in P$ such that $xey - xy = (x+p)y - xy = xy + p^I - xy = p^{II} \in P$. Hence, we obtain $xey - xy \in P$, that is e is a P-internal multiplier in N.

Proposition 4.11 (i) Let N be a P-left permutable near-ring. If there exists an $e \in N - P$ such that e is P-idempotent and e is not P-central, then P is not a P-completely prime ideal of N.

(ii) Let N be a P-right permutable near-ring. If there exists an $e \in N - P$ such that e is P-idempotent and e is not a P-internal multiplier in N, then P is not a P-completely prime ideal of N.

Proof. **(i)** By assumption we have $e \notin P$ and there exists an $n \in N$ such that $ne - en \in P$. But we also have $(ne - en)e \in P$. In fact, there exist $p, p^I, p^{II}, p^{III} \in P$ such that

$$\begin{aligned}(en - en)e + P &= (nee - ene) + P \\ &= (n(e+p) + (nee + p^I)) + P \\ &= (\mathrm{n}(\mathrm{e}+\mathrm{p}) - \mathrm{ne} + \mathrm{ne} + (\mathrm{n}(\mathrm{e}+\mathrm{p}) + \mathrm{p}^{\mathrm{I}})) + \mathrm{P} \\ &= \mathrm{p}^{\mathrm{II}} + \mathrm{ne} - \mathrm{p}^{\mathrm{I}} - \mathrm{ne} + \mathrm{p}^{\mathrm{II}} + \mathrm{P} \subseteq \mathrm{P}.\end{aligned}$$

Hence, P is not a P-completely prime ideal.

(ii) Under assumptions we have $e \notin P$ and there exist $x, y \in N$ such that $xey - xy \notin P$. But there exist $p, p^I, p^{II}, p^{III} \in P$ such that

$$\begin{aligned}(xey - xy)e + P &= (xeye - xye) + P \\ &= (xyee + p - xye) + P \\ &= (\mathrm{xy}(\mathrm{e} + \mathrm{p}^{\mathrm{I}}) + \mathrm{p} - \mathrm{xye}) + \mathrm{P} \\ &= (\mathrm{p}^{\mathrm{II}} - \mathrm{xye} + \mathrm{p} - \mathrm{xye}) + \mathrm{P} \subseteq \mathrm{P}.\end{aligned}$$

Then P is not a P-completely prime ideal of N.

Corollary 4.12 (i) Let N be a P-left permutable near-ring, if there exists an $e \in N - P$ such that e is P-idempotent and e is not P-central, then N is not a P-completely prime near-ring.
(ii) Let N be a P-right permutable near-ring. If there exists an $e \in N - P$ such that e is P-idempotent and e is not a P-internal multiplier in N, then N is not a P-completely prime near-ring.

Corollary 4.13 Let N be a P-completely prime near-ring such that $N_d - P \neq \emptyset$. If N is a P-right permutable, then all P-idempotents of N are P-central.

Proof. Since $N_d - P \neq \emptyset$, there exists an $n_d \notin N_d - P$. Let $e \in N$ be P-idempotent. Since N is P-right permutable, for all $n \in N$ there exists a $p \in P$ such that $n_d(ne - en) + P = (n_d en + p + n_d en) + P \subseteq P$. Since N is P-completely prime and $n_d \notin P$, then $ne - en \in P$. Thus, all P-idempotents of N are P-central.

Proposition 4.14 Let $N = N_d$ be a P-completely semiprime near-ring and $e \in N$ be P-idempotent. Then e is a P-central P-idempotent element in N.

Proof. Since $N = N_d$ and $e \in N$ is P-idempotent, for all $n \in N$ there exist $p, p^I, p^{II}, p^{III}, p^{IV}, p^V \in P$ such that

$$\begin{aligned}(en - ene)^2 + P &= (enen - enene - eneen + eneene) + P \\ &= (enen - enene - en(e + p) + en(ene + p^I)) + P \\ &= (enen - enene + p^{II} - enen + p^{III} + enene) + P \\ &= (e(nen - nene) - e(nen + nene) + p^{IV}) + P \\ &= p^V + P \subseteq P.\end{aligned}$$

Since N is P-completely semiprime, we obtain $en - ene \in P$. In a similar way, $ne - ene \in P$ can also be demonstrated. Thus, we have proven that $ne - en \in P$, that is e is P-central P-idempotent.

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