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Some Basic Properties of Completely Prime Ideals in Near Rings

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Abstract. In this investigation we studied completely prime, weakly completely prime, quasi completely prime and weakly quasi completely prime ideals in near-rings. Some characterizations of completely prime and weakly completely prime ideals were obtained. Moreover, we investigated relationships between completely prime and weakly completely prime ideals in near-rings. Finally, we obtained necessary and sufficient conditions for a weakly completely prime ideal to be a completely prime ideal.

Keywords: *completely prime ideal; near-ring; quasi completely prime; weakly completely prime ideal; weakly quasi completely prime.*

1 Introduction

Throughout this paper, by a near-ring N we mean a zero-symmetric near-ring with identity 1. For basic definitions in near-rings one may refer to [1]. In 1970 W.L.M. Holcombe introduced the notion of (0, 1, 2)-prime ideals of a near-ring [2]. In 1977 G. Pilz introduced the notion of prime ideals of a near-ring [1]. In 1988 N.J. Groenewald introduced the notion of completely (semi) prime ideals of a near-ring [3]. In 1991 N.J. Groenewald introduced the notion of 3-(semi) prime ideals of a near-ring [4]. In [5] D.D. Anderson and E. Smith defined weakly prime ideals in commutative rings; an ideal P of a ring R is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$.

In 2012 H.H. Abbass and S.M. Ibrahem introduced the concept of a completely semi prime ideal with respect to an element of a near-ring and the completely prime ideals in near-rings with respect to an element of a near-ring [6].

In this investigation we studied completely prime, weakly completely prime, quasi completely prime and weakly quasi completely prime ideals in near-rings. Some characterizations of completely prime and weakly completely prime ideals were obtained. Moreover, we investigated relationships between completely prime and weakly completely prime ideals in near-rings. Finally, we

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obtained necessary and sufficient conditions for a weakly completely prime ideal to be a completely prime ideal.

2 Basic Results

In this section we refer to [1,4,6,7] for some elementary aspects and quote a number of theorems and lemmas that are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1 [6] A near-ring is a triple $(N,+,\cdot)$ of a nonempty set N together with two binary operations "+" and "$\cdot$" (called addition and multiplication respectively) defined on N such that the following holds:

(i) $(N,+)$ is a group.

(ii) $(N,\cdot)$ is a semigroup.

(iii) $(b+c)a = ba + ca,$ for all $a,b,c \in N.$

Definition 2.2 [1] An ideal P of a near-ring N is called a completely prime ideal if for $a,b \in N$ such that $ab \in P$ implies that $a \in P$ or $b \in P.$

Definition 2.3 An ideal P of a near-ring N is called a weakly completely prime ideal if for $a,b \in N$ such that $0 \neq ab \in P$ implies that $a \in P$ or $b \in P.$

Clearly, every completely prime ideal is weakly completely prime and $\{0\}$ is always a weakly completely prime ideal of N. The following example shows that a weakly completely prime ideal need not be a completely prime ideal in general.

Example 2.4 [4] Let $N = \{0, a, b, c, d, 1, 2, 3\}$. Define addition and multiplication in N as follows:

+	0	1	2	3	a	b	c	c
0	0	1	2	3	a	b	c	d
1	1	2	3	0	d	c	a	b
2	2	3	0	1	b	a	d	c
3	3	0	1	2	c	d	b	a
a	a	d	b	c	2	0	1	3
b	b	c	a	d	0	2	3	1
c	c	a	d	b	1	3	0	2
d	d	b	c	a	3	1	2	0

$\cdot$	0	1	2	3	a	b	c	d
0	0	0	0	0	0	0	0	0
1	0	1	2	3	a	b	c	d
2	0	2	0	2	2	2	0	0
3	0	3	2	1	b	a	c	d
a	0	a	2	b	a	b	c	d
b	0	b	2	a	b	a	c	d
c	0	c	0	c	0	0	0	0
d	0	d	0	d	2	2	0	0

Then $(N,+,\cdot)$ is a near-ring but $(N,+,\cdot)$ is not a ring. Here $\{0,\ c\}$ is a weakly completely prime ideal but not completely prime, since $2\cdot 2=0\in\{0,\ c\}$.

Lemma 2.5 [7] Let A be an ideal of $(N,+,\cdot)$. Then N/A is a near-ring under the operations: For all $a,b\in N$

$$(a+A)+(b+A)=(a+b)+A \text{ and } (a+A)(b+A)=(ab)+A.$$

Lemma 2.6 [7] Let A and B be ideals of $(N,+,\cdot)$. Then

$$(A+B)/A\approx B/(A\cap B).$$

Furthermore, if $A\subseteq B$, then $(N/A)/(B/A)\approx N/B$.

3 Main Results

We start with the following theorem that gives a relation between completely prime and weakly completely prime ideals in a near-ring. Our starting point is the following lemma:

Lemma 3.1 Let N be a near-ring and let A be a left ideal of N. Then $(A:B)$ is a left ideal in N, where $(A:B)=\{n\in N : nB\subseteq A\}$.

Proof. Let N be a near-ring and let A be a left ideal of N. Suppose that $n\in N$ and $m,n\in(A:B)$. Then $mB\subseteq A$ and $nB\subseteq A$ so that

$$(n-m)B=nB-mB\subseteq A.$$

Therefore $n-m\in(A:B)$. For $a\in(A:B)$ and $n\in N$,

$$\begin{aligned}(n+a-n)B &= nB+aB-nB\\ &\subseteq nB+A-nB\\ &\subseteq A\end{aligned}$$

since A is a left ideal of N. Therefore, $n+a-n\in(A:B)$. Thus $(A:B)$ is a normal subgroup of N. Let $m,n\in N, a\in(A:B)$. Then

$$\begin{aligned}(m(n-a)-mn)B &= (m(n-a))B-(mn)B\\ &= m((n-a)B)-(mn)B\\ &= m(nB-aB)-(mn)B\\ &= m(nB-aB)-(mn)B\\ &\subseteq A\end{aligned}$$

Thus $m(n-a)-mn\in(A:B)$. Hence $(A:B)$ is a left ideal in N. □

Definition 3.2 [1] A left ideal P of a near-ring N is called a quasi completely prime ideal if for $a,b\in N$ such that $a,b\in P$ implies that $a\in P$ or $b\in P$.

Definition 3.3 A left ideal P of a near-ring N is called a weakly quasi completely prime ideal if for $a,b\in N$ such that $0\neq ab\in P$ implies that $a\in P$ or $b\in P$.

Theorem 3.4 Let N be a near-ring, and let A be an ideal of N. If A is a weakly quasi completely prime (quasi completely prime) ideal of N, then $(A:B)$ is a weakly quasi completely prime (quasi completely prime) ideal in N, where $B\not\subset A$.

Proof. Let N be a near-ring, and let A be a weakly quasi completely prime ideal of N. Suppose that $0\neq mn\in(A:B)$ and $m\notin(A:B)$. Then

$$0\neq m(nB)=(mn)B\subseteq A.$$

By the definition of weakly quasi completely prime ideal we get $m\in A$ or $nB\subseteq A$ so that $n\in(A:B)$. Hence $(A:B)$ is a weakly quasi completely prime ideal in N. □

Corollary 3.5 Let N be a near-ring and let A be a weakly quasi completely prime (quasi completely prime) ideal of N. Then $(A:m)$ is a weakly quasi completely prime (quasi completely prime) ideal in N, where $m\in N-A$.

Proof. This follows from Theorem 3.4. □

Theorem 3.6 Let N be a near-ring and let P be an ideal of N. If P is a weakly completely prime ideal that is not completely prime, then $P^2=0$.

Proof. Let N be a near-ring. Suppose that $P^2 \neq 0$ we show that P is a completely prime ideal in N. Let $ab \in P$, where $a,b \in N$. If $ab \neq 0$, then either

$$a \in P \text{ or } b \in P$$

since P is a weakly completely prime ideal. So suppose that $ab = 0$. If $Pb \neq 0$, then there is an element p' of P such that $p'b \neq 0$, so that

$$0 \neq p'b = p'b + 0 = p'b + ab = (p' + a)b \in P,$$

and hence P as a weakly completely prime ideal gives either $p' + a \in P$ or $b \in P$. As $p' + a \in P$ and $p' \in P$ we have either $a \in P$ or $b \in P$. So we can assume that $Pb = 0$. Similarly, we can assume that $Pa = 0$. Since $P^2 \neq 0$, there exist $c, d \in P$ such that $cd \neq 0$. Then

$$0 \neq (a + c)(b + d) \in P,$$

so either $a + c \in P$, or $b + d \in P$, and hence either $a \in P$ or $b \in P$. Thus P is a completely prime ideal. Clearly, $0 \subseteq P^2$. Hence, $P^2 = 0$, as required. □

Corollary 3.7 Let N be a near-ring and let P an ideal of N. If $P^2 \neq 0$, then P is completely prime if and only if P is weakly completely prime.

Proof. This follows from Theorem 3.6. □

Lemma 3.8 Let N be a near-ring, and let P be a proper ideal of N. If P is a weakly completely prime ideal of N, then

$$(P : Na) = P \cup (0 : Na),$$

where $a \in N - P$.

Proof. Let N be a near-ring, and let P be a weakly completely prime ideal of N. Clearly,

$$P \cup (0 : Na) \subseteq (P : Na).$$

For the other inclusion, suppose that $m \in (P : Na)$, so that

$$m(Na) \subseteq P.$$

If $0 \neq m(Na)$ and P is a weakly completely prime ideal of N then $Na \subseteq P$. If $0 = m(Na)$, then $m \in (0 : Na)$. So we have the equality. □

Corollary 3.9 Let N be a near-ring and let P be a proper ideal of N. If P is a weakly completely prime ideal of N, then $(P : a) = P \cup (0 : a)$, where $a \in N - P$.

Proof. This follows from Lemma 3.8. □

Corollary 3.10 Let N be a near-ring with identity, P be a proper ideal of N and $a \in N - P$. If $(P : Na) = P \cup (0 : Na)$ and $(P : Na) = P$, then $(P : Na) = (0 : Na)$.

Proof. This follows from Lemma 3.8. □

Theorem 3.11 Let N be a near-ring with identity and let P be a proper ideal of N. If $(P : n) = P$ or $(P : n) = (0 : n)$, then P is a weakly completely prime ideal of N, where $n \in N - P$.

Proof. Let N be a near-ring with identity and let P be a proper ideal of N. Suppose that Let $0 \neq mn \in P$, where $m \in N - P$. Then

$$m \in (P : n) = P \cup (0 : n)$$

by Corollary 3.10, hence $m \in P$ since $mn \neq 0$, as required. □

Theorem 3.12 Let $N = N_1 \times N_2$, where each N_i is a near-ring with identity. If P is a weakly completely prime (completely prime) ideal of N_1, then $P \times N_2$ is a weakly completely prime (completely prime) ideal of N.

Proof. Suppose that $N = N_1 \times N_2$, where each N_i is a near-ring with identity and P is a weakly completely prime ideal of N_1. Let

$$0 \neq (a,b)(c,d) = (ac,bd) \in P \times \mathrm{N}_2,$$

where $(a,b),(c,d) \in N$ so either $a \in P$ *or* $c \in P$ since P is weakly completely prime. It follows that either

$$(a,b) \in P \times N_2 \text{ or } (c,d) \in P \times N_2$$

By the definition of weakly completely prime ideal, we have $P \times N_2$ is a weakly completely prime ideal of N. □

Corollary 3.13 Let $N = N_1 \times N_2$, where each N_i is a near-ring with identity. If P is a weakly completely prime (completely prime) ideal of N_2, then $N_1 \times P$ is a weakly completely prime (completely prime) ideal of N.

Proof. This follows from Lemma 3.12. □

Corollary 3.14 Let $N = \prod_{i=1}^{n} N_i$, where each N_i is a near-ring with identity. If P is a weakly completely prime (completely prime) ideal of N_j, then $N_1 \times N_2 \times \ldots \times P_j \times N_{j+1} \times \ldots \times N_n$ is a weakly completely prime (completely prime) ideal of N.

Proof. This follows from Theorem 3.12 and Corollary 3.13. □

Theorem 3.15 Let $N = N_1 \times N_2$, where each N_i is a near-ring with identity. If P is a weakly completely prime ideal of N, then either $P = 0$ or P is completely prime.

Proof. Let $N = N_1 \times N_2$, where each N_i is a near-ring with identity and let $P = P_1 \times N_2$ be a weakly completely prime ideal of N. We can assume that $P \neq 0$. So there is an element (a,b) of P with $(a,b) \neq (0,0)$. Then

$$(0,0) \neq (a,1)(1, b) \in P,$$

gives either

$$(a,1) \in P \text{ or } (1, b) \in P = P_1 \times N_2.$$

If $(a,1) \in P$, then $P = P_1 \times N_2$. We will show that P_1 is completely prime hence P is weakly completely prime by Theorem 3.12. Let $cd \in P_1$, where $c, d \in N_1$. Then

$$(0,0) \neq (c,1)(d,1) = (cd,1) \in P,$$

so either $(c,1) \in P$ or $(d,1) \in P$ and hence either $c \in P_1$ or $d \in P_1$. By a similar argument, $N_1 \times P_2$ is completely prime. □

Proposition 3.16 Let $A \subseteq P$ be proper ideals of a near-ring N. Then the following holds:

(i) If P is weakly completely prime (completely prime), then P / A is weakly completely prime (completely prime).

(ii) If A and P / A are weakly completely prime (completely prime), then P is weakly completely prime (completely prime).

Proof. (i) Let $0 \neq (a + A)(b + A) = ab + A \in P / A,$ where $a, b \in N,$ so $ab \in P.$ If $ab = 0 \in A,$ then

$$(a + A)(b + A) = 0,$$

a contradiction. So if P is weakly completely prime, then either $a \in P$ or $b \in P,$ hence either $a + A \in P / A$ or $b + A \in P / A,$ as required.

(ii) Let $0 \neq ab \in P,$ where $a, b \in N,$ so $(a + A)(b + A) \in P / A.$ For $ab \in A,$ if A is weakly completely prime, then either

$$a \in A \subseteq P \text{ or } b \in A \subseteq P.$$

So we may assume that $ab \notin A.$ Then either $a + A \in P / A$ or $b + A \in P / A.$ It follows that either $a \in P$ or $b \in P$ as needed. □

Theorem 3.17 Let P and Q be weakly completely prime ideals of a near-ring N that are not completely prime. Then $P + Q$ is a weakly completely prime ideal of N.

Proof. Since $(P + Q) / Q \approx Q / (P \cap Q),$ we get that $(P + Q) / Q$ is weakly completely prime by Proposition 3.16 (i). Now the assertion follows from Proposition 3.16 (ii). □

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