**g- Inverses of Interval Valued Fuzzy Matrices**

Arunachalam R. Meenakshi & Muniasamy Kaliraja

Department of Mathematics, Karpagam University, Coimbatore-641 021, India

Email: mkr.maths009@gmail.com

**Abstract.** In this paper, we have discussed the g- Inverses of Interval Valued Fuzzy Matrices (IVFM) as a generalization of g- inverses of regular fuzzy matrices. The existence and construction of g-inverses, \{1, 2\} inverses, \{1, 3\} inverses and \{1, 4\} inverses of Interval valued fuzzy matrix are determined in terms of the row and column spaces.

**Keywords:** g-Inverses of fuzzy matrix; g-inverses of Interval valued fuzzy matrix.

1 Introduction

A fuzzy matrix is a matrix over the max-min fuzzy algebra \(\mathcal{F} = [0,1] \) with operations defined as \(a+b = \max\{a,b\}\) and \(a \cdot b = \min\{a,b\}\) for all \(a,b \in \mathcal{F}\) and the standard order \(\geq\) of real numbers over \(\mathcal{F}\). A matrix \(A \in \mathcal{F}_{mn}\) is said to be regular if there exists \(X \in \mathcal{F}_{mn}\) such that \(AXA = A\). \(X\) is called a generalized inverse of \(A\) and is denoted by \(A^{-}\). In [1], Thomason has studied the convergence of powers of a fuzzy matrix. In [2], Kim and Roush have developed a theory for fuzzy matrices analogous to that for Boolean matrices [3]. A finite fuzzy relational equation can be expressed in the form of a fuzzy matrix equation as \(x.A = b\) for some fuzzy coefficient matrix \(A\). If \(A\) is regular, then \(x.A=b\) is consistent and \(bX\) is a solution for some g-inverse \(X\) of \(A\) [4]. For more details on fuzzy matrices one may refer to [5, 6]. Recently, the concept of the interval valued fuzzy matrix (IVFM) as a generalization of fuzzy matrix has been introduced and developed by Shyamal and Pal [7]. In earlier work, we have studied the regularity of IVFM [8] and analogous to that for complex matrices [9].

In this paper, we discuss the g-inverses of interval valued fuzzy matrices (IVFM) as a generalization of the g-inverses of regular fuzzy matrices studied in [2, 6], and as an extension of the regularity of the IVFM discussed in [8]. In section 2, we present the basic definition, notation of the IVFM and required results of g-inverses of regular fuzzy matrices. In Section 3, the existence and construction of g-inverses, \{1, 2\} inverses, \{1, 3\} inverses and \{1, 4\} inverses of interval-valued fuzzy matrices are determined in terms of the row and column spaces of IVFM.
2 Preliminaries

In this section, some basic definitions and results needed are given. Let IVFM denote the set of all interval-valued fuzzy matrices, that is, fuzzy matrices whose entries are all subintervals of the interval \([0, 1]\).

**Definition 2.1.** For a pair of fuzzy matrices \(E = (e_{ij})\) and \(F = (f_{ij})\) in \(F_{mn}\) such that \(E \leq F\), the interval valued fuzzy matrix \([E, F] = ([e_{ij}, f_{ij}]\) is the matrix, whose \(ij\)th entry is the interval with lower limit \(e_{ij}\) and upper limit \(f_{ij}\).

In particular for \(E = F\), IVFM \([E, E]\) reduces to the fuzzy matrix \(E \in F_{mn}\).

For \(A = (a_{ij}) = ([a_{ijL}, a_{ijU}]\) \(\in (IVFM)_{mn}\), let us define \(A_L = (a_{ijL})\) and \(A_U = (a_{ijU})\). Clearly, the fuzzy matrices \(A_L\) and \(A_U\) belong to \(F_{mn}\) such that \(A_L \leq A_U\).

Therefore, by Definition (2.1), \(A\) can be written as

\[
A = [A_L, A_U]
\]

where \(A_L\) and \(A_U\) are called the lower and upper limits of \(A\) respectively.

Here we shall follow the basic operation on IVFM as given in [8].

For \(A = (a_{ij}) = ([a_{ijL}, a_{ijU}]\) and \(B = (b_{ij}) = ([b_{ijL}, b_{ijU}]\) of order \(mn\), their sum, denoted as \(A + B\), is defined as

\[
A + B = (a_{ij} + b_{ij}) = [a_{ijL} + b_{ijL}, a_{ijU} + b_{ijU}]
\]

For \(A = (a_{ij})_{mn}\) and \(B = (b_{ij})_{np}\) their product, denoted as \(AB\), is defined as

\[
AB = (C_{ij}) = [\sum_{k=1}^{n} a_{ikL} b_{kjL}, \sum_{k=1}^{n} a_{ikU} b_{kjU}]
\]

If \(A = [A_L, A_U]\) and \(B = [B_L, B_U]\) then \(A + B = [A_L + B_L, A_U + B_U]\)

\[
AB = [A_L B_L, A_U B_U]
\]

\(A \geq B\) if and only if \(a_{ijL} \geq b_{ijL}\) and \(a_{ijU} \geq b_{ijU}\) if and only if \(A + B = A\)

In particular if \(a_{ijL} = a_{ijU}\) and \(b_{ijL} = b_{ijU}\) then by Eq. (3) reduces to the standard max. min. composition of fuzzy matrices [2, 6].

For \(A \in (IVFM)_{mn}\), \(A^T, R(A), C(A), A^\dagger, A \{1\}\) denotes the transpose, row space, column space, g-inverses and set of all g-inverses of \(A\), respectively.

**Lemma 2.2.** (Lemma 2 [5]) For \(A, B \in F_{mn}\) if \(A\) is regular, then

(i) \(R(B) \subseteq R(A) \iff B = BA\) for each \(A \in A \{1\}\)
(ii) \( \mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AA^{-1}B \) for each \( A \in \mathbb{A} \). 

**Lemma 2.3.** If \( A \in \mathbb{F}_{mn} \) with \( \mathcal{R}(A) = \mathcal{R}(A^T A) \), then \( A^T A \) is regular fuzzy matrix if and only if \( A \) is a regular fuzzy matrix. If \( A \in \mathbb{F}_{mn} \) with \( \mathcal{C}(A) = \mathcal{C}(AA^T) \), then \( AA^T \) is a regular fuzzy matrix if and only if \( A \) is a regular fuzzy matrix.

In the following, we will make use of the following results proved in our earlier work [8]. For the sake of completeness we will provide the proof.

**Lemma 2.4.** (Theorem 3.3 [8])

Let \( A = [A_L, A_U] \in (\text{IVFM})_{mn} \)

Then the following holds:

(i) \( A \) is regular IVFM \( \Leftrightarrow A_L \) and \( A_U \) are regular

(ii) \( \mathcal{R}(A) = [\mathcal{R}(A_L), \mathcal{R}(A_U)] \) and \( \mathcal{C}(A) = [\mathcal{C}(A_L), \mathcal{C}(A_U)] \).

**Proof.**

(i) Since \( A \in (\text{IVFM})_{mn} \), any vector \( x \in \mathcal{R}(A) \) is of the form \( x = y.A \) for some \( y \in (\text{IVFM})_{1n} \), that is, \( x \) is an interval valued vector with \( n \) components.

Let us compute \( x \in \mathcal{R}(A) \) as follows:

\[
x = \begin{bmatrix}
\sum_{i=1}^{m} a_{i} \cdot A_i & \\
\end{bmatrix}
\]

where \( A_i \) is the \( i^{th} \) row of \( A \). Equating the \( j^{th} \) component on both sides yields

\[
x_j = \sum_{i=1}^{m} a_{i} \cdot a_i.
\]

Since, \( a_i = [a_{ijL}, a_{ijU}] \)

\[
x = \begin{bmatrix}
\sum_{i=1}^{m} a_{ijL} \cdot a_{ijU} & \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\sum_{i=1}^{m} (a_{ijL} \cdot a_{ijU}) & \\
\end{bmatrix}
\]

\[
= [x_{jL}, x_{jU}].
\]
world is the 4th component of x and x is the 6th component of x. Therefore, R(A) = [R(A), R(A)]

(ii) For A = [A, A], the transpose of A is AT = [A, A]T. By using (i) we get, C(A) = R(A) = [R(A), R(A)] = [C(A), C(A)].

Lemma 2.5. (Theorem 3.7 [8])

For A and B E (IVFM)m
(i) R(B) C R(A) A = XA for some X E (IVFM)m
(ii) C(B) C C(A) A = AY for some Y E (IVFM)n

Proof.

(i) Let A = [A, A] and B = [B, B]. Since, B = XA, for some X E (IVFM), put X = [X, X]. Then, by Equation (3), B = XA, A and B = XA, A.

Hence, by( Lemma (2.2)), R(B) C R(A) and R(B) C R(A).

By Lemma (2.4)(ii), R(B) = [R(B), R(B)] C [R(A), R(A)] = R(A). Thus R(B) C R(A). Conversely, R(B) C R(A).

⇒ R(B) C R(A) and R(B) C R(A) (By Lemma (2.4) (ii))

⇒ B = Y A and B = ZA

Then B = [B, B]

= [Y A, ZA]

= [Y, Z] [A, A] (By Eq. (3))

= X[A, A], where X = [Y, Z] E (IVFM)m

= X A

B = X A

(ii) This can be proved along the same lines as that of (i) and hence omitted.

3 g- Inverses of Interval Valued Fuzzy Matrices

In this section, we will discuss the g-inverses of an IVFM and their relations in terms of the row and column spaces of the matrix as a generalization of the results available in the literature on fuzzy matrices [2, 6] as a development of our earlier work [8] on regular IVFMs and analogous to that for complex matrices [9].
Definition 3.1. For $A \in (\text{IVFM})_{mn}$ if there exists $X \in (\text{IVFM})_{nm}$ such that

1. $AXA = A$
2. $XAX = X$
3. $(AX)^T = (AX)$
4. $(XA)^T = (XA)$, then $X$ is called a $g$-inverse of $A$.

$X$ is said to be a $\lambda$-inverse of $A$ and $X \in A\{\lambda\}$ if $X$ satisfies $\lambda$ equation where $\lambda$ is a subset of $\{1, 2, 3, 4\}$. $A\{\lambda\}$ denotes the set of all $\lambda$- inverses of $A$. In particular if $\lambda = \{1, 2, 3, 4\}$ then $X$ unique and is called the Moore Penrose inverse of $A$, denoted as $A^\dagger$.

Remark 3.2. From Definition (3.1) of $\lambda$-inverses for $A \in (\text{IVFM})$, by applying Eq. (3) for $A = [A_L, A_U]$ and $X = [X_L, X_U]$ it can be verified that the existence and construction of $\lambda$-inverses of $A \in (\text{IVFM})_{mn}$ reduces to that of the $\lambda$-inverses of $A_L, A_U \in F_{mn}$.

Theorem 3.3. Let $A \in (\text{IVFM})_{mn}$ and $X \in A\{1\}$, then $X \in A\{2\}$ if and only if $\mathcal{R}(AX) = \mathcal{R}(X)$.

Proof.

Since $A = [A_L, A_U]$ and $X = [X_L, X_U]$

$X \in A\{2\} \Rightarrow XAX = X$, then by Eq. (3),

$X_L A_L X_L = X_L$ and $X_U A_U X_U = X_U; X_L \in A_L\{2\}$ and $X_U \in A_U\{2\}$

$\Rightarrow A_L \in X_L\{1\}$ and $A_U \in X_U\{1\}$

$\Rightarrow \mathcal{R}(X_L) = \mathcal{R}(A_L X_L)$ and $\mathcal{R}(X_U) = \mathcal{R}(A_U X_U)$

$\Rightarrow \mathcal{R}(AX) = \mathcal{R}(X).$ (By Lemma (2.4))

Conversely,

Let $\mathcal{R}(AX) = \mathcal{R}(X)$, then by Lemma (2.4), $\mathcal{R}(X) \subseteq \mathcal{R}(AX)$ implies $X = YAX$ for some $Y \in (\text{IVFM})_m$. $X(AX) = (YAX)(AX)$

$XAX = Y(AXA)X$

$= YAX$ (By Definition (3.1))

$= X$

Thus $X \in A\{2\}$. 
Remark 3.4. In the above Theorem (3.3), the condition \( X \in A \{1\} \) is essential. This is illustrated in the following example.

Example 3.5.

Let

\[
A = \begin{pmatrix} [0,1] & [1,1] \\ [1,1] & [0,0] \end{pmatrix}, \quad X = \begin{pmatrix} [1,1] & [0,1] \\ [0,0] & [0,1] \end{pmatrix}
\]

Then by representation (1) we have,

\[
A_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_U = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}
\]

\[
X_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

\[
A_L X_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq A_L \quad \text{implies} \quad X_L \not\in A_L \{1\} \quad \text{and} \quad A_U X_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq A_U \quad \text{implies} \quad X_U \not\in A_U \{1\}
\]

\[
A_L X_L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_U X_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

But

\[
X_L A_L X_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq X_L \quad \text{and} \quad X_U A_U X_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq X_U.
\]
Hence $X_L \notin A_L \{2\}$ and $X_U \notin A_U \{2\}$. Then by Eq. (3) we have, $AXA \neq A$, therefore $X \notin A \{1\}$. Here $R (X_L) = R (A_L X_L)$ and $R (X_U) = R (A_U X_U)$. Therefore by Lemma (2.4), $R (X) = R (AX)$, but $XAX \neq X$. Hence $X \notin A \{2\}$.

**Theorem 3.6.** For $A \in (IVFM)_{mn}$, $A$ has a $\{1, 3\}$inverse if and only if $A^T A$ is a regular IVFM and $R (A^T A) = R (A)$.

**Proof.** Since $A$ is regular, Lemma (2.4), $A_L$ and $A_U$ are regular. Let $A$ has a $\{1, 3\}$ inverse $X$ (say) then by Eq. (3), $A_L$ has a $\{1, 3\}$ inverse $X_L$ and $A_U$ has a $\{1, 3\}$ inverse $X_U$.

Then $A_L X_L A_L = A_L$ and $(A_L X_L)^T = A_L X_L$

$$A_L^T (A_L X_L A_L) = A_L^T A_L$$

$$(A_L^T A_L X_L) A_L = A_L^T A_L$$

$$R (A_L^T A_L) \subseteq R (A_L) \quad \text{(By Lemma (2.2))}$$

Similarly, $R (A_U^T A_U) \subseteq R (A_U)$

Therefore by Equation (3) we have, $R (A^T A) \subseteq R (A)$

Also $(A_L X_L)^T A_L = A_L X_L A_L$

$$\Rightarrow X_L^T A_L^T A_L = A_L$$

$$\Rightarrow X_L^T (A_L^T A_L) = A_L$$

$$R (A_L) \subseteq R (A_L^T A_L) \quad \text{(By Lemma (2.2))}$$

Similarly, $R (A_U) \subseteq R (A_U^T A_U)$. By Equation (3) we have, $R (A) \subseteq R (A^T A)$. Thus, $R (A) = R (A^T A)$. Since $X \in A \{1\}$, $R (A) = R (XA)$. Hence, $R (A^T A) = R (A) = R (XA)$ (By Lemma (2.5)), $YA^T A = XA$ let $Y = [Y_L, Y_U]$ then, $A_L^T A_L (Y_L A_L^T A_L) = A_L^T A_L (X_L A_L)$

$$(A_L^T A_L) Y_L (A_L^T A_L) = A_L^T (A_L X_L A_L)$$

$$= A_L^T A_L$$

Similarly, $A_U^T A_U (Y_U A_U^T A_U) = A_U^T A_U$. By (3) we have, $A^T A (Y A^T A) = A^T A$

Thus $A^T A$ is a regular interval valued fuzzy matrix. Conversely, let $A^T A$ be a regular interval-valued fuzzy matrix and $R (A) = R (A^T A)$. By Lemma (2.3),
A is a regular IVFM. Let us take $Y = (A^T)^{-1}A^T \in (IVFM)$. We claim that $Y \in A \{1, 3\}$.

$\mathcal{R}(A) = \mathcal{R}(A^T)$ and $A^TA$ is regular, by Lemma (2.3) $A = A(A^T)^{-1}A^T = AYA$, $Y \in A \{1\}$ and since $\mathcal{R}(A) = \mathcal{R}(A^T)$, $A = XA^T$, by Lemma (2.4), $A_L = XLA_L^TA_L$ and $A_U = XUA_U^TA_U$. Let $Y = [Y_L, Y_U]$.

Then, $A_L Y_L = X_L A_L^T A_L (A_L^TA_L)^{-1}A_L^T$

$= X_L A_L^T A_L (A_L^TA_L)^{-1}A_L^T X_L^T$

$= X_L (A_L^TA_L)(A_L^TA_L)^{-1}(A_L^TA_L)X_L^T$

$= X_L (A_L^TA_L^T)$

$= X_L A_L^T$

Similarly, $A_U Y_U = X_U A_U^T$. Then by Eq. (3) we have, $AY = XA^T$

$(A_L Y_L)^T = (X_L A_L^T)^T$

$= A_L X_L^T$

$= X_L A_L^T A_L X_L^T$

$= X_L A_L^T = A_L Y_L$

Similarly, $(A_U Y_U)^T = X_U A_U^T = A_U Y_U$. Then by Equation (3) we have, $(AY)^T = AY, Y \in A \{3\}$. Since $\mathcal{R}(A) = \mathcal{R}(A^T)$ by Lemma (2.4) and regularity of $A^TA$ we get

$A = A(A^T)^{-1}A^T = AYA, Y \in A \{1\}$. Thus A has a \{1, 3\} inverse.

**Theorem 3.7.** For $A \in (IVFM)_{mn}$, A has \{1, 4\} inverse if and only if $AA^T$ is regular and $\mathcal{C}(AA^T) = \mathcal{C}(A)$.

**Proof.** This can be proved in the same manner as that of Theorem (3.6).

**Corollary 3.8.** Let $A \in (IVFM)_{mn}$ be a regular IVFM with $A^TA$ is a regular IVFM and $\mathcal{R}(A^TA) = \mathcal{R}(A)$, then $Y = (A^TA)^{-1}A^T \in A \{1, 2, 3\}$.

**Proof.** $Y \in A \{1, 3\}$ follows from Theorem (3.6), it is enough verify $Y = [Y_L, Y_U] \in A \{2\}$ that is, $Y_L A_L Y_L = Y_L$ and $Y_U A_U Y_U = Y_U$.

$Y_L A_L Y_L = Y_L (X_L A_L^T A_L) (A_L T A_L)^{-1} A_L^T$
Inverses of Interval Valued Fuzzy Matrices


Similarly, \( Y_U A_U Y_U = Y_U \). Then by Eq. (3), \( Y A Y = Y \).

Thus \( Y \in A\{1, 2, 3\} \).

**Theorem 3.9.** Let \( A \in (IVFM)_{mn} \) be a regular IVFM with \( A A^T \) is a regular IVFM and \( \mathcal{R}(A^T) = \mathcal{R}(AA^T) \) then \( Z = A^T (AA^T)^{-1} \in A\{1, 2, 4\} \).

**Proof.** Similar to the proof of Theorem (3.7) and Corollary (3.8) hence omitted.

4 Conclusion

The main results of the present paper are the generalization of the results on \( g \)-inverses of regular fuzzy matrices found in [2, 6] and the extension of our earlier work on regular IVFMs [8].

**References**