Hankel Inequalities for a Subclass of Bi-Univalent Functions based on Salagean type $q$-Difference Operator

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Abstract. In this investigation a new subclass of bi-univalent functions is established that are defined in the open unit disk $\Delta= \{ z \in \mathbb{C} ; |z| < 1 \}$ and are endowed with the Salagean type $q$-difference operator. Then, Hankel inequalities for the new function class are obtained and several related consequences of the results are also stated.

Keywords: bi-univalent; coefficient bounds; convex functions; Hankel inequalities; Starlike; univalent.

1 Introduction

Let $\mathcal{A}$ indicate the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

normalized by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ indicate the subclass of $\mathcal{A}$ comprising of functions of the form Eq. (1) and also univalent in $\Delta$.

For the function $f \in \mathcal{A}$, Jackson’s $q$-derivative [1] $(0 < q < 1)$ is expressed by:

$$\mathcal{D}_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \tag{2}$$

and $\mathcal{D}_q^2 f(z) = \mathcal{D}_q (\mathcal{D}_q f(z))$. Thus, from Eq. (2), we deduce that

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$ 

If $q \to 1^-$, we get $[n]_q \to n$. 

Received October 28th, 2017, Revised May 13th, 2020, Accepted for publication June 2nd, 2020
Copyright © 2020 Published by ITB Institute for Research and Community Services, ISSN: 2337-5760,
DOI: 10.5614/j.math.fund.sci.2020.52.2.4
Lately, in [2] the Şalagean type $q$-differential operator has been introduced as given by
\[
\mathcal{D}_q^0 f(z) = f(z)
\]
\[
\mathcal{D}_q^1 f(z) = z \mathcal{D}_q f(z)
\]
\[
\mathcal{D}_q^k f(z) = z \mathcal{D}_q (\mathcal{D}_q^{k-1} f(z))
\]
\[
\mathcal{D}_q^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \Delta).
\]
(3)

For $q \to 1^-$, we get
\[
\mathcal{D}_q^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \Delta)
\]
the familiar Şalagean derivative [3].

Noonan and Thomas [4] introduced the $q^{th}$ Hankel determinant of function $f$ by
\[
H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix} \quad (q \geq 1).
\]

In particular,
\[
H_2(1) = \begin{vmatrix}
a_1 & a_2 \\
a_2 & a_3
\end{vmatrix} = a_1 a_3 - a_2^2 = a_3 - a_2^2
\]
and
\[
H_2(2) = \begin{vmatrix}
a_2 & a_3 \\
a_3 & a_4
\end{vmatrix} = a_2 a_4 - a_3^2.
\]

Then, Fekete and Szegö [5] obtained estimates of $|H_2(1)| = |a_3 - \theta a_2^2|$ for $\theta$ is real. That is, if $f \in \mathcal{A}$, then
\[
|a_3 - \theta a_2^2| \leq \begin{cases}
4\theta - 3 & \theta \geq 1 \\
1 + 2 \exp\left(\frac{-2\theta}{1-\theta}\right) & 0 \leq \theta \leq 1.
\end{cases}
\]

Furthermore, Keogh and Merkes [6] derived sharp estimates for $|H_2(1)|$ when $f$ is starlike, convex and close-to-convex in $\Delta$.

Next, according to the Koebe One Quarter Theorem [7], every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z, (z \in \Delta)$ and $f(f^{-1}(w)) = w \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4}).$ A function $f \in \mathcal{A}$ is said to be bi-
univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ indicate the class of bi-univalent functions defined in the unit disk $\Delta$. Since $f \in \Sigma$ has the Taylor representation given by Eq. (1), computation shows that $g = f^{-1}$ has the following representation:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \ldots. \quad (4)$$

Several researchers have introduced new subclasses of bi-univalent functions and derived non-sharp the initial coefficients (see [8-18]).

Now, by using the Sălăgean type $q$-differential operator for functions $g$ of the form Eq. (4), we define:

$$D_g^k g(w) = w - a_2 [2]^k w^2 + (2a_2^2 - a_3)[3]^k w^3 + \ldots \quad (5)$$

and introduce a new subclass of $\Sigma$ to acquire the estimates of the initial Taylor-Maclaurin coefficients. Then, by using the values of $a_2$ and $a_3$, we derive the Fekete-Szegő and Hankel inequalities.

2 Bi-Univalent Function Class $FS_q^k(\lambda, \beta)$

In this section, we will give the following new subclass involving the Sălăgean type $q$-difference operator and also its related classes.

**Definition 2.1.** A function $f \in \Sigma$ given by Eq. (1) is said to be in the class

$$FS_q^k(\lambda, \beta) \quad (0 \leq \beta < 1, 0 \leq \lambda \leq 1, z, w \in \Delta)$$

if the following conditions hold:

$$\Re \left( (1 - \lambda) \frac{D_q^k f(z)}{z} + \lambda (D_q^k f)(z) \right) > \beta$$

and

$$\Re \left( (1 - \lambda) \frac{D_q^k g(w)}{w} + \lambda (D_q^k g)(w) \right) > \beta.$$  

**Example 2.2.** A function $f \in \Sigma$, members of which are given by Eq. (1) and

1. for $\lambda = 0$, let $FS_q^k(0, \beta) =: R_q^k(\beta)$ denote the subclass of $\Sigma$ and the following conditions hold

$$\Re \left( \frac{D_q^k f(z)}{z} \right) > \beta \quad \text{and} \quad \Re \left( \frac{D_q^k g(w)}{w} \right) > \beta$$

2. for $\lambda = 1$, let $FS_q^k(1, \beta) =: H_q^k(\beta)$ denote the subclass of $\Sigma$ and satisfy the following conditions
\[\Re \left[ (D_q^k f(z))^\beta \right] > \beta \quad \text{and} \quad \Re \left[ (D_q^k g(w))^\beta \right] > \beta.\]

3 \textbf{ Hankel Inequalities for } f \in \mathcal{FS}_\Sigma^k(\lambda, \beta)\]

In this section, we will determine the functional \(|a_2a_4 - a_3^2|\) for the functions \(f \in \mathcal{FS}_\Sigma^k(\lambda, \beta)\) due to Altunkaya and Yalçın [19]. Now, we recall the following lemmas:

**Lemma 3.1.** (See [4]) Let \(\mathcal{P}\) be the well-known class of Carathéodory functions, that is \(c(z) \in \mathcal{A}\) with the power series expansion
\[
c(z) = 1 + \sum_{n=1}^\infty c_n z^n \quad (z \in \Delta)
\]
and \(\Re(c(z)) > 0\). Then
\[
|c_n| \leq 2 \quad (n = 1, 2, 3, \ldots)
\]
and is sharp for each \(n\). Indeed,
\[
c(z) = \frac{1 + z}{1 - z} = 1 + \sum_{n=1}^\infty 2 z^n \quad (\forall n \geq 1).
\]

**Lemma 3.2.** (See [20]) If \(c \in \mathcal{P}\), then
\[
2c_2 = c_1^2 + x(4 - c_1^2),
\]
\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2z)
\]
for some complex numbers \(x, z\) with \(|x| \leq 1\) and \(|z| \leq 1\).

**Lemma 3.3.** (See [5]) The power series for \(c\) converges in \(\Delta\) to a function in \(\mathcal{P}\) if and only if the Toeplitz determinants
\[
T_n = \begin{vmatrix}
2 & c_1 & c_2 & \cdots & c_n \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
c_{-n} & c_{-n+1} & \cdots & \cdots & 2
\end{vmatrix} \quad (n = 1, 2, 3, \ldots)
\]
and \(c_{-k} = 2\) are all nonnegative. They are exactly positive except for
\[
c(z) = \sum_{k=1}^m \rho_k c_0(e^{t_kx^2}), \quad \rho_k > 0, \ t_k \text{ real}
\]
and \(t_k \neq t_j \quad (k \neq j)\). In this case \(T_n > 0 \quad (n < m - 1)\) and \(T_n = 0 \quad (n \geq m)\).

Next, we designate the second Hankel coefficient estimates for \(f \in \mathcal{FS}_\Sigma^k(\lambda, \beta)\).
Theorem 3.4. Let $f \in \mathcal{F}_q^{k}(\lambda, \beta)$. Then

\[
\frac{|a_2 a_4 - a_3^2|}{H(2)}, \quad A(\beta, \lambda, k, q) \geq 0, B(\beta, \lambda, k, q) \geq 0
\]

\[
\max \left\{ \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2 k^3_q}, H(2) \right\}, \quad A(\beta, \lambda, k, q) > 0, B(\beta, \lambda, k, q) < 0
\]

\[
\frac{4(1 - \beta)^2}{(1 + 2\lambda)^2 k^2_q}, \quad A(\beta, \lambda, k, q) \leq 0, B(\beta, \lambda, k, q) \leq 0
\]

\[
\max \{ H(\varepsilon_0), H(2) \}, \quad A(\beta, \lambda, k, q) < 0, B(\beta, \lambda, k, q) > 0
\]

where

\[
H(2) = \frac{16(1 - \beta)^4}{(1 + \lambda)^4 k^4_q} + \frac{4(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2^k_q][4^k_q]},
\]

\[
H(\varepsilon_0) = \frac{-B(\beta, \lambda, k, q)}{A(\beta, \lambda, k, q)} = \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2 k^2_q} \frac{-B^2(\beta, \lambda, k, q)}{4A(\beta, \lambda, k, q)}
\]

\[
A(\beta, \lambda, k, q) = \frac{(1 - \beta)^4}{(1 + \lambda)^4 k^4_q} - \frac{(1 - \beta)^3}{4(1 + \lambda)^2(1 + 2\lambda)[2^k_q][3^k_q]} + \frac{(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2^k_q][4^k_q]}
\]

\[
B(\beta, \lambda, k, q) = \frac{(1 - \beta)^3}{2(1 + \lambda)(1 + 3\lambda)[2^k_q][3^k_q]} + \frac{3(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2^k_q][4^k_q]}
\]

\[
- \frac{(1 - \beta)^2}{(1 + 2\lambda)^2 k^2_q}
\]

Proof. Suppose that $f \in \mathcal{F}_q^{k}(\beta, \lambda)$. There are two functions $\phi, \psi \in \mathcal{P}$ satisfying the conditions of Lemma 3.1 such that

\[
(1 - \lambda) \frac{\Delta^k_q f(z)}{z} + \lambda \left( \mathcal{D}_q^k f(z) \right)' = \beta + (1 - \beta) \phi(z), \quad (8)
\]

\[
(1 - \lambda) \frac{\Delta^k_q g(w)}{w} + \lambda \left( \mathcal{D}_q^k g(w) \right)' = \beta + (1 - \beta) \psi(z), \quad (9)
\]

where

\[
\phi(z) = 1 + c_2 z + c_3 z^2 + c_3 z^3 + \cdots
\]
\( \psi(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \ldots \).

Now, by comparing the corresponding coefficients in Eq. (8) and Eq. (9), we get

\[
(1 + \lambda)[2]_q^5 a_2 = (1 - \beta)c_1, \\
(1 + 2\lambda)[3]_q^5 a_2 = (1 - \beta)c_2, \\
(1 + 3\lambda)[4]_q^5 a_4 = (1 - \beta)c_3
\]

and

\[
-(1 + \lambda)[2]_q^5 a_2 = (1 - \beta)d_1, \\
(1 + 2\lambda)[3]_q^5 (2a_2^2 - a_3) = (1 - \beta)d_2, \\
-(1 + 3\lambda)[4]_q^5 (5a_2^3 - 5a_2a_3 + a_4) = (1 - \beta)d_3.
\]

From Eq. (10) and Eq. (13), we get

\[
a_2 = \frac{1 - \beta}{(1 + \lambda)[2]_q^5} c_1 = -\frac{1 - \beta}{(1 + \lambda)[2]_q^5} d_1,
\]

which implies

\[
c_1 = -d_1.
\]

Now from Eq. (11) and Eq. (14), we obtain

\[
a_3 = \frac{(1 - \beta)^2}{(1 + \lambda)^2 [2]_q^5} c_1^2 + \frac{(1 - \beta)}{2(1 + 2\lambda)[3]_q^5} (c_2 - d_2).
\]

On the other hand, subtracting Eq. (15) from Eq. (12) and using Eq. (16), we get

\[
a_4 = \frac{5(1 - \beta)^2}{4(1 + \lambda)(1 + 2\lambda)[2]_q^5 [3]_q^5} c_1 (c_2 - d_2) + \frac{(1 - \beta)}{2(1 + 3\lambda)[4]_q^5} (c_3 - d_3).
\]

Thus, we establish that

\[
\begin{align*}
|a_2a_4 - a_3^2| &= \left| -\frac{(1 - \beta)^3}{(1 + \lambda)^5 [2]_q^6} c_1^4 \\
&\quad + \frac{(1 - \beta)}{4(1 + \lambda)^2(1 + 2\lambda)[2]_q^5 [3]_q^5} c_1^2 (c_2 - d_2) \\
&\quad + \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^5 [4]_q^5} c_1 (c_3 - d_3) - \frac{(1 - \beta)^2}{4(1 + 2\lambda)[3]_q^5} (c_2 - d_2)^2 \right|.
\end{align*}
\]

Now, by Lemma 3.2, we get

\[
2c_2 = c_1^2 + x(4 - c_1^2) \quad \text{and} \quad 2d_2 = d_1^2 + y(4 - d_1^2),
\]

and hence, by Eq. (18), we have
Further, we get

\[4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,\]

\[4d_3 = d_1^3 + 2(4 - d_1^2)d_1 y - d_1(4 - d_1^2)y^2 + 2(4 - d_1^2)(1 - |y|^2)w\]

and thus, we acquire

\[c_3 - d_3 = \frac{c_1^3}{2} + \frac{c_1(4-c_1^2)}{2}(x+y) - \frac{c_1(4-c_1^2)}{4}(x^2 + y^2)\]

\[+ \frac{4 - c_1^2}{2}[(1 - |x|^2)z - (1 - |y|^2)w].\]

Using Eq. (19) – Eq. (20) in Eq. (17), we get

\[
|a_2a_4 - a_2^2| = \left|\frac{-(1 - \beta)^4}{(1 + \lambda)^4[2]_q^4} c_1^4 + \frac{(1 - \beta)^2}{4(1 + \lambda)(1 + 3\lambda)[2]_q^k[4]_q} c_1^4 \right|
\]

\[+ \frac{(1 - \beta)^2}{4(1 + \lambda)^2(1 + 2\lambda)[2]_q^2[3]_q} \frac{c_1(4-c_1^2)}{2} (x - y)
\]

\[+ \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^2[4]_q} \frac{c_1(4-c_1^2)}{2} (x + y)
\]

\[+ \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^2[4]_q} \frac{c_1(4-c_1^2)}{4} (x^2 + y^2)
\]

\[= + \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^2[4]_q} \frac{c_1(4-c_1^2)}{2} [(1 - |x|^2)z - (1 - |y|^2)w]
\]

\[- \frac{(1 - \beta)^2}{4(1 + 2\lambda)[2]_q^2[3]_q} \frac{(4-c_1^2)^2}{4} (x - y)^2|.
\]

Since \(c \in P\), we find that \(|c_1| \leq 2\). Thus, letting \(|c_1| = \varepsilon \in [0,2]\) and applying triangle inequality on Eq. (21), we get

\[
|a_2a_4 - a_2^2| \leq \frac{(1 - \beta)^4}{(1 + \lambda)^4[2]_q^4} \varepsilon^4 + \frac{(1 - \beta)^2}{4(1 + \lambda)(1 + 3\lambda)[2]_q^k[4]_q} \varepsilon^4
\]

\[+ \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^2[4]_q} \varepsilon(4 - \varepsilon^2)
\]

\[+ \left(\frac{(1 - \beta)^3}{4(1 + \lambda)^2(1 + 2\lambda)[2]_q^2[3]_q} + \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^2[4]_q}\right) \frac{\varepsilon^2(4 - \varepsilon^2)}{2} (|x| + |y|)
\]

\[+ \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^2[4]_q} \frac{\varepsilon(4 - \varepsilon^2)}{4} (|x|^2 + |y|^2).
\]
\[-\frac{(1-\beta)^2}{4(1+2\lambda)^2[3]_q^{2k}} \frac{(4-\epsilon^2)^2}{4}(|x| + |y|)^2. \quad (21)\]

For $\delta = |x| \leq 1$ and $\theta = |y| \leq 1$, we get

\[|a_2a_4 - a_2^2| \leq C_1 + C_2(\delta + \theta) + C_3(\delta^2 + \theta^2) \quad \text{(22)}\]

\[C_4(\delta + \theta)^2 = \Psi(\delta, \theta),\]

where

\[C_1 = C_1(\epsilon) = \frac{(1 - \beta)^4}{(1 + \lambda)^4[2]_q^{4k}} \frac{\epsilon^4}{4(1 + \lambda)(1 + 3\lambda)[2]_q^{6k}[4]_q^{k}} + (1 - \beta)^2 \frac{2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^{6k}[4]_q^{k}} \epsilon(4 - \epsilon^2) \geq 0,\]

\[C_2 = C_2(\epsilon) = \frac{(1 - \beta)^3}{4(1 + \lambda)(1 + 2\lambda)[2]_q^{4k}[3]_q^{k}} + (1 - \beta)^2 \frac{2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^{6k}[4]_q^{k}} \epsilon^2(4 - \epsilon^2) \geq 0,\]

\[C_3 = C_3(\epsilon) = \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^{6k}[4]_q^{k}} \frac{\epsilon^2(\epsilon - 2)(4 - \epsilon^2)}{4} \leq 0,\]

\[C_4 = C_4(\epsilon) = \frac{(1 - \beta)^2}{4(1 + 2\lambda)^2[3]_q^{2k}} \frac{(4 - \epsilon^2)^2}{4} \geq 0.\]

Next, we will find the maximum of $(\Psi(\delta, \theta))$ in $\Gamma = \{(\delta, \theta): 0 \leq \delta \leq 1, 0 \leq \theta \leq 1\}$. Since the coefficients of $\Psi(\delta, \theta)$ have dependent variable $\epsilon$, we should maximize $\Psi(\delta, \theta)$ for the cases $\epsilon = 0, 2 \epsilon = 2$ and $\epsilon \in (0,2)$.

1. Let $\epsilon = 0$. Thus, from (22), we may write

\[\Psi(\delta, \theta) = \frac{(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}} (\delta + \theta)^2.\]

2. We can find that the maximum of $\Psi(\delta, \theta)$ occurs at $\delta = \theta = 1$ and we find

\[
\max \{ \Psi(\delta, \theta): 0 \leq \delta \leq 1, 0 \leq \theta \leq 1 \} = \frac{4(1-\beta)^2}{(1+2\lambda)^2[3]_q^{2k}}.
\]

3. Let $\epsilon = 2$. Thus, $\Psi(\delta, \theta)$ is a constant function

\[\Psi(\delta, \theta) = \frac{16(1-\beta)^4}{(1+\lambda)^4[2]_q^{4k}} + \frac{4(1-\beta)^2}{(1+\lambda)(1+3\lambda)[2]_q^{6k}[4]_q^{k}}.
\]

4. Let $\epsilon \in (0,2)$. If we change $\delta + \theta = \zeta$ and $\delta, \theta = \eta$, then

\[\Psi(\delta, \theta) = C_1(\zeta) + C_2(\epsilon)\zeta + [C_3(\epsilon) + C_4(\epsilon)]\zeta^2 - 2C_3(\epsilon)\eta = \mathcal{G}(\zeta, \eta), \quad 0 \leq \zeta \leq 2, 0 \leq \eta \leq 1.
\]
Presently, we try to get maximum of $G(\zeta, \eta)$ in
$$\{ (\zeta, \eta): 0 \leq \zeta \leq 2, 0 \leq \eta \leq 1 \}.$$  
From the definition of $G(\zeta, \eta)$, we get
$$G'_{\zeta}(\zeta, \eta) = C_2(\epsilon) + 2[C_3(\epsilon) + C_4(\epsilon)]\zeta = 0,$$
$$G'_{\eta}(\zeta, \eta) = -2C_3(\epsilon) = 0.$$  
We deduce that the function doesn’t have any critical point in $\Gamma$. Thus, $\Psi(\delta, \theta)$ doesn’t have any critical point in square $\Gamma$ and so the function doesn’t get maximum value in $\Gamma$.

Next, we inspect the maximum of $\Psi(\delta, \theta)$ on the boundary of $\Gamma$. Firstly, let $\delta = 0, 0 \leq \theta \leq 1$ (or let $\delta = 0, 0 \leq \delta \leq 1$). Then, we may write
$$\Psi(0, \theta) = C_1(\epsilon) + C_2(\epsilon)\theta + [C_3(\epsilon) + C_4(\epsilon)]\theta^2 = \varphi_1(\theta).$$  
Thus,
$$\varphi_1'(\theta) = C_2(\epsilon) + 2[C_3(\epsilon) + C_4(\epsilon)]\theta.$$  
**Case (i):** If $C_3(\epsilon) + C_4(\epsilon) \geq 0$, then $\varphi_1'(\theta) > 0$. The function is increasing and so the maximum occurs at $\theta = 1$.

**Case (ii):** Let $C_2(\epsilon) + C_4(\epsilon) < 0$. Since $C_2(\epsilon) + 2[C_3(\epsilon) + C_4(\epsilon)] > 0$, $C_2(\epsilon) + 2[C_3(\epsilon) + C_4(\epsilon)]\theta \geq C_2(\epsilon) + 2[C_3(\epsilon) + C_4(\epsilon)]$ holds for all $\theta \in [0,1]$. So, $\varphi_1'(\theta) > 0$. Hence, $\varphi_1(\theta)$ is an increasing function. Thus, the maximum occurs at $\theta = 1$,
$$\max \{ \Psi(0, \theta): 0 \leq \theta \leq 1 \} = C_1(\epsilon) + C_2(\epsilon) + C_3(\epsilon) + C_4(\epsilon).$$  
Secondly, let $\delta = 1, 0 \leq \theta \leq 1$ (similarly, $\theta = 1, 0 \leq \delta \leq 1$). Then
$$\Psi(1, \theta) = C_1(\epsilon) + C_2(\epsilon) + C_3(\epsilon) + C_4(\epsilon) + [C_2(\epsilon) + 2C_4(\epsilon)]\theta + [C_3(\epsilon) + C_4(\epsilon)]\theta^2 = \varphi_2(\theta).$$  
It can be stated that $\varphi_2(\theta)$ is an increasing function like case (i). In that way,
$$\max \{ \Psi(1, \theta): 0 \leq \theta \leq 1 \} = C_1(\epsilon) + 2[C_2(\epsilon) + C_3(\epsilon)] + 4C_4(\epsilon).$$  
Also, for every $\epsilon \in (0,2)$, we can easily see that
$$C_1(\epsilon) + 2[C_2(\epsilon) + C_3(\epsilon)] + 4C_4(\epsilon) > C_1(\epsilon) + C_2(\epsilon) + C_3(\epsilon) + C_4(\epsilon).$$  
Therefore, we find that
$$\max \{ \Psi(\delta, \theta): 0 \leq \delta \leq 1, 0 \leq \theta \leq 1 \} = C_1(\epsilon) + 2[C_2(\epsilon) + C_3(\epsilon)] + 4C_4(\epsilon).$$
Since \( \varphi_1(1) \leq \varphi_2(1) \) for \( \epsilon \in [0,2] \), \( \max \Psi(\delta,\vartheta) = \Psi(1,1) \) on the boundary of \( \Gamma \). So, the maximum of \( \Psi \) occurs at \( \delta = 1 \) and \( \vartheta = 1 \) in the \( \Gamma \).

Let us define \( \mathcal{H} : (0,2) \to \mathbb{R} \) as
\[
\mathcal{H}(\epsilon) = \max \Psi(\delta,\vartheta) = \Psi(1,1) = 2[C_2(\epsilon) + C_3(\epsilon)] + C_1(\epsilon) + 4C_4(\epsilon).
\]
Therefore, from Eq. (23), we obtain
\[
\mathcal{H}(\epsilon) = \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^k} + A(\beta, \lambda, k, q) \epsilon^4 + 2B(\beta, \lambda, k, q) \epsilon^2,
\]
where
\[
A(\beta, \lambda, k, q) = \frac{(1 - \beta)^4}{(1 + \lambda)^4[2]_q^k} - \frac{(1 - \beta)^3}{4(1 + \lambda)^2(1 + 2\lambda)[2]_q^{2k}[3]_q^k} - \frac{(1 - \beta)^2}{2(1 + \lambda)(1 + 3\lambda)[2]_q^k[4]_q^k} + \frac{(1 - \beta)^2}{4(1 + 2\lambda)^2[3]_q^{2k}}
\]
\[
B(\beta, \lambda, k, q) = \frac{(1 - \beta)^3}{(1 + \lambda)^2(1 + 2\lambda)[2]_q^{2k}[3]_q^k} + \frac{3(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2]_q^k[4]_q^k} - \frac{2(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}}.
\]

Now, we try to get the maximum value of \( \mathcal{H}(\epsilon) \) in \((0,2)\). After some basic calculations, we have
\[
\mathcal{H}'(\epsilon) = 4A(\beta, \lambda, k, q) \epsilon^3 + 2B(\beta, \lambda, k, q) \epsilon.
\]

Next, we examine the different cases of \( A(\beta, \lambda, k, q) \) and \( B(\beta, \lambda, k, q) \) as follows:

**Case 1**: Let \( A(\beta, \lambda, k, q) \geq 0 \) and \( B(\beta, \lambda, k, q) \geq 0 \), then \( \mathcal{H}'(\epsilon) \geq 0 \). Hence, the maximum point has to be on the boundary of \( \epsilon \in [0,2] \), that is \( \epsilon = 2 \). Thus,
\[
\max \{ \Psi(\delta,\vartheta) : 0 \leq \delta \leq 1, 0 \leq \vartheta \leq 1 \} = \mathcal{H}(2)
\]
\[
= \frac{16(1 - \beta)^4}{(1 + \lambda)^4[2]_q^k} + \frac{4(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2]_q^k[4]_q^k}
\]
Case 2: If \( A(\beta, \lambda, k, q) > 0 \) and \( B(\beta, \lambda, k, q) < 0, e_0 = \sqrt{-\frac{B(\beta, \lambda, k, q)}{2A(\beta, \lambda, k, q)}} \) is a critical point of \( \mathcal{H}(\varepsilon) \). Since \( \mathcal{H}''(e_0) < 0 \), the maximum value of function \( \mathcal{H}(\varepsilon) \) occurs at \( \varepsilon = e_0 \) and

\[
\mathcal{H}(e_0) = \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}} + A(\beta, \lambda, k, q) e_0^4 + 2B(\beta, \lambda, k, q) e_0^2
\]

\[
= \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}} - \frac{3B^2(\beta, \lambda, k, q)}{4A(\beta, \lambda, k, q)}.
\]

In this case, \( \mathcal{H}(e_0) < \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}} \). Therefore,

\[
\max \{ \Psi(\delta, \vartheta) \colon 0 \leq \delta \leq 1, 0 \leq \vartheta \leq 1 \} = \max \left\{ \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}}, \frac{16(1 - \beta)^4}{(1 + \lambda)^4[2]_q^{2k}} + \frac{4(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2]_q^{2k}[4]_q^{2k}} \right\}. \tag{25}
\]

Case 3: If \( A(\beta, \lambda, k, q) \leq 0 \) and \( B(\beta, \lambda, k, q) \leq 0, \mathcal{H}(\varepsilon) \) is decreasing in \((0, 2)\). Therefore,

\[
\max \{ \Psi(\delta, \vartheta) \colon 0 \leq \delta \leq 1, 0 \leq \vartheta \leq 1 \} = \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}}. \tag{26}
\]

Case 4: If \( A(\beta, \lambda, k, q) < 0 \) and \( B(\beta, \lambda, k, q) > 0, e_0 \) is a critical point of \( \mathcal{H}(\varepsilon) \). Since \( \mathcal{H}''(e_0) < 0 \), the maximum value of \( \mathcal{H}(\varepsilon) \) occurs at \( \varepsilon = e_0 \) and

\[
\mathcal{H}(e_0) < \frac{4(1 - \beta)^2}{(1 + 2\lambda)^2[3]_q^{2k}}. \]

Therefore,

\[
\max \{ \Psi(\delta, \vartheta) \colon 0 \leq \delta \leq 1, 0 \leq \vartheta \leq 1 \} = \max \left\{ \mathcal{H}(e_0), \frac{16(1 - \beta)^4}{(1 + \lambda)^4[2]_q^{2k}} + \frac{4(1 - \beta)^2}{(1 + \lambda)(1 + 3\lambda)[2]_q^{2k}[4]_q^{2k}} \right\}. \tag{27}
\]

Thus, from Eqs. (24-26) and Eq. (27), the proof is completed.

**Remark 3.5.** For \( \lambda = 0 \) (and \( \lambda = 1 \)) in Theorem 3.4, we can confirm the Hankel inequalities for the function classes \( \mathcal{R}_2(\phi), \mathcal{H}_2(\phi) \), respectively.

**Acknowledgement**

The authors are grateful to the referees of this article for their valuable comments and advice.
References


