# A Fractional-Order Food Chain Model with Omnivore and Anti-Predator

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**Abstract**

A fractional-order food chain model is proposed in this article. The model is built by prey, intermediate predator, and omnivore. It is assumed that intermediate predator only eat prey and omnivore can consume prey and intermediate predator. But, prey has the ability called as anti-predator behavior to escape from both predators. For the first discussion, it is found that all solutions are existential, uniqueness, boundedness, and non-negative. Further, we analyze the existence condition and local stability of all points, that is point for the extinction of all populations, both predators, intermediate predator, omnivore, and point for the existence of all populations. We also investigate the global stability of all points, except point for the extinction of all populations and both predators. Finally, we preform several numerical solutions by using the nonstandard Grunwald-Letnikov approximation to demonstrate the our analytical results.

*Keywords: food chain, fractional-order, Grunwald-Letnikov approximation, stability*

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## 1. Introduction

In this decades, the mathematical biology with its various models have been well studied by many researchers to understand the dynamics of population interactions [22],[23]. The tropic interactions among the various species that form complex networks are called as food webs. This interactions shape the pattern of food webs. In consequent, the design of food webs and the strength of interactions affect the pattern of tropic dynamics in food webs [4]. There are at least three species involved, namely species $x$, species $y$, and species $z$. According to [10], all food webs predator-prey system with three species are separated into four types in 34 cases, that is food chain (see Figure 1(a)), two predators competing for one prey (see Figure 1(b)), one predator acting on two preys (see Figure 1(c)), and loops. In the case loop, it is divided into two cases, that is food chain with omnivore (see Figure 1(d)) and cycle (see Figure 1(e)). Here, Species $z$ can be called as specialist predators that have a limited diet and generalist predator that use a variety of resources other than two tropic levels [9].

Since 1970's, authors have provides interesting and impressive results in studying the dynamics of three species predator-prey systems. Many natural phenomena are described by authors to obtain a formula that represent real events such as protection of prey [28], harvesting [32],[29] , alternative food on predator [30], and the existence of omnivore [16],[7],[4],[8],[9],[31] In this paper, we focus on three species food web predator-prey systems with omnivore. There are three species involved, namely species $y$ are intermediate predator, species $z$ are omnivore, and species $x$ are resource or prey consumed by both of them [5]. The existence of omnivore is an important topology that gives the natural characteristics of tropic networks [6]. In ecosystem, omnivore is defined as predator that eat more than one tropic level. Furthermore, omnivore can complicate the structure of tropic webs and exert indirect effect of predator on basal resource through intermediate predators [4],[7].

Many authors study the food chain model with omnivore and its modified version. Generally, a mathematical model describing the food chain with omnivore has been proposed by Holt and Polis [16]. In his observation, the model is constructed into Lotka-Volterra model with linear functional response without intraspecific competition and showed that instability of equilibrium point occurs due to omnivorous predation. Tanabe and Namba [7] have used the similar model as Holt and Polis [16] and have proven that the omnivorous predation can destabilize the equilibrium point and create chaos in the system. Then, Namba et al. [4] have considered

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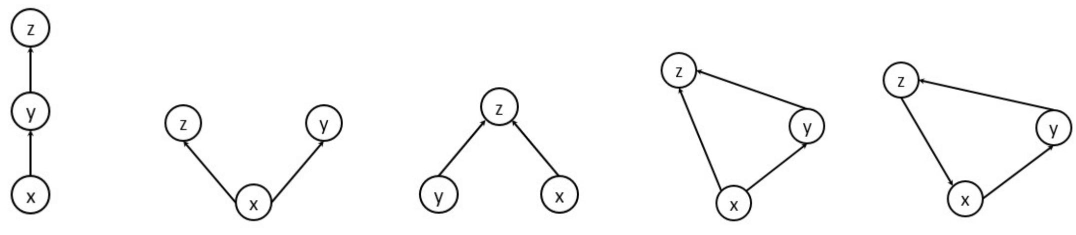



Figure 1: All possible schematics of predator-prey systems for the interaction between three species, where the arrows show interactions between species: (a) general food chain process, (b) one prey-two predators, (c) two preys-one predator, (d) food chain process involving omnivore, (e) cycle.

the similar model as Tanabe and Namba [7] by adding the intraspecific competition for both predators and find the bifurcation diagrams with all parameters as bifurcation parameter. Kang and Wedekin [8] consider the Holling type III between intermediate predator and omnivore in their food chain model and separate it into two models, one with a generalist omnivore and another with a specialist omnivore. In this model, the intraspecific competition for both predators is negligible. Sen et al. [31] develop Holt and Polis model by adding the intraspecific competition and Holling type II function response between intermediate predator and omnivore. Since their model considers Holling type II function response for predators, omnivore predation may play the role in stabilizing the system [11].

Biologically, prey populations have the ability to run away from predators such as hiding from predator, running fast ability, or showing fearful behavior so predator will think that eating it can get them bitten. The ability in prey is called as anti-predator behavior that helps prey in fighting predators [23]. In fact, it is a common part of marine or terrestrial food web ecological systems. Some interesting examples of our case are given by the interaction between Tetranychus as species $x$, Phytoseiulus as species $y$, and Stethorus as species $z$. Rosenheim and Corbett [12] report that many arthropods including Tetranychus relatively settle on host plants to meet their nutritional needs. They involve a high degree of concealment within the host plant with minimal opportunities for locomotion. In addition, Phytoseiulus and Stethorus are the natural predators of Tetranychus.

Based on the previous success works, authors use the first-order derivative predator-prey systems which is limited by its ability to involve the previous conditions on the growth of species. In fact, it must take into account all conditions both past and current states which is called as memory effects [25]. This effect is formed into the fractional-order model. Currently, the model has grown rapidly and becomes the popular study in investigating the dynamic behavior of interactions between species as in [33], [38], [27], [36], [35], [34], [25], [26], [24], [39]. It is known that the order of fractional derivative has a significant effect in the dynamic behavior of models. This is different from the first-order predator-prey model which only depends on the parameter values. Here, we formulate the model of Holt and Polis [16] by assuming the intraspesific competition, Holling type II for intermediate predator and omnivore, and the anti-predator behavior on prey. Then, we replace the first-order model to the fractional-order model. There are various operators of fractional-order differential equations. However, we choose the Caputo operator because it can be applied on the classic initial conditions as in the integer order differential equation. This operator has rich analytical tools in identifying the dynamic behavior of predator-prey models.

In this research, we aim to observe the dynamics behavior of a food chain model with omnivore assuming that these prey have the ability to escape from predatory attacks. To archive our purpose, we present several discussion which are arranged as follows. First, the mathematical model is separated into two sections: Model formulation on section 2.1 and Model with the Caputo operator on section 2.2. In section 3 and 4, we show that the solutions of system exist and unique as well as they are uniformly bounded and non-negative. In section 5 and 6, we investigate the existence and stability of equilibrium point, both locally and globally. In section 7, we conduct several numerical simulations to support our analytical results. Finally, the conclusion is given in section 8.

## 2. Mathematical Model

### 2.1. Model Formulation

The three species food chain models consisting one prey, one intermediate predator, and one omnivore as apex predator is modeled by adopting a Lotka-Volterra food chain model proposed by Holt and Polis [16]. We symbolize $x(t), y(t)$, and $z(t)$ as the population density for prey (e.g. Tetranychus), intermediate predator (e.g. Phytoseiulus), and omnivore (e.g. Stethorus), respectively at time $t$. Their model is shown as follows.

$$\begin{aligned} \frac{dx}{dt} &= (r - a_1 x - \xi_1 y - \xi_2 z)\, x, \\ \frac{dy}{dt} &= (-\delta_1 + \beta_1 x - \eta_1 z)\, y, \\ \frac{dz}{dt} &= (-\delta_2 + \beta_2 x + \eta_2 y)\, z, \end{aligned} \tag{1}$$

where $r$ is the natural growth rate of prey. $a_1$ are the competition rate for prey. $\xi_1, \xi_2$ are the capture rate of intermediate predator and omnivore respectively. $\delta_1, \delta_2$ are the natural death rate of intermediate predator and omnivore respectively. $\beta_1, \beta_2$ are the conversion rate of prey into intermediate predator and omnivore respectively. $\eta_1, \eta_2$ represent the capture rate of omnivore in preying intermediate predator and the conversion rate of intermediate predator into omnivore.

Based on the model proposed by Holt and Polis [16], intermediate predator can only consume prey while omnivore can eat prey and intermediate predator. Both predators have to compete with each other to survive in the community. Moreover, all populations satisfy the following ecological assumptions.

- We include the intraspecific competition for intermediate predators and omnivores denoted with $a_i, i = 2, 3$ respectively.
- Species $y$ and $z$ consume $x$ by following Holling type I function because they depend on search time in preying species $x$ where handling time and other more dynamics don't apply [14].
- Species $z$ consume $y$ by following Holling type II function because species $z$ spends some time for searching and capturing species $y$ [17].
- Species $x$ has anti-predator behavior such as hiding, foraging, and escaping [23].

Based on the above assumption, the model (1) can be rewritten as a continuous time food chain model as follows.

$$\begin{aligned} \frac{dx}{dt} &= (r - a_1 x - \xi_1 y - \xi_2 z)\, x, \\ \frac{dy}{dt} &= \left(-\delta_1 + (\beta_1 - \varphi_1)\, x - a_2 y - \frac{\eta_1 z}{1 + \sigma y}\right) y, \\ \frac{dz}{dt} &= \left(-\delta_2 + (\beta_2 - \varphi_2)\, x - a_3 z + \frac{\eta_2 y}{1 + \sigma y}\right) z, \end{aligned} \tag{2}$$

where $\varphi_1, \varphi_2$ denote the anti-predator behavior of prey towards intermediate predator and omnivore. $\sigma$ is the half saturation constant. We also confirm that all parameters are positive values and the solution of system lies in $\mathbb{R}^3_+$, where $\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0\right\}$.

### 2.2. Model with Caputo Operator

First, We define the Caputo fractional operator (CFO) as follows.

**Definition 2.1.** (See [3]). The CFO derivative with order-$\alpha$ is defined as follows.

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\, d\tau,$$

with $\Gamma(.)$ is Gamma function and $\alpha \in (0, 1]$.

By using the similar manner as done in [24], [25], [26], the first order derivative of Model (2) is replaced with CFO order derivative as given in Definition 2.1. Therefore, the model can be written as follows.

$$\begin{aligned} D_t^\alpha x &= (r - a_1 x - \xi_1 y - \xi_2 z)\, x, \\ D_t^\alpha y &= \left(-\delta_1 + (\beta_1 - \varphi_1)\, x - a_2 y - \frac{\eta_1 z}{1+\sigma y}\right) y, \\ D_t^\alpha z &= \left(-\delta_2 + (\beta_2 - \varphi_2)\, x - a_3 z + \frac{\eta_2 y}{1+\sigma y}\right) z. \end{aligned} \tag{3}$$

When the operator is replaced with the CFO order derivative, the time's dimension of the first order derivative is formed from $t$ to $t^\alpha$. As a result, the model becomes inconsistent because some parameters such as $r, a_1, a_2, a_3, \xi_1, \xi_2, \delta_1, \delta_2, \beta_1, \beta_2, \varphi_1, \varphi_2, \eta_1, \eta_2$ have the dimension of time $t^1$. We can adjust it by changing the scale of all favorable parameters. Thus, the model (3) transforms into the following model.

$$\begin{aligned} D_t^\alpha x &= \left(\bar{r} - \bar{a}_1 x - \bar{\xi}_1 y - \bar{\xi}_2 z\right) x, \\ D_t^\alpha y &= \left(-\bar{\delta}_1 + \left(\bar{\beta}_1 - \bar{\varphi}_1\right) x - \bar{a}_2 y - \frac{\bar{\eta}_1 z}{1+\sigma y}\right) y, \\ D_t^\alpha z &= \left(-\bar{\delta}_2 + \left(\bar{\beta}_2 - \bar{\varphi}_2\right) x - \bar{a}_3 z + \frac{\bar{\eta}_2 y}{1+\sigma y}\right) z, \end{aligned} \tag{4}$$

where $\bar{r} = r^\alpha, \bar{a}_1 = a_1^\alpha, \bar{a}_2 = a_2^\alpha, \bar{a}_3 = a_3^\alpha, \bar{\xi}_1 = \xi_1^\alpha, \bar{\xi}_2 = \xi_2^\alpha, \bar{\delta}_1 = \delta_1^\alpha, \bar{\delta}_2 = \delta_2^\alpha, \bar{\beta}_1 = \beta_1^\alpha, \bar{\beta}_2 = \beta_2^\alpha, \bar{\varphi}_1 = \varphi_1^\alpha, \bar{\varphi}_2 = \varphi_2^\alpha, \bar{\eta}_1 = \eta_1^\alpha, \bar{\eta}_2 = \eta_2^\alpha$. For simplicity, we re-symbolize by eliminating bar $\bar{\cdot}$ on each parameter. From the model (4), we obtain the final model as follows.

$$\begin{aligned} D_t^\alpha x &= (r - a_1 x - \xi_1 y - \xi_2 z)\, x, \\ D_t^\alpha y &= \left(-\delta_1 + (\beta_1 - \varphi_1)\, x - a_2 y - \frac{\eta_1 z}{1+\sigma y}\right) y, \\ D_t^\alpha z &= \left(-\delta_2 + (\beta_2 - \varphi_2)\, x - a_3 z + \frac{\eta_2 y}{1+\sigma y}\right) z. \end{aligned} \tag{5}$$

## 3. Existence and Uniqueness

In this section, it is seen that all solutions of model exist and unique. We start by introducing the following lemma.

**Lemma 3.1.** *(See [1]). Consider the CFO system*

$$D_t^\alpha x(t) = f(t, x), t > 0, x(0) \geq 0, \alpha \in (0, 1] \tag{6}$$

*with $f : [0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \in \mathbb{R}^n$. The equation (6) has a unique and existing solution on $[0, \infty) \times \Omega$ when $f(t, x)$ fits the locally Lipschitz condition to $x$.*

By applying Lemma 3.1, we obtain the following theorem, where this ensures that the solutions of System (5) exist and unique.

**Theorem 3.2.** *Assume that System (5) has $X(0) = (x(0), y(0), z(0))$ and $t \in [0, \infty]$ in the region $\Omega_M \times [0, \infty]$, where $\Omega_M = \left\{(x, y, z) \in \mathbb{R}_+^3 : \max\{|x|, |y|, |z|\} \leq M, M > 0\right\}$ for sufficiently large $M$. Thus, the solution of System (5) is exists and unique.*

*Proof:* To prove the existence and uniqueness of solution in the region $\Omega_M \times [0, \infty]$ for sufficiently large $M$, we consider the existence of $M$ which is ensured by the boundedness of solution as shown below. First, let $X = (x, y, z)^T$ and $\bar{X} = (\bar{x}, \bar{y}, \bar{z})^T$. The system (5) can be written as Equation (7) .

$$D_t^{\alpha} X = H(X), \tag{7}$$

where

$$H(X) = \begin{pmatrix} rx - a_1 x^2 - \xi_1 xy - \xi_2 xz \\ -\delta_1 y + (\beta_1 - \varphi_1) xy - a_2 y^2 - \frac{\eta_1 yz}{1+\sigma y} \\ -\delta_2 z + (\beta_2 - \varphi_2) xz - a_3 z^2 + \frac{\eta_2 yz}{1+\sigma y} \end{pmatrix} = \begin{pmatrix} H_1(X) \\ H_2(X) \\ H_3(X) \end{pmatrix}.$$

By applying Equation (7) for any $X, \bar{X} \in \Omega_M$, we have

$$\begin{aligned} \left\| H(X) - H(\bar{X}) \right\| &= \sum_{i=1}^{3} \left| H_i(X) - H_i(\bar{X}) \right|, \\ &= \left| r(x - \bar{x}) - a_1 \left(x^2 - \bar{x}^2\right) - \xi_1 (xy - \bar{x}\bar{y}) - \xi_2 (xz - \bar{x}\bar{z}) \right| + \\ &\quad \left| -\delta_1 (y - \bar{y}) + (\beta_1 - \varphi_1)(xy - \bar{x}\bar{y}) - a_2 \left(y^2 - \bar{y}^2\right) - \eta_1 \left( \frac{yz}{1+\sigma y} - \frac{\bar{y}\bar{z}}{a + \sigma \bar{y}} \right) \right| + \\ &\quad \left| -\delta_2 (z - \bar{z}) + (\beta_2 - \varphi_2)(xz - \bar{x}\bar{z}) - a_3 \left(z^2 - \bar{z}^2\right) + \eta_2 \left( \frac{yz}{1+\sigma y} - \frac{\bar{y}\bar{z}}{1 + \sigma \bar{y}} \right) \right|. \end{aligned}$$

By using the triangle inequality $|v_1 \pm v_2| \leq |v_1| \pm |v_2|$ and considering that $\max\{|x|, |y|, |z|\} \leq M$, we get

$$\begin{aligned} \left\| H(X) - H(\bar{X}) \right\| &\leq r|x - \bar{x}| + a_1 \left|x^2 - \bar{x}^2\right| + (\xi_1 + \beta_1 - \varphi_1)|xy - \bar{x}\bar{y}| + \\ &\quad (\xi_2 + \beta_2 - \varphi_2)|xz - \bar{x}\bar{z}| + \delta_1 |y - \bar{y}| + a_2 \left|y^2 - \bar{y}^2\right| + \\ &\quad (\eta_1 + \eta_2)|\bar{z}(y - \bar{y})| + (\eta_1 + \eta_2)|(\bar{y} + \sigma y \bar{y})(z - \bar{z})| + \\ &\quad \delta_2 |z - \bar{z}| + a_3 \left|z^2 - \bar{z}^2\right|, \\ &\leq L_1 |x - \bar{x}| + L_2 |y - \bar{y}| + L_3 |z - \bar{z}|, \\ &\leq L \left\| X - \bar{X} \right\|, \end{aligned}$$

where

$$\begin{aligned} L_1 &= r + (2a_1 + \xi_1 + \beta_1 - \varphi_1 + \xi_2 + \beta_2 - \varphi_2) M, \\ L_2 &= \delta_1 + (2a_2 + \xi_1 + \beta_1 - \varphi_1 + \eta_1 + \eta_2) M, \\ L_3 &= \delta_2 + (2a_3 + \xi_2 + \beta_2 - \varphi_2 + \eta_1 + \eta_2) M + (\eta_1 + \eta_2) \sigma M^2, \\ L &= \max\{L_1, L_2, L_3\}. \end{aligned}$$

Therefore, $H(X)$ satisfies the Lipschitz condition with respect to $X$. According to Lemma 3.1, the solution $X(t) \in \Omega_M$ of System (5) with initial conditions $X(0) = (x(0), y(0), z(0))$ is exist and unique. ■

## 4. Boundedness and Non-negative

To describe that the boundedness and non-negative of solution as well as ensure the biological significance of System (5), the following lemma are needed.

**Lemma 4.1.** *(See [33]). Suppose $x(t)$ is a continuous function on $[0, +\infty)$. If $x(t)$ satisfies $D_t^{\alpha} x(t) + \mu x(t) \leq \vartheta, x(0) \geq 0$, where $\alpha \in (0, 1], (\mu, \vartheta) \in \mathbb{R}^2$, and $\mu \neq 0$, then $x(t) \leq \left(x(0) - \frac{\vartheta}{\mu}\right) E_{\alpha}[-\mu t^{\alpha}] + \frac{\vartheta}{\mu}$.*

By using the above lemma, the boundedness and non-negative of solution is ensured by the following theorem.

**Theorem 4.2.** *Suppose that $\beta_1 < \varphi_1 + \eta_1 \xi_1$ and $\beta_2 < \varphi_2 + \eta_2 \xi_2$. Consider System (5) with initial conditions $x(0), y(0), z(0) \geq 0$, then all solutions are uniformly bounded and non-negative.*

*Proof:* First, we want to show that all solutions with non-negative initial condition of System (5) are uniformly bounded. By defining a function $V\left(t\right)=x+\frac{y}{\eta_1}+\frac{z}{\eta_2}$, we get

$$\begin{aligned} D_t^{\alpha}V\left(t\right)+\mu V\left(t\right) &= \left(r+\mu\right)x-a_1x^2+\frac{\left(\mu-\delta_1\right)}{\eta_1}y-\frac{a_2}{\eta_1}y^2+\frac{\left(\mu-\delta_2\right)}{\eta_2}z-\frac{a_3}{\eta_2}z^2+ \\ & \frac{\left(\beta_1-\varphi_1-\eta_1\xi_1\right)}{\eta_1}xy+\frac{\left(\beta_2-\varphi_2-\eta_2\xi_2\right)}{\eta_2}xz, \end{aligned}$$

■

By taking $\beta_1<\varphi_1+\eta_1\xi_1$ and $\beta_2<\varphi_2+\eta_2\xi_2$, we have

$$D_t^{\alpha}V\left(t\right)+\mu V\left(t\right)\leq\frac{\left(r+\mu\right)^2}{4a_1}+\frac{\left(\mu-\delta_1\right)^2}{4a_2\eta_1}+\frac{\left(\mu-\delta_2\right)^2}{4a_3\eta_2}\equiv H.$$

By using Lemma 4.1, we obtain

$$V\left(t\right)\leq V\left(0\right)E_{\alpha}\left[-\mu t^{\alpha}\right]+\frac{H}{\mu}\left(1-E_{\alpha}\left[-\mu t^{\alpha}\right]\right),$$

Notice that $E_{\alpha}\left(S\right)=\sum_{k=0}^{\infty}\frac{S^k}{\Gamma(\alpha k+1)}$ is Mittag-Leffler function [2], $\Gamma\left(S\right)=\int_0^{\infty}x^{S-1}e^{-S}dx$ is Euler's Gamma function, and $0<E_{\alpha}\left[-\mu t^{\alpha}\right]\leq 1$. For $t\to\infty$, we have $0\leq V\left(t\right)\leq V\left(0\right)+\frac{H}{\mu}$. Thus, by using non-negative initial condition, all solutions of System (5) are limited to $\Omega$, that is

$$\Omega=\left\{\left(x,y,z\right)\in\mathbb{R}_+^3: x+\frac{y}{\eta_1}+\frac{z}{\eta_2}\leq\frac{H}{\mu}\right\}. \tag{8}$$

Now, we will prove that by employing the initial condition, all solutions are also non-negative. If we use the inequality (8), then

$$x+\frac{y}{\eta_1}+\frac{z}{\eta_2}\leq\frac{H}{\mu}. \tag{9}$$

Based on Equation (5) and Inequality (9), we get

$$\begin{aligned} D_t^{\alpha}x &\geq \left(r-\frac{a_1H}{\mu}-\frac{\xi_1\eta_1H}{\mu}-\frac{\xi_2\eta_2H}{\mu}\right)x, \\ &= \left(r-\left(a_1+\xi_1\eta_1+\xi_2\eta_2\right)\frac{H}{\mu}\right)x, \\ &= h_1x, \end{aligned}$$

where $h_1=r-\left(a_1+\xi_1\eta_1+\xi_2\eta_2\right)\frac{H}{\mu}$. By using $E_{\alpha,1}\left(t\right)>0$ as shown in [20], [21], we obtain $x\left(t\right)\geq x\left(0\right)E_{\alpha,1}\left(h_1t^{\alpha}\right)$. Thus, we have

$$x\left(t\right)\geq 0,\forall t\geq 0. \tag{10}$$

From Equation (5), Inequality (9) and (10), we get

$$\begin{aligned} D_t^{\alpha}y &\geq -\left(\delta_1+\frac{a_2\eta_1H}{\mu}+\frac{\eta_1^2H}{\mu+\sigma\eta_1H}\right)y, \\ &= -h_2y, \end{aligned}$$

where $h_2=\delta_1+\frac{a_2\eta_1H}{\mu}+\frac{\eta_1^2H}{\mu+\sigma\eta_1H}$. Therefore, we obtain $y\left(t\right)\geq y\left(0\right)E_{\alpha,1}\left(-h_2t^{\alpha}\right)$. Thus, we have

$$y(t) \geq 0, \forall t \geq 0. \tag{11}$$

By considering Equation (5), Inequality (10) and (11), we obtain

$$\begin{aligned} D_t^\alpha z &\geq -\left(\delta_2 + \frac{a_3\eta_2 H}{\mu}\right) z, \\ &= -h_3 z, \end{aligned}$$

with $h_3 = \delta_2 + \frac{a_3\eta_2 H}{\mu}$. Therefore, $z(t) \geq z(0) E_{\alpha,1}(-h_3 t^\alpha)$. Thus, we have $z(t) \geq 0, \forall t \geq 0$. Hence, all non-negative real numbers in $\mathbb{R}^3$ lie in the region $\Omega_+$, that is

$$\Omega_+ = \{(x, y, z) \,|x \geq 0, y \geq 0, z \geq 0\}.$$

## 5. Equilibrium Point and Local Stability

In this section, we determine the equilibrium points and their existence conditions using the following definition.

**Definition 5.1.** (See [25]). Consider the CFO system

$$D_t^\alpha \vec{x} = \vec{f}(\vec{x}), \vec{x}(0) \geq 0, \alpha \in (0, 1]. \tag{12}$$

A equilibrium point $\vec{x}^*$ in System (12) is obtained when $\vec{f}(\vec{x}^*) = 0$. Biologically, a point $\vec{x}^*$ is the biological point when it fits the condition $\vec{x}^* \geq 0$.

The equilibrium point of System (5) is obtained by solving $D_t^\alpha x = D_t^\alpha y = D_t^\alpha z = 0$. Therefore, we have

1) $E_0(0, 0, 0)$ is all populations extinction point that always exist.
2) $E_1\left(\frac{r}{a_1}, 0, 0\right)$ is the both predator extinction point that always exist.
3) $E_2(\tilde{x}, 0, \tilde{z})$ is the intermediate predator extinction point, where

$$\begin{aligned} \tilde{x} &= \frac{\delta_2\xi_2 + ra_3}{\beta_2\xi_2 + a_1a_3 - \varphi_2\xi_2}, \\ \tilde{z} &= \frac{r - a_1\tilde{x}}{\xi_2}. \end{aligned}$$

   This point exist when $\beta_2\xi_2 + a_1a_3 > \varphi_2\xi_2$ and $r > a_1\tilde{x}$.
4) $E_3(\hat{x}, \hat{y}, 0)$ is the omnivore extinction point, where

$$\begin{aligned} \hat{x} &= \frac{\delta_1\xi_1}{\beta_1\xi_1 + a_1a_2 - \varphi_1\xi_1}, \\ \hat{y} &= \frac{r - a_1\hat{x}}{\xi_1}. \end{aligned}$$

   It can be confirmed that the point $E_3$ exist when $\beta_1\xi_1 + a_1a_2 > \varphi_1\xi_1$ and $r > a_1\hat{x}$.
5) $E_4(x^*, y^*, z^*)$ is the all populations survive point, where

$$\begin{aligned} x^* &= \frac{\sigma a_2\xi_2 (y^*)^2 + (\sigma\delta_1\xi_2 + a_2\xi_2 - \eta_1\xi_1) y^* + \delta_1\xi_2 + r\eta_1}{\sigma\xi_2 y^* (\beta_1 - \varphi_1) + a_1\eta_1 + \xi_2(\beta_1 - \varphi_1)}, \\ z^* &= \frac{-(1 + \sigma y^*)\left[y^*(a_1a_2 + \xi_1(\beta_1 - \varphi_1)) + a_1\delta_1 - r(\beta_1 - \varphi_1)\right]}{\sigma\xi_2 y^*(\beta_1 - \varphi_1) + a_1\eta_1 + \xi_2(\beta_1 - \varphi_1)}. \end{aligned}$$

   Meanwhile, $y^*$ is obtained by solving the cubic equation $A(y^*)^3 + B(y^*)^2 + Cy + D = 0$ with

$$
\begin{aligned}
A &= \sigma^2 \left[a_1 a_2 a_3 + a_2 \xi_2 (\beta_2 - \varphi_2) + a_3 \xi_1 (\beta_1 - \varphi_1)\right], \\
B &= \sigma \left[a_1 a_3 (\sigma \delta_1 + 2a_2) + (\beta_1 - \varphi_1)(2a_3 \xi_1 + \eta_2 \xi_2 - \delta_2 \sigma \xi_2 - a_3 \sigma r)\right] + \\
&\quad \sigma \left[(\beta_2 - \varphi_2)(\sigma \delta_1 \xi_2 + 2a_2 \xi_2 - \eta_1 \xi_1)\right], \\
C &= a_1 a_3 (a_2 + 2\sigma \delta_1) + a_1 \eta_1 (\eta_2 - \sigma \delta_2) - (\beta_1 - \varphi_1)(2a_3 \sigma r + 2\delta_2 \sigma \xi_2 - a_3 \xi_1 - \eta_2 \xi_2) + \\
&\quad (\beta_2 - \varphi_2)(2\sigma \delta_1 \xi_2 + \sigma \eta_1 r + a_2 \xi_2 - \eta_1 \xi_1), \\
D &= a_1 (a_3 \delta_1 - \delta_2 \eta_1) - (\beta_1 - \varphi_1)(a_3 r + \delta_2 \xi_2) + (\beta_2 - \varphi_2)(\delta_1 \xi_2 + \eta_1 r).
\end{aligned}
$$

To obtain explicit form and existence condition, we solve the cubic equation by applying the Cardan's method as in [13].

Furthermore, we will analyze the local stability of each point by employing the following theorem.

**Theorem 5.1.** *(See [3], [37]). A point $\vec{x}^*$ of System (12) is the equilibrium point which is locally asymptotically stable when all eigenvalues $\lambda_i$ of Jacobian matrix $J = \frac{\partial \vec{f}}{\partial \vec{x}}$ at $\vec{x}^*$ fit $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$ for all $i \in n$.*

From System (5), we have the Jacobian matrix $J$ evaluated at any points as follows.

$$
J(E) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{13}
$$

where

$$
\begin{aligned}
b_{11} &= -2a_1 x - \xi_1 y - \xi_2 z + r, & b_{12} &= -\xi_1 x, \\
b_{13} &= -\xi_2 x, & b_{21} &= (\beta_1 - \varphi_1) y, \\
b_{22} &= -2a_2 y + (\beta_1 - \varphi_1) x - \frac{\eta_1 z}{(1+\sigma y)^2} - \delta_1, & b_{23} &= -\frac{\eta_1 y}{1+\sigma y}, \\
b_{31} &= (\beta_2 - \varphi_2) z, & b_{32} &= \frac{\eta_2 z}{(1+\sigma y)^2}, \\
b_{33} &= -2a_3 z + (\beta_2 - \varphi_2) x + \frac{\eta_2 y}{1+\sigma y} - \delta_2. & &
\end{aligned}
$$

In this article, the Jacobian matrix (13) is denoted as $J(E_n) = \left(b_{ij}^{[n]}\right)$ at $E_n$, for $n = 0, 1, \ldots, 4$. By substituting the equilibrium points, we can investigate their local stability condition, which is presented as follows.

**Theorem 5.2.** *$E_0$ is a saddle point and $E_1$ is locally asymptotically stable when $\beta_1 < \frac{a_1 \delta_1}{r} + \varphi_1$ and $\beta_2 < \frac{a_2 \delta_2}{r} + \varphi_2$.*

*Proof:* First, the Jacobian matrix $J(E_0)$ and $J(E_1)$ is obtained as follows.

$$
J(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -\delta_1 & 0 \\ 0 & 0 & -\delta_2 \end{bmatrix}, \tag{14}
$$

$$
J(E_1) = \begin{bmatrix} -r & \frac{-\xi_1 r}{a_1} & \frac{-\xi_2 r}{a_1} \\ 0 & \frac{r}{a_1}(\beta_1 - \varphi_1) - \delta_1 & 0 \\ 0 & 0 & \frac{r}{a_2}(\beta_2 - \varphi_2) - \delta_2 \end{bmatrix}. \tag{15}
$$

From Equation (14), we obtain the eigenvalues $\lambda_1 = r, \lambda_2 = -\delta_1, \lambda_3 = -\delta_2$. Thus, $|\arg(\lambda_1)| = 0 < \frac{\alpha\pi}{2}$ and $|\arg(\lambda_{2,3})| = \pi > \frac{\alpha\pi}{2}$. Therefore, the point $E_0$ is a saddle point. Based on Equation (15), we get the eigenvalues $\lambda_1 = -r, \lambda_2 = \frac{r}{a_1}(\beta_1 - \varphi_1) - \delta_1, \lambda_3 = \frac{r}{a_2}(\beta_2 - \varphi_2) - \delta_2$. It is clear that $\lambda_1 < 0, \lambda_2 < 0$ if $\beta_1 < \frac{a_1 \delta_1}{r} + \varphi_1, \lambda_3 < 0$ if $\beta_2 < \frac{a_2 \delta_2}{r} + \varphi_2$. Thus, we have $|\arg(\lambda_{1,2,3})| = \pi > \frac{\alpha\pi}{2}$. Base on the result, $E_1$ is locally asymptotically stable.

■

**Theorem 5.3.** *Suppose that $\beta_1 > \varphi_1$ as well as the following case.*

$$\begin{aligned}\gamma_1 &= b_{11}^{[2]} + b_{33}^{[2]},\\ \gamma_2 &= b_{11}^{[2]} b_{33}^{[2]} - b_{13}^{[2]} b_{31}^{[2]},\\ \alpha^* &= \frac{2}{\pi}\left|\tan^{-1}\frac{\sqrt{4\gamma_2-\gamma_1^2}}{\gamma_1}\right|.\end{aligned}$$

*The point $E_2$ is locally asymptotically stable if it follows $\tilde{x} < \frac{\delta_1+\eta_1\tilde{z}}{\beta_1-\varphi_1}$ and one of the following conditions.*

1) $\gamma_1^2 \geq 4\gamma_2, \gamma_1 < 0$, *and* $\gamma_2 > 0$,
2) $\gamma_1^2 < 4\gamma_2$, *and if* $\gamma_1 < 0$, *or* $\gamma_1 > 0$ *and* $\alpha < \alpha^*$.

*Proof:* At the point $E_2$, the Jacobian matrix $J(E_2) = \left(b_{ij}^{[2]}\right)$ is presented as follows.

$$J(E_2) = \begin{bmatrix} b_{11}^{[2]} & b_{12}^{[2]} & b_{13}^{[2]} \\ 0 & b_{22}^{[2]} & 0 \\ b_{31}^{[2]} & b_{32}^{[2]} & b_{33}^{[2]} \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} b_{11}^{[2]} &= -2a_1\tilde{x} - \xi_2\tilde{z} + r, & b_{12}^{[2]} &= -\xi_1\tilde{x},\\ b_{13}^{[2]} &= -\xi_2\tilde{x}, & b_{22}^{[2]} &= (\beta_1-\varphi_1)\tilde{x} - \eta_1\tilde{z} - \delta_1,\\ b_{31}^{[2]} &= (\beta_2-\varphi_2)\tilde{z}, & b_{32}^{[2]} &= \eta_2\tilde{z},\\ b_{33}^{[2]} &= -2a_3\tilde{z} + (\beta_2-\varphi_2)\tilde{x} - \delta_2. \end{aligned}$$

From Equation (16), one of the eigenvalues is $\lambda_1 = b_{22}^{[2]}$ and the other eigenvalues is the roots of quadratic equation $\lambda^2 - \gamma_1\lambda + \gamma_2 = 0$, where $\gamma_1 = b_{11}^{[2]} + b_{33}^{[2]}$ and $\gamma_2 = b_{11}^{[2]} b_{33}^{[2]} - b_{13}^{[2]} b_{31}^{[2]}$. It is clear that $\lambda_1 < 0$ when $\tilde{x} < \frac{\delta_1+\eta_1\tilde{z}}{\beta_1-\varphi_1}$ with $\beta_1 > \varphi_1$. Thus, we have $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. From the quadratic equation, we obtain the eigenvalues $\lambda_{2,3} = \frac{\gamma_1 \pm \sqrt{\Lambda}}{2}$ with $\Lambda = \gamma_1^2 - 4\gamma_2$. We notice that if $\gamma_2 > 0$ and $\gamma_1 < 0$, then $\Lambda \geq 0$. Obviously, $\gamma_1^2 \geq 4\gamma_2$ and $\lambda_{2,3} < 0$. Thus, $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$. In the other word, $E_2$ is locally asymptotically stable. Next, suppose $\Lambda < 0$. Obviously, $\gamma_1 < 4\gamma_2$. Thus, $\lambda_{2,3}$ are a pair of complex conjugate eigenvalues. By using Theorem 5.1, $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$ is attained when $\alpha < \alpha^*$ for both $\gamma_1 > 0$ or $\gamma_1 < 0$. Thus, $E_2$ is locally asymptotically stable. The stability condition for $E_2$ is proven. ■

**Theorem 5.4.** *Suppose that $\sigma\delta_2 > \eta_2$ and $\beta_2 > \varphi_2$. By considering the following case.*

$$\begin{aligned}\theta_1 &= b_{11}^{[3]} + b_{22}^{[3]},\\ \theta_2 &= b_{11}^{[3]} b_{22}^{[3]} - b_{12}^{[3]} b_{21}^{[3]},\\ \alpha^* &= \frac{2}{\pi}\left|\tan^{-1}\frac{\sqrt{4\theta_2-\theta_1^2}}{\theta_1}\right|.\end{aligned}$$

*The point $E_3$ is locally asymptotically stable when it follows $\hat{x} < \frac{\delta_2(1+\sigma\hat{y})-\eta_2\hat{y}}{(\beta_2-\varphi_2)(1+\sigma\hat{y})}$ and one of the following conditions.*

1) $\theta_1^2 \geq 4\theta_2, \theta_1 < 0$, *and* $\theta_2 > 0$,
2) $\theta_1^2 < 4\theta_2$, *and if* $\theta_1 < 0$, *or* $\theta_1 > 0$ *and* $\alpha < \alpha^*$.

*Proof:* First, we identify the Jacobian matrix $J(E_3) = \left(b_{ij}^{[3]}\right)$ as follows.

$$J(E_3) = \begin{bmatrix} b_{11}^{[3]} & b_{12}^{[3]} & b_{13}^{[3]} \\ b_{21}^{[3]} & b_{22}^{[3]} & b_{23}^{[3]} \\ 0 & 0 & b_{33}^{[3]} \end{bmatrix}, \tag{17}$$

where

$$
\begin{aligned}
b_{11}^{[3]} &= -2a_1\hat{x} - \xi_1\hat{y} + r, & b_{12}^{[3]} &= -\xi_1\hat{x},\\
b_{13}^{[3]} &= -\xi_2\hat{x}, & b_{21}^{[3]} &= (\beta_1 - \varphi_1)\,\hat{y},\\
b_{22}^{[3]} &= -2a_2\hat{y} + (\beta_1 - \varphi_1)\,\hat{x} - \delta_1, & b_{23}^{[3]} &= -\frac{\eta_1\hat{y}}{1+\sigma\hat{y}},\\
b_{33}^{[3]} &= (\beta_2 - \varphi_2)\,\hat{x} + \frac{\eta_2\hat{y}}{1+\sigma\hat{y}} - \delta_2.
\end{aligned}
\tag{18}
$$

Based on Equation (17), we obtain that one of the eigenvalue is $\lambda_1 = b_{33}^{[3]}$ and the other is quadratic equations $\lambda^2 - \theta_1\lambda + \theta_2 = 0$, where $\theta_1 = b_{11}^{[3]} + b_{22}^{[3]}$ and $\theta_2 = b_{11}^{[3]}b_{22}^{[3]} - b_{12}^{[3]}b_{21}^{[3]}$. It is known $\lambda_1 < 0$ when $\hat{x} < \frac{\delta_2(1+\sigma\hat{y})-\eta_2\hat{y}}{(\beta_2-\varphi_2)(1+\sigma\hat{y})}$ with $\sigma\delta_2 > \eta_2$ and $\beta_2 > \varphi_2$. Thus, it confirms that $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$. The other eigenvalues is solved by investigating the negative roots of quadratic equations. We have $\lambda_{2,3} = \frac{\theta_1 \pm \sqrt{\triangle}}{2}$ with $\triangle = \theta_1^2 - 4\theta_2$. If $\theta_1 < 0$ and $\theta_2 > 0$, then $\triangle \geq 0$. Obviously, $\theta_1^2 \geq 4\theta_2$. Therefore, $\lambda_{2,3} < 0$ and $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$. Thus, $E_3$ is locally asymptotically stable. However, suppose $\triangle < 0$. We have $\theta_1^2 < 4\theta_2$. Therefore, $\lambda_{2,3}$ and its complex conjugate are eigenvalues. By applying Theorem 5.1, $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$ if $\alpha < \alpha^*$ for both $\theta_1 > 0$ or $\theta_1 < 0$. Hence, $E_3$ is locally asymptotically stable. Therefore, the stability condition for $E_3$ is proven. ■

**Theorem 5.5.** *Suppose that*

$$
\begin{aligned}
\chi_1 &= -\left(b_{11}^{[4]} + b_{22}^{[4]} + b_{33}^{[4]}\right).\\
\chi_2 &= b_{11}^{[4]}b_{22}^{[4]} - b_{12}^{[4]}b_{21}^{[4]} + b_{11}^{[4]}b_{33}^{[4]} - b_{13}^{[4]}b_{31}^{[4]} + b_{22}^{[4]}b_{33}^{[4]} - b_{23}^{[4]}b_{32}^{[4]}.\\
\chi_3 &= -\left(b_{11}^{[4]}b_{22}^{[4]}b_{33}^{[4]} + b_{12}^{[4]}b_{23}^{[4]}b_{31}^{[4]} + b_{13}^{[4]}b_{21}^{[4]}b_{32}^{[4]} - b_{12}^{[4]}b_{21}^{[4]}b_{33}^{[4]} - b_{11}^{[4]}b_{23}^{[4]}b_{32}^{[4]} - b_{13}^{[4]}b_{22}^{[4]}b_{31}^{[4]}\right).\\
D(P) &= 18\chi_1\chi_2\chi_3 + (\chi_1\chi_2)^2 - 4\chi_3\chi_1^3 - 4\chi_2^3 - 27\chi_3^2.
\end{aligned}
$$

*The point $E_4$ is called locally asymptotically stable if it satisfies one of the following conditions.*

1) $D(P) > 0,\ \chi_1, \chi_3 > 0,\ \chi_1\chi_2 > \chi_3,$
2) $D(P) < 0,\ \chi_1, \chi_1 \geq 0,\ \chi_3 > 0,\ \alpha < \frac{2}{3},$
3) $D(P) < 0,\ \chi_1, \chi_2, \chi_3 > 0,\ \chi_1\chi_2 = \chi_3,\ \alpha \in [0, 1).$

*Proof:* At the point $E_4$, we get the Jacobian matrix $J(E_4) = \left(b_{ij}^{[4]}\right)$ as follows.

$$
J(E_4) = \begin{bmatrix} b_{11}^{[4]} & b_{12}^{[4]} & b_{13}^{[4]}\\ b_{21}^{[4]} & b_{22}^{[4]} & b_{23}^{[4]}\\ b_{31}^{[4]} & b_{32}^{[4]} & b_{33}^{[4]} \end{bmatrix},
\tag{19}
$$

where

$$
\begin{aligned}
b_{11}^{[4]} &= -a_1x^*, & b_{12}^{[4]} &= -\xi_1x^*,\\
b_{13}^{[4]} &= -\xi_2x^*, & b_{21}^{[4]} &= (\beta_1 - \varphi_1)\,y^*,\\
b_{22}^{[4]} &= -a_2y^* + \frac{\sigma\eta_1y^*z^*}{(1+\sigma y^*)^2}, & b_{23}^{[4]} &= -\frac{\eta_1y^*}{1+\sigma y^*},\\
b_{31}^{[4]} &= (\beta_2 - \varphi_2)\,z^*, & b_{32}^{[4]} &= \frac{\eta_2z^*}{(1+\sigma y^*)^2},\\
b_{33}^{[4]} &= -a_3z^*.
\end{aligned}
$$

Based on Equation (19), all eigenvalues of $J(E_4)$ is the negative roots of cubic equations $P(\lambda) = \lambda^3 + \chi_1\lambda^2 + \chi_2\lambda + \chi_3 = 0$ with $\chi_1 = -\mathrm{tr}(J(E_4)), \chi_2 = M_{11} + M_{22} + M_{33}$ with $M_{ii}, i = 1, 2, 3$ are the minor matrix of $J(E_4)$ after removing the row $i$ and column $i$, and $\chi_3 = -\det(J(E_4))$. By using the same criterion as in [15], the stability condition for $E_4$ is proven. ■

## 6. Global Stability

By employing the lemma below, we analyze the global stability of each point.

**Lemma 6.1.** *(See [18]). For any $t > 0$, $D_t^\alpha \left[x(t) - x^* - x^* \ln \frac{x(t)}{x^*}\right] \leq \left(1 - \frac{x^*}{x(t)}\right) D_t^\alpha x(t)$, where $x(t) \in \mathbb{R}_+$ is a continuous and derivable function, $x^* \in \mathbb{R}_+$, and $\forall \alpha \in (0, 1]$.*

**Lemma 6.2.** *(See [19]). If a continuous and derivable function $V(x) : \Psi \to \mathbb{R}$ satisfies $D_t^\alpha V(x) \leq 0$, then the solution of $D_t^\alpha x(t) = f(x(t))$ goes from $\Psi$ and remains in $\Psi$ for all time, where $\Psi$ is a bounded closed set. It is known that $E := \{x | D_t^\alpha V(x) = 0\}$ and $M$ is the biggest number set of $E$. Thus, the solution of $x(t)$ departing from $\Psi$ tends to $M$ when $t \to \infty$.*

Suppose $V_i, i = 1, 2, 3$ are the Lyapunov functions. The global stability condition of equilibrium point is guaranteed by the following theorems.

**Theorem 6.3.** *Suppose that $\beta_1 > \varphi_1$ and $\beta_2 > \varphi_2$. The point $E_2$ is globally asymptotically stable when $\left(\tilde{z} + \frac{\delta_1}{\eta_1}\right) > \left(\frac{\xi_1(\beta_2 - \varphi_2)\tilde{x}}{\eta_2 \xi_2}\right)$ and $\left(\frac{\xi_1(\beta_2 - \varphi_2)}{\eta_2 \xi_2}\right) > \left(\frac{\beta_1 - \varphi_1}{\eta_1}\right)$.*

*Proof:* By considering $V_1$ as follows.

$$V_1(x, y, z) = \left(\frac{\beta_2 - \varphi_2}{\eta_2 \xi_2}\right)\left(x - \tilde{x} - \tilde{x} \ln \frac{x}{\tilde{x}}\right) + \frac{1}{\eta_1} y + \frac{1}{\eta_2}\left(z - \tilde{z} - \tilde{z} \ln \frac{z}{\tilde{z}}\right).$$

We investigate that $V_1(E_2) = 0$. Then, the first condition is satisfied. Furthermore, by using Lemma 6.1, we obtain

$$\begin{aligned}
D_t^\alpha V_1 &= \left(\frac{\beta_2 - \varphi_2}{\eta_2 \xi_2}\right)\left(1 - \frac{\tilde{x}}{x}\right) D_t^\alpha x + \frac{1}{\eta_1} D_t^\alpha y + \frac{1}{\eta_2}\left(1 - \frac{\tilde{z}}{z}\right) D_t^\alpha z, \\
&= -\frac{a_1(\beta_2 - \varphi_2)}{\eta_2 \xi_2}(x - \tilde{x})^2 - \frac{a_3}{\eta_2}(z - \tilde{z})^2 + \frac{\xi_1(\beta_2 - \varphi_2)\tilde{x}}{\eta_2 \xi_2} y - \frac{\tilde{z} y}{1 + \sigma y} - \frac{\delta_1}{\eta_1} y \\
&\quad - \frac{a_2}{\eta_1} y^2 + \frac{\beta_1 - \varphi_1}{\eta_1} xy - \frac{\xi_1(\beta_2 - \varphi_2)}{\eta_2 \xi_2} xy, \\
&\leq -\frac{a_1(\beta_2 - \varphi_2)}{\eta_2 \xi_2}(x - \tilde{x})^2 - \frac{a_3}{\eta_2}(z - \tilde{z})^2 - \left(\tilde{z} + \frac{\delta_1}{\eta_1} - \frac{\xi_1(\beta_2 - \varphi_2)\tilde{x}}{\eta_2 \xi_2}\right) y \\
&\quad - \left(\frac{\xi_1(\beta_2 - \varphi_2)}{\eta_2 \xi_2} - \frac{\beta_1 - \varphi_1}{\eta_1}\right) xy.
\end{aligned}$$

It is easy to confirm that $D_t^\alpha V_1 \leq 0$ when $\left(\tilde{z} + \frac{\delta_1}{\eta_1}\right) > \left(\frac{\xi_1(\beta_2 - \varphi_2)\tilde{x}}{\eta_2 \xi_2}\right)$ and $\left(\frac{\xi_1(\beta_2 - \varphi_2)}{\eta_2 \xi_2}\right) > \left(\frac{\beta_1 - \varphi_1}{\eta_1}\right)$ with $\beta_1 > \varphi_1$ and $\beta_2 > \varphi_2$. Based on Lemma 6.2, the non-negative solutions tend to $E_2$. Thus, the point $E_2$ is globally asymptotically stable. ■

**Theorem 6.4.** *Let $\beta_1 > \varphi_1$ and $\beta_2 > \varphi_2$. The point $E_3$ is globally asymptotically stable if $\frac{\delta_2}{\eta_2} > \left(\hat{y} + \frac{\xi_2(\beta_1 - \varphi_1)\hat{x}}{\eta_1 \xi_1}\right)$ and $\left(\frac{\xi_2(\beta_1 - \varphi_1)}{\eta_1 \xi_1}\right) > \left(\frac{\beta_2 - \varphi_2}{\eta_2}\right)$.*

*Proof:* First, we define $V_2$ as follows.

$$V_2(x, y, z) = \left(\frac{\beta_1 - \varphi_1}{\eta_1 \xi_1}\right)\left(x - \hat{x} - \hat{x} \ln \frac{x}{\hat{x}}\right) + \frac{1}{\eta_1}\left(y - \hat{y} - \hat{y} \ln \frac{y}{\hat{y}}\right) + \frac{1}{\eta_2} z.$$

We can confirm that $V_2(E_3) = 0$ so that the first condition is proven. By considering Lemma 6.1,

$$
\begin{aligned}
D_t^\alpha V_2 &= \left(\frac{\beta_1-\varphi_1}{\eta_1\xi_1}\right)\left(1-\frac{\hat{x}}{x}\right)D_t^\alpha x+\frac{1}{\eta_1}\left(1-\frac{\hat{y}}{y}\right)D_t^\alpha y+\frac{1}{\eta_2}D_t^\alpha z,\\
&= -\frac{a_1\left(\beta_1-\varphi_1\right)}{\eta_1\xi_1}\left(x-\hat{x}\right)^2-\frac{a_2}{\eta_1}\left(y-\hat{y}\right)^2+\frac{\xi_2\left(\beta_1-\varphi_1\right)\hat{x}}{\eta_1\xi_1}z+\frac{\hat{y}z}{1+\sigma y}-\frac{\delta_2}{\eta_2}z\\
&\quad -\frac{a_3}{\eta_2}z^2+\frac{\beta_2-\varphi_2}{\eta_2}xz-\frac{\xi_2\left(\beta_1-\varphi_1\right)}{\eta_1\xi_1}xz,\\
&\leq -\frac{a_1\left(\beta_1-\varphi_1\right)}{\eta_1\xi_1}\left(x-\hat{x}\right)^2-\frac{a_2}{\eta_1}\left(y-\hat{y}\right)^2-\left(\frac{\delta_2}{\eta_2}-\hat{y}-\frac{\xi_2\left(\beta_1-\varphi_1\right)\hat{x}}{\eta_1\xi_1}\right)z\\
&\quad -\left(\frac{\xi_2\left(\beta_1-\varphi_1\right)}{\eta_1\xi_1}-\frac{\beta_2-\varphi_2}{\eta_2}\right)xz.
\end{aligned}
$$

If $\frac{\delta_2}{\eta_2}>\left(\hat{y}+\frac{\xi_2(\beta_1-\varphi_1)\hat{x}}{\eta_1\xi_1}\right)$ and $\left(\frac{\xi_2(\beta_1-\varphi_1)}{\eta_1\xi_1}\right)>\left(\frac{\beta_2-\varphi_2}{\eta_2}\right)$, then $D_t^\alpha V_2\leq 0$. According to Lemma 6.2, the non-negative solutions tend to $E_3$. Thus, the point $E_3$ is globally asymptotically stable. ■

**Theorem 6.5.** *Suppose that $\beta_1>\varphi_1$ and $\beta_2>\varphi_2$. By noticing some conditions as follows.*

$$
\begin{aligned}
\psi_1 &= \frac{1}{r}\left(\frac{\delta_1 y^*}{\eta_1}+\frac{\delta_2 z^*}{\eta_2}\right),\\
\psi_2 &= \min\left\{\frac{\eta_1 z^*+\delta_1-a_2y^*}{\eta_1\xi_1},\frac{\delta_2-\eta_2y^*-a_3}{\eta_2\xi_2},\frac{\eta_2\left(\beta_1-\varphi_1\right)+\eta_1\left(\beta_2-\varphi_2\right)-\eta_1\eta_2 r}{a_1\eta_1\eta_2}\right\}.
\end{aligned}
$$

*The point $E_4$ is globally asymptotically stable when it satisfies the following conditions, that is $\xi_1>\left(\frac{\beta_1-\varphi_1}{\eta_1}\right),\xi_2>\left(\frac{\beta_2-\varphi_2}{\eta_2}\right)$, and $\psi_1<x^*<\psi_2$.*

*Proof:* By defining $V_3$ as follows.

$$
V_3\left(x,y,z\right)=\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\frac{1}{\eta_1}\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\frac{1}{\eta_2}\left(z-z^*-z^*\ln\frac{z}{z^*}\right).
$$

It is clear that $V_3\left(x^*,y^*,z^*\right)=0$. Thus, the first requirement is satisfies. By following Lemma 6.1, we have

$$
\begin{aligned}
D_t^\alpha V_3 &= \left(1-\frac{x^*}{x}\right)D_t^\alpha x+\frac{1}{\eta_1}\left(1-\frac{y^*}{y}\right)D_t^\alpha y+\frac{1}{\eta_2}\left(1-\frac{z^*}{z}\right)D_t^\alpha z,\\
&= -\left(\frac{\beta_1-\varphi_1}{\eta_1}+\frac{\beta_2-\varphi_2}{\eta_2}-r-a_1x^*\right)-\left(\frac{z^*}{1+\sigma y}+\frac{\delta_1}{\eta_1}-\xi_1x^*-\frac{a_2}{\eta_1}y^*\right)y\\
&\quad -\left(\frac{\delta_2}{\eta_2}-\xi_2x^*-\frac{y^*}{1+\sigma y}-\frac{a_3}{\eta_2}\right)z-\left(\xi_1-\frac{\beta_1-\varphi_1}{\eta_1}\right)xy-\left(\xi_2-\frac{\beta_2-\varphi_2}{\eta_2}\right)xz\\
&\quad -a_1x^2-\frac{a_2}{\eta_1}y^2-\frac{a_3}{\eta_2}z^2-\left(rx^*-\frac{\delta_1}{\eta_1}y^*-\frac{\delta_2}{\eta_2}z^*\right),\\
&\leq -\left(\frac{\beta_1-\varphi_1}{\eta_1}+\frac{\beta_2-\varphi_2}{\eta_2}-r-a_1x^*\right)-\left(z^*+\frac{\delta_1}{\eta_1}-\xi_1x^*-\frac{a_2}{\eta_1}y^*\right)y\\
&\quad -\left(\frac{\delta_2}{\eta_2}-\xi_2x^*-y^*-\frac{a_3}{\eta_2}\right)z-\left(\xi_1-\frac{\beta_1-\varphi_1}{\eta_1}\right)xy-\left(\xi_2-\frac{\beta_2-\varphi_2}{\eta_2}\right)xz\\
&\quad -\left(rx^*-\frac{\delta_1}{\eta_1}y^*-\frac{\delta_2}{\eta_2}z^*\right).
\end{aligned}
$$

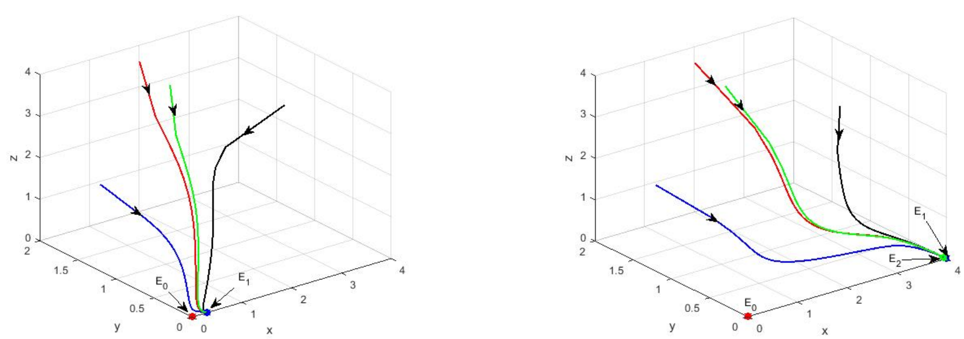
Therefore, if $\xi_1>\left(\frac{\beta_1-\varphi_1}{\eta_1}\right),\xi_2>\left(\frac{\beta_2-\varphi_2}{\eta_2}\right)$, and $\psi_1<x^*<\psi_2$, then $D_t^\alpha V_3\leq 0$. In consequence of Lemma 6.2, the non-negative solutions tend to $E_4$. Thus, the point $E_4$ is globally asymptotically stable. ■

## 7. NUMERICAL SIMULATIONS

To support our analytical results and show the behavior of model, we perform the numerical solution for System (5). In our work, we use the nonstandard Grunwald-Letnikov approximation method for nonlinear fractional-order differential equations which is combination from Grunwald-Letnikov approximation method developed by [40] and the nonstandard finite difference presented in [42], [43], [44]. This method has been applied by several researcher as in [41], [27], [36]. To construct the numerical schemes for System (5), we apply the same way in [41], [36]. Thus, we have the nonstandard Grunwald-Letnikov schemes as follows.

$$\begin{aligned} x_{n+1} &= \frac{\sum_{j=1}^{n+1} c_j^{\alpha} x_{n+1-j} + w_{n+1}^{\alpha} x_0 + \Delta t^{\alpha} r x_n}{1+\Delta t^{\alpha}\left(a_1 x_n + \xi_1 y_n + \xi_2 z_n\right)}, \\ y_{n+1} &= \frac{\sum_{j=1}^{n+1} c_j^{\alpha} y_{n+1-j} + w_{n+1}^{\alpha} y_0 - \Delta t^{\alpha} \delta_1 y_n}{1+\Delta t^{\alpha}\left(-\left(\beta_1-\varphi_1\right)x_n + a_2 y_n + \frac{\eta_1 z_n}{1+\sigma y_n}\right)}, \\ z_{n+1} &= \frac{\sum_{j=1}^{n+1} c_j^{\alpha} z_{n+1-j} + w_{n+1}^{\alpha} z_0 - \Delta t^{\alpha} \delta_2 z_n}{1+\Delta t^{\alpha}\left(-\left(\beta_2-\varphi_2\right)x_n + a_3 z_n - \frac{\eta_2 y_n}{1+\sigma y_n}\right)}, \end{aligned} \tag{20}$$

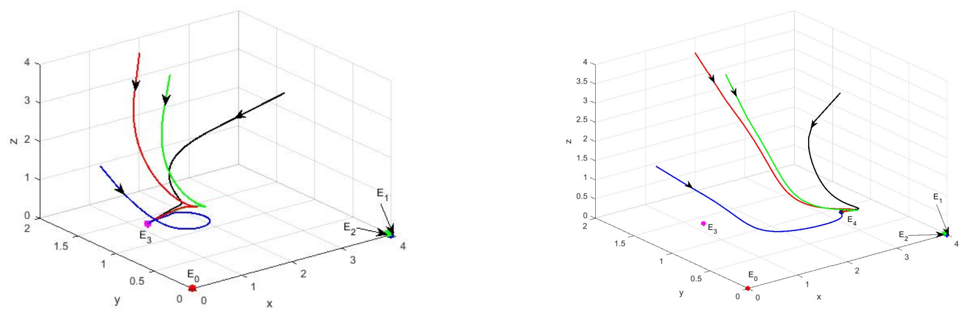
where $c_j^{\alpha} = \left(1-\frac{(\alpha+1)}{j}\right)c_{j-1}^{\alpha}; c_1^{\alpha} = \alpha$; and $w_{n+1}^{\alpha} = \frac{(n+1)^{-\alpha}}{\Gamma(1-\alpha)}$. It is known that $\Delta t$ means the time step of numerical integration and $c_j^{\alpha}$ is the positive values and follows a condition, that is $0 < c_{n+1}^{\alpha} < c_n^{\alpha} < \cdots < c_1^{\alpha} = \alpha$ with $n \leq 1$ [40]. The form of our scheme (20) is explicit so that it is easy to be applied.




(a) $r = 0.15$ and $\delta_1 = 0.2$ (b) $r = 2$ and $\delta_1 = 2.2$

Figure 2: 3-D Phase portraits for $E_1$ and $E_2$ with $\alpha = 0.8$ and $\Delta t = 0.1$.

To verify the stability analysis and numerical scheme obtained in the previous discuss, we do several numerical solutions. It is known that we don't have the actual data so that we use the hypothetical values as our parameters where it corresponds to the stability conditions. First, we select the following parameters, that is $r = 0.15; \xi_1 = 1; \xi_2 = 0.5; a_1 = 0.5; \delta_1 = 0.2; \beta_1 = 1.2; a_2 = 0.3; \eta_1 = 1.3; \varphi_1 = 0.7; \delta_2 = 0.3; \beta_2 = 0.1; a_3 = 0.3; \eta_2 = 1; \varphi_2 = 0.02; \sigma = 0.3$. We have two equilibrium points, that is $E_0\,(0,0,0)$ as a saddle point and $E_1\,(0.3,0,0)$ is locally asymptotically stable. This condition fits to Theorem 5.2, where it is proven by the solutions that converges to $E_1$ (see Figure 2(a)). Here, species $x$ exist and both predators become extinct. Since the natural growth of prey is small, the species $y$ and $z$ undergo extinction due to decreased predation on prey and increased intraspecific competition caused by limited food. However, the species $x$ can survive even though its population density is small. When the natural growth rate of prey and death of intermediate predator are raised to $r = 2$ and $\delta_1 = 2.2$, Theorem 5.3 and 6.3 are satisfied. Therefore, we have three equilibrium points, that is $E_0\,(0,0,0)\,; E_1\,(4,0,0)\,;$ and $E_2\,(3.947, 0, 0.053)$. Here, the point $E_2$ is stable (both locally and globally) but $E_1$ becomes a saddle point. This can be proven by all solutions that converge to $E_2$ (see Figure 2(b)). This indicates that species $x$ and $z$ survive but species $y$ become extinct. The species $y$ became extinct due to a high natural mortality. However, the species $z$ can survive together with species $x$ due to abundant food and no competition within the community.




(a) $r = 2$ and $\sigma = 5.3$ (b) $r = 2$ and $\sigma = 0.3$

Figure 3: 3-D Phase portraits for $E_3$ and $E_4$ with $\alpha = 0.8$ and $\Delta t = 0.1$.

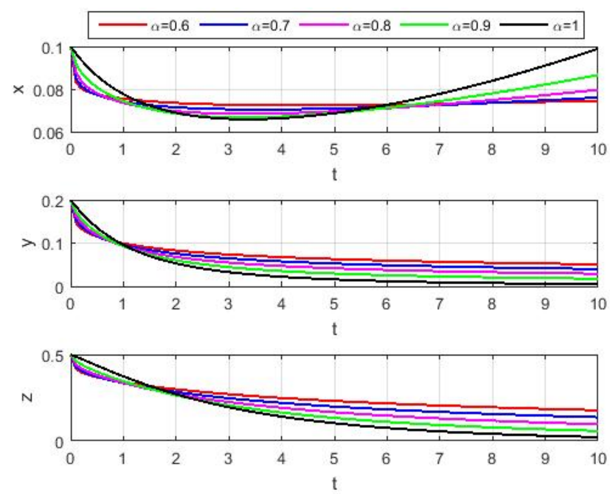



Figure 4: Solution curves for System (5) by taking various of $\alpha$ values and $\Delta t = 0.1$.

By considering the previous parameters except $r = 2$ and $\sigma = 5.3$, we have four existing equilibrium points which fit to Theorem 5.4 and 6.4. Therefore, the point $E_3$ is asymptotically stable (both locally and globally) but the other points are a saddle point. This is shown from all solutions which converge to $E_3\,(1.231, 1.385, 0)$ (see Figure 3(a)). Thus, we can conclude that species $x$ and $y$ exist but species $z$ is stopped. Since the natural growth of prey is huge, the species $y$ and $z$ have abundant food. However, the species $z$ undergo extinct due to high environment protection from intermediate predator. Therefore, the intermediate predator can consume prey easily but omnivores are not. They need great effort to survive. When we take $r = 2$ and $\sigma = 0.3$, all equilibrium points exist and Theorem 5.5 and 6.5 are satisfied. Thus, the point $E_3$ becomes a saddle point and $E_4$ is asymptotically stable, both locally and globally. This is proven from all solutions which converge to $E_4\,(2.461, 0.377, 0.785)$ (see Figure 3(b)). In this case, all populations can survive in the community. Here, both predators have abundant food and can eat prey but they can still survive the attack of predator. To show the effect of memory denoted by $\alpha$ as in System (5), we perform numerical simulation using all parameters as in the first experiment with various orders of $\alpha$. When order $\alpha$ is close to $\alpha = 1$, the Caputo fractional-order system solution is also close to the first order system solutions (see Figure 4). From the graphical analysis, we observe that the population in species $x$ decrease significantly by decreasing the fractional order. Meanwhile, The population in species $y$ and $z$ increase significantly by decreasing the fractional order. Therefore, since the memory effect of all populations is small, the prey density decreases but the predator density for both intermediate predators and omnivores increases.

## 8. Conclusion

A fractional-order food chain model has presented in the previous discussion. This model explains the food chain process of three species built by prey, intermediate predator, and omnivore. It is known that our model has five equilibrium points, where their stability analysis (both locally and globally) is obtained conditionally. These dynamic conditions are confirmed by our nonstandard Grunwald-Letnikov schemes. Our scheme can fit on the obtained analytical results. In addition, when the order of derivative is reduced, then the solution convergence of each point will decrease. In this case, we can interpret that the density of species $x$ is directly proportional to the fractional order. Meanwhile, the density of species $y$ and $z$ is inversely proportional to the fractional order.

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