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A Reconstruction Method for Compressed Sampling in Shift-Invariant Spaces

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Abstract. A traditional sampling method is that the signal should be sampled at a rate exceeding twice the highest frequency. This is based on the assumption that the signal occupies the entire bandwidth. In practice, however, many signals are sparse so that only part of the bandwidth is used. Compressed sampling has been developed for low-rate sampling of continuous time sparse signals in shift-invariant spaces generated by *m* kernels with period *T*. However, in general the reconstruction of compressed sampling signals is unstable. To reconstruct the signal, continuous reconstruction is replaced by generalized inverse. In this paper, periodic non-uniform sampling and the reconstruction of functions in shift-invariant spaces are discussed, the unique sparse expression is obtained by using the minimal L_1 norm. Also, necessary condition and error of reconstruction were analyzed. Finally, the method was validated via simulation and it was shown that the method was effective.

Keywords: *error analysis; generalized inverse; minimum L_1 normal; periodic non-uniform sampling; shift-invariant spaces; sparse signals.*

1 Introduction

The traditional assumption underlying most analog-to-digital converters is that the sample must be acquired at the Shannon-Nyquist rate, corresponding to twice the highest frequency. In fact, radio frequency technology enables the modulation of narrow-band signals by high carrier frequencies, which cannot be sampled using the Shannon theorem anymore [1,2]. Chen Meng and Jamal Tuqan introduced the concept of retaining periodic nonuniform samples from a group of samples selected from a larger set, obtained by L times oversampling a continuous signal [3]. Ha T. Nguyen addressed a method to reconstruct the original signal from periodic nonuniform samples by utilizing optimal $H\infty$ [4]. But most of these approaches can only completely recover the original signals on the condition that it is known where those signals are located and those signals occupy almost the whole bandwidth.

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However, many man-made signals are sparse, consisting of a relatively small number of narrowband transmissions spread across a wide spectrum range. If we use a traditional periodic non-uniform sampling approach, the sampling system will be inefficient. David L. Donoho [5] and Emamnuel J. Candès [6] have pointed out that if a signal has a sparse expansion, one can discard small coefficients without much perceptual loss. There are a lot of reconstruction algorithms that can recover sampled sparse signals, such as multiple measurement vectors [7], minimum L_1 norm [5], orthogonal matching pursuit [8], etc. In this study, we brought compressive sensing into the periodic nonuniform sampling scheme, which is a promising way to achieve sampling and reconstruction of blind sparse signals.

This paper is organized as follows. In Section 2, periodic nonuniform sampling theory and the architecture of the system are introduced. Section 3 presents the proposed approach of recovering the sampled sparse signals. Section 4 deals with simulations and performance analysis in terms of the empirical recovery rate. In Section 5, we analyze the reconstruction errors. Finally, in Section 6 the conclusions of this work are outlined.

2 Theory of Periodic Nonuniform Sampling

Traditional nonuniform sampling is used to convert a continuous analogue signal $x(t)\in L_2$-space into its discrete representation. The architecture of the sampling system is shown in Figure 1.

In Eq. (1), let $a_i(t)$ be a nonuniform sample sequence of the i-th channel defined by

$$a_i(t) = \sum_{n=-\infty}^{+\infty} \delta(t - Tn - i\tau)(0 \le i \le s-1) \tag{1}$$

where, T is sampling period and τ is sequence.

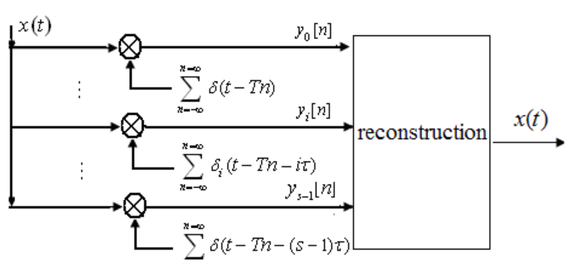

Figure 1 Mode of periodic nonuniform sampling.

Then, as shown in Figure 1, input channels to the reconstruction box can be written as in Eq. (2):

$$y_i(n) = \sum_{n=-\infty}^{n=\infty} x(nT + i\tau)\delta(t - nT - i\tau) \quad (2)$$

where $0 \le i \le s - 1$

In Eq. (3), the corresponding discrete-time Fourier transform is given by:

$$Y_i(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X(\omega - 2\pi n / T)e^{-j2\pi ni\tau/T} \quad (3)$$

In order to reconstruct $x(t)$ from the samples $y[n](y[n]=[y_0[n], y_1[n],\ldots,y_{s-1}[n]])$, we assume that $x(t)$ lies in a subspace $V(\varphi)$ of L_2. In this paper, we define that $V(\varphi)$ is generated by m space functions $\varphi(t)$:

$$V(\varphi) = \{\sum_{p=0}^{m-1}\sum_{n\in z} r_p[n]\varphi_p(t - nT) : r_p[n] \in L_2\}$$

We can represent any $x(t) \in V(\varphi)$ as in Eq. (4):

$$x(t) = \sum_{p=0}^{m-1}\sum_{n\in z} r_p[n]\varphi_p(t - nT) \quad (4)$$

In order to guarantee a unique stable representation of any signal in $V(\varphi)$ by sequence $\{r_p[n]\}$, the generators $\varphi(t)$ must form a Riesz basis of L_2. In other words, there exist two constants, $0<A \le B<\infty$, such that in Eq. (5).

$$A\|r[n]\|_2^2 \le \left\|\sum_{p=0}^{m-1}\sum_{n\in z} r_p[n]\varphi_p(t - nT)\right\|_2^2 \le B\|r[n]\|_2^2 \quad (5)$$

where, $\|r[n]\|_2^2 = \sum_{p=0}^{m-1}\sum_{n\in z}|r_p[n]|^2$, $\|\bullet\|_2$ is L_2 norm.

Proposition: if and only if $\alpha I \le W(\omega) \le \beta I$, generator $\varphi_p(t-nT)$ forms a Riesz basis.

Where, I is the identity matrix

$$W(\omega) = \begin{bmatrix} w_{1,1} & \cdots & w_{1,m} \\ \vdots & \ddots & \vdots \\ w_{m,1} & \cdots & w_{m,m} \end{bmatrix}$$

$$w_{a,b} = \frac{1}{T}\sum_{n\in z}\psi_a^{\,*}(\omega-2\pi n/T)\psi_b(\omega-2\pi n/T)$$

Here, $\psi(\omega)$ is the Fourier transform of $\varphi(t)$, a,b is subscript.

Proof: Eq. (5) can be rewritten as shown in Eq. (6):

$$\alpha\left\|\mathrm{r}[n]\right\|_2^2 \le \int_{-\infty}^{\infty}\left|\sum_{n\in z}\mathrm{r}^*[n]\varphi(t-nT)\right|^2 dt \le \beta\left\|\mathrm{r}[n]\right\|_2^2 \tag{6}$$

In Eq. (7), from the theory of Parseval:

$$\int_{-\infty}^{\infty}\left|\sum_{n\in z}\mathrm{r}^*[n]\varphi(t-nT)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\sum_{n\in z}\mathrm{R}^*(\omega)\psi(\omega)\right|^2 dt \tag{7}$$

where, is $\mathrm{R}(\omega)$ is the discrete-time Fourier transform of $r[n]$, and $\mathrm{R}(\omega)$ is 2π-periodic. Then Eq. (7) can be rewritten as in Eq. (8).

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\sum_{n\in z}\mathrm{R}^*(\omega)\psi(\omega-2\pi n/T)\right|^2 dt\\
&=\frac{1}{2\pi}\sum_{n\in z}\int_0^{2\pi}\mathrm{R}^*(\omega)\psi(\omega-2\pi n/T)\psi^*(\omega-2\pi n/T)\mathrm{R}(\omega)d\omega\\
&=\frac{T}{2\pi}\int_0^{2\pi}\mathrm{R}^*(\omega)W(\omega)\mathrm{R}(\omega)d\omega
\end{aligned} \tag{8}$$

From Parseval's theorem then we have Eq. (9) as follows:

$$\left\|\mathrm{r}[n]\right\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{R}^*(\omega)\mathrm{R}(\omega)d\omega \tag{9}$$

It is easy to see that $W(\omega)$ is a positive self-adjoint that has real nonnegative eigenvalues. Let α be the minimal eigenvalues and β the maximal eigenvalues. Then we have:

$$AI\mathrm{R}^*(\omega)\mathrm{R}(\omega) \le \mathrm{R}^*(\omega)\mathrm{R}(\omega)W(\omega) \le BI\mathrm{R}^*(\omega)\mathrm{R}(\omega)$$

The conclusion can be obtained that *φp(t-nT)* form a Riesz basis if and only if $\alpha I \le W(\omega) \le \beta I$.

The above-mentioned subspace $V(\varphi)$ is a single space. A more interesting scenario is when $x(t)$ lies in a union of subspaces $\bigcup V_p(\varphi)$ ($0 \le p \le m$-1) (see Eq. (10)):

$$x(t) \in \bigcup V_p \varphi) \tag{10}$$

In the frequency domain, Eq.(4) can be represented as in Eq. (11):

$$X(\omega) = \sum_{p=0}^{m-1} R_p(\omega) \psi_p(\omega) \tag{11}$$

where, $R_P(\omega)$ is the discrete-time Fourier transform of $r_p[n]$, and $\psi_p(\omega)$ is a Fourier transform of $\varphi_p(t)$.

Substitute Eq. (11) into Eq. (3), then we have Eq. (12).

$$\begin{aligned} Y_i(\omega) &= \frac{1}{T} \sum_{n=-\infty}^{+\infty} \sum_{p=0}^{m-1} R_p(\omega + 2\pi n / T) \psi_p(\omega + 2\pi n / T) e^{-j2\pi n i \tau / T} \\ &= \frac{1}{T} \sum_{p=0}^{m-1} R_p(\omega) \sum_{n=-\infty}^{+\infty} \psi_p(\omega + 2\pi n / T) e^{-j2\pi n i \tau / T} \end{aligned} \tag{12}$$

An appropriate matrix representation of Eq. (12) is given by Eq. (13).

$$Y(\omega) = H(\omega) R(\omega) \tag{13}$$

Where, $Y(\omega) = (Y_0(\omega), Y_1(\omega), \cdots Y_{s-1}(\omega))'$

$$R(\omega) = (R_0(\omega), R_1(\omega), \cdots R_{m-1}(\omega))'$$

$$H(\omega) = \begin{bmatrix} h_{0,0} & h_{0,1} & \dots & h_{0,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{s-1,0} & h_{s-1,1} & \dots & h_{s-1,m-1} \end{bmatrix}$$

$$h_{i,p}(\omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \psi_p(\omega + 2\pi n / T) e^{-j2\pi n i \tau / T}$$

3 Analysis of Reconstruction Algorithm

In this work, our interest is to take sparse signals as an example for analyzing the reconstruction system. We suppose that *x(t)* is a *k*-sparse signal, which means that the union of subspaces over at most *k* elements or only *k* trains $\{r_p[n]\}_{p=0,1,\ldots,m-1}$ 1 is not zero. In order to improve efficiency and to ensure the complete reconstruction of the sampled sparse signal, our sampling scheme has to recover the sparse signal from a set of $k < s << m$ sampling sequences. We cannot recover *R(ω)* via solving Eq. (13) directly when Eq. (13) is a underdetermined equation.

A unique *R(ω)* can be recovered by solving the optimization problem (see Eq. (14)).

$$s.t\ Y(\omega) = H(\omega)R(\omega)\ \min\|R(\omega)\|_0\ s.t\ Y(\omega) = H(\omega)R(\omega) \tag{14}$$

This optimization will recover a *k*-sparse signal exactly with high probability if it meets a certain condition, which is given by Theorem 1.

Theorem 1: if *R*(*ω*) is a solution of Eq. (9) and $\|R(\omega)\|_0 \leq kr(H(\omega))/2$, then *R*(*ω*) is the unique *k*-sparse solution. $kr(H(\omega))$ is the Kruskal rank of *H*(*ω*). The Kruskal rank is the maximal *q* such that every set of *q* columns of *H*(*ω*) is linearly independent.

Proof: Suppose $R_a(\omega), R_b(\omega)$ are solutions to Eq.(9), then we have $H(\omega)(R_b(\omega) - R_a(\omega)) = 0$.

Because *H*(*ω*) has $kr(H(\omega))$ linearly independent columns. We have Eq. (15).

$$\|R_b(\omega) - R_a(\omega)\|_0 \geq kr(H(\omega)) \tag{15}$$

On the other hand, we have:

$$\max\left\{\|R_b(\omega)\|_0, \|R_a(\omega)\|_0\right\} \leq kr(H(\omega))/2$$
$$\therefore \|R_b(\omega)\|_0 + \|R_a(\omega)\|_0 \leq kr(H(\omega))$$

Therefore,

$$,\|R_b(\omega) - R_a(\omega)\|_0 \leq \|R_b(\omega)\|_0 + \|R_a(\omega)\|_0 \leq kr(H(\omega)),$$

which contradicts with Eq. (15). Therefore, we have $R_b(\omega) = R_a(\omega)$.

But unfortunately the solution of Eq.(10) is both numerically unstable and an NP-hard problem, which leads to difficulty recovering *R*(*ω*) using L_0 norm. There are several alternative algorithms that can replace L_0 norm for recovering *R*(*ω*). In this work, the minimum L_1 normal approach algorithm was selected.

3.1 Minimum L_1 Normal Approach

The aim of the minimum L_1 normal algorithm is to find the unique sparse representation of $R(\omega)$. The approach is to recover $R(\omega)$ in three steps: 1) we find support set B, which is a set of indices corresponding to the non-zero elements of $\{R(\omega))\}$; 2) once B is found, we have a new matrix, $H_B(\omega)$, which consists of columns of $\boldsymbol{H}(\omega)$ whose indices correspond to set B. Then, the k non-zero elements of $\{)\}$ $R(\omega)$)} can be recovered; 3) all m sequences $\{r[n]\}$ can be obtained by using an interpolator, the structure of which is shown in Figure 2.

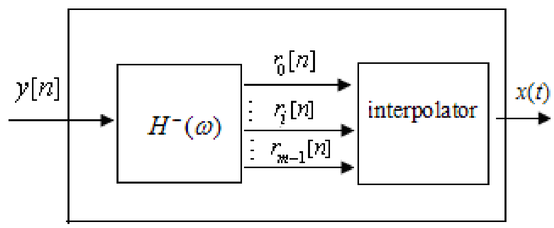

Figure 2 Block of reconstruction bank.

If index s in sensing matrix $\boldsymbol{H}(\omega)$ is smaller than m, then Eq. (13) has unbounded solutions, which is in a space with (m-s) dimensions:

$$T = N(\boldsymbol{H}(\omega)) + R^*(\omega)$$

where, is $N(\boldsymbol{H}(\omega))$ is null space. Eq. (8) can be rewritten as:

$$\boldsymbol{Y}(\omega) = \boldsymbol{H}(\omega)(\boldsymbol{R}^*(\omega) + \boldsymbol{O}(\omega))$$

where, $\boldsymbol{O}(\omega)$ is the vector in null space $N(\boldsymbol{H}(\omega))$

Therefore, our aim is to obtain solution $R^*(\omega)$ in null space $N(\boldsymbol{H}(\omega))$ through minimum L_1 normal (see Eq. (16)).

$$\min \|\boldsymbol{R}(\omega)\|_1 \text{ s.t } \boldsymbol{Y}(\omega) = \boldsymbol{H}(\omega)\boldsymbol{R}(\omega) \tag{16}$$

Suppose $R_{\text{opt}}(\omega)$ meets the optimal solution of Eq. (13), then we have Eq. (17):

$$Y(\omega) = H(\omega)R_{\text{opt}}(\omega) \tag{17}$$

If set B is the position of nonzero elements of $R_{\mathrm{opt}}(\omega)$ and the new matrix $H_B(\omega)$ is formed by the column vectors corresponding to nonzero elements in $\boldsymbol{H}(\omega)$ and $R_{\mathrm{opt}}(\omega)$, then gives Eq. (18):

$$Y(\omega) = H_B(\omega)R^B(\omega) \tag{18}$$

where, $R_B(\omega)$ is a vector formed by nonzero elements in $R_{\mathrm{opt}}(\omega)$.

In $R_{\mathrm{opt}}(\omega)$, there are at most *k* nonzero elements, then $H_B(\omega)$ has *k* column vectors at most. According to analysis of Theorem 1, a necessary condition for complete reconstruction of the *k*-sparse signal is $kr(H(\omega)) \geq 2k$, so $H_B(\omega)$ must be a column non-singular matrix, otherwise the number of nonzero elements in $R_{\mathrm{opt}}(\omega)$ will drop, which is contradictory to optimality. Therefore, the Penrose generalized inverse matrix of $H_B(\omega)$ is defined as in Eq. (19).

$$\boldsymbol{H}_B^-(\omega) = (\boldsymbol{H}_B^{\mathrm{H}}(\omega)\boldsymbol{H}(\omega))^{-1}\boldsymbol{H}_B^{\mathrm{H}}(\omega) \tag{19}$$

Where, $(\cdot)^{\mathrm{H}}$ is conjugate transpose.

We can obtain the solution of Eq.(18) in Eq. (20).

$$\boldsymbol{R}^B(\omega) = (\boldsymbol{H}_B^{\mathrm{H}}(\omega)\boldsymbol{H}(\omega))^{-1}\boldsymbol{H}_B^{\mathrm{H}}(\omega)\boldsymbol{Y}(\omega) \tag{20}$$

When set B is found, all *m* reconstructed sequences $r[n]$ are obtained, that is shown in Eq. (21).

$$\begin{aligned} &r_B[n] = H_B^- y[n] \\ &r_i[n] = 0 \qquad i \notin B \end{aligned} \tag{21}$$

where, $r_B[n] = [r_0[n];\ r_1[n], \cdots,\ r_{k\text{-}1}[n]]$.

3.2 Analysis on Reconstruction Condition of Approach

In this work, L_1 normal was used to find the unique sparse expression. The L_1 normal approach given in Theorem 2 fully meets the complete reconstruction conditions.

Theorem 2: If the number of $\boldsymbol{R}(\omega)$'s nonzero elements is *k*, let B be an index set that contains nonzero elements of $\boldsymbol{R}(\omega)$, then the matrix $H_B(\omega)$ is formed by column vectors of $\boldsymbol{H}(\omega)$ corresponding to set B.

When
$\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} \left|\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)\right|_1\right) < 1$, minimum L_1 normal can achieve complete reconstruction.
Where, $\boldsymbol{H}_B^-(\omega)$ is generalized inverse of $\boldsymbol{H}_B(\omega)$, $\boldsymbol{h}_i(\omega)$, $i \notin B$ is the rest column vector of $\boldsymbol{H}(\omega)$ after removing $\boldsymbol{H}_B(\omega)$.

Proof: Suppose the optimal solution is $\boldsymbol{R}^B(\omega)$, $\boldsymbol{R}_r(\omega)$ can be any solution of $\boldsymbol{H}(\omega)\boldsymbol{R}(\omega) = \boldsymbol{Y}(\omega)$, then:

$$\left\|\boldsymbol{R}^B(\omega)\right\|_1 = \left\|\boldsymbol{H}_B^-(\omega)\boldsymbol{H}_B(\omega)\boldsymbol{R}^B(\omega)\right\|_1 = \left\|\boldsymbol{H}_B^-(\omega)\boldsymbol{Y}(\omega)\right\|_1 = \left\|\boldsymbol{H}_B^-(\omega)\boldsymbol{H}(\omega)\boldsymbol{R}_r(\omega)\right\|_1 <$$

$$\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} |\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)|_1\right)\left\|\boldsymbol{R}_r(\omega)\right\|_1$$

When $\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} |\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)|_1\right) < 1$, then

$$\left\|\boldsymbol{R}^B(\omega)\right\|_1 < \left\|\boldsymbol{R}_r(\omega)\right\|_1.$$

Therefore, when $\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} |\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)|_1\right) < 1$, the sparse signal can be completely reconstructed through minimum L_1 normal. Therefore, the minimum L_1 normal approach can completely reconstruct signals.

Defined:

$$|D(\omega)|_1 = \sum_{i=0}^{N-1} |d_i(\omega)|$$
$$D(\omega) = (d_0(\omega), d_1(\omega), \cdots, d_{N-1}(\omega))$$

where, $\|\bullet\|_1$ is L_1 normal.

Theorem 3 gives conditions for complete reconstruction through the minimum L_1 normal approach.

Theorem 3: When $\left\|\boldsymbol{R}(\omega)\right\|_0 < \frac{1}{2}(\mu^{-1}+1)$, minimum L_1 normal can realize complete reconstruction. Where, $\mu = \max_{j \neq g}(\max_{-\pi/T \leqslant \omega < \pi/T} |\boldsymbol{h}_j^{\mathrm{H}}(\omega)\boldsymbol{h}_g(\omega)|)$.

Proof: According to Theorem 2, when $\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} |\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)|_1\right) < 1$, minimum L_1 normal can realize complete reconstruction of the sparse signal (see Eq. (22)).

$$\begin{aligned}
&\max_{i \notin B}\left(\max_{-\pi/T \le \omega \le \pi/T} \left|H_B^-(\omega)h_i(\omega)\right|_1\right) \\
&= \max_{i \notin B}\left(\max_{-\pi/T \le \omega < \pi/T} \left|\left(H_B^-(\omega)H_B(\omega)\right)^{-1} H_B^H(\omega)h_i(\omega)\right|_1\right) \\
&\le \max_{i \notin B}\left(\left|\left(H_B^H(\omega)H_B(\omega)\right)^{-1}\right|_{1,1} \max_{-\pi/T \le \omega < \pi/T} \left|H_B^H(\omega)h_i(\omega)\right|_1\right)
\end{aligned} \tag{22}$$

Supposed $\mu = \max_{j \neq g}\left(\max_{-\pi/T \leqslant \omega < \pi/T} \| \left|\boldsymbol{h}_j^{\mathrm{H}}(\omega)\boldsymbol{h}_g(\omega)\right|\right)$, then:

$$\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega < \pi/T} \left|\boldsymbol{H}_B^{\mathrm{H}}(\omega)\boldsymbol{h}_i(\omega)\right|_1\right) \leqslant \mu k$$

where, k is signal sparsity.

Suppose $\boldsymbol{H}_B^{\mathrm{H}}(\omega)\boldsymbol{H}_B(\omega) = I_k + E$, I_k is unit matrix, then: $|E|_{1,1} < \mu(k-1)$.
Then in Eq. (23) we have:

$$\left|\left(H_B^H(\omega)H_B(\omega)\right)^{-1}\right|_{1,1} = \left|\left(I_k + E\right)^{-1}\right|_{1,1} \le \frac{1}{1-|E|_{1,1}} \le \frac{1}{1-\mu(k-1)} \tag{23}$$

From Eq. (23), we can obtain Eq. (24):

$$\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} \left|\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)\right|_1\right) < \frac{\mu k}{1-\mu(k-1)} \tag{24}$$

When $\max_{i \notin B}\left(\max_{-\pi/T \leqslant \omega \leqslant \pi/T} \left|\boldsymbol{H}_B^-(\omega)\boldsymbol{h}_i(\omega)\right|_1\right) < 1$, minimum L1 normal can completely reconstruct signals. Then, $k < \frac{1}{2}(\mu^{-1}+1)$. Therefore, when $\|\boldsymbol{R}(\omega)\|_0 < \frac{1}{2}(\mu^{-1}+1)$, minimum L_1 normal can realize complete reconstruction.

3.3 Interpolator

As soon as $r[n]$ has been obtained, we have recovered $x(t)$ through an interpolator.

T_N is defined as the oversampling periodic that satisfies $T_N = T / M$.

We can rewrite Eq.(3) as in Eq. (25).

$$x[nMT_N] = \sum_{p=0}^{m-1}\sum_{c\in z} r_p[c]\varphi_p(nT_N - cT) \tag{25}$$

Upsampling the sequence ($x[nT_N]$:n∈Z) by a factor of M, the *d*-th sub-sequence is given by Eq. (26).

$$x[nMT_N + dT_N] = \sum_{p=0}^{m-1}\sum_{c\in z} r_p[c]\varphi_p(nMT_N + dT_N - cT) \tag{26}$$

DTFT of Eq.(25) results Eq. (27).

$$X_d(\omega) = \sum_{p=0}^{m-1} R_p(\omega)\psi_{p,d}(\omega) \tag{27}$$

Finally, we have the reconstructed signals in the frequency domain as shown in Eq. (28).

$$\begin{aligned} x(\omega) &= \sum_{d=0}^{M-1} e^{j\omega d} x_d(M\omega) \\ &= \sum_{d=0}^{M-1} e^{j\omega d} \sum_{p=0}^{m-1} R_p(M\omega)\psi_{p,d}(M\omega) \\ &= \sum_{p=0}^{m-1} R_p(M\omega) \sum_{d=0}^{M-1} e^{j\omega d} \psi_{p,d}(M\omega) \end{aligned} \tag{28}$$

4 Error Analysis

We defined an angle between two closed subspaces A and B of a Hilbert space V in Eqs. (29) and (30) as follows:

$$\cos(A, B) = \inf_{f\in A, \|f\|=1} \|P_B f\| \tag{29}$$

$$\sin(A, B) = \sup_{f\in A, \|f\|=1} \|P_{B^\perp} f\| \tag{30}$$

When reconstructed signal $\hat{x}(t) \in V$, we can conclude sampling error *e(x(t))* in Eq. (31) as follows:

$$\begin{aligned}\|e(x(t))\|^2 &= \|x(t)-\hat{x}(t)\|^2 \\ &= \|P_V x(t)-\hat{x}(t)\|^2 + \|P_{V^\perp} x(t)\|^2 \\ &\geq \|P_{V^\perp} x(t)\|^2\end{aligned} \tag{31}$$

From Eq.(31), we can have Eq. (32).

$$P_{V^\perp} x(t) = P_{V^\perp} e(x(t)) \tag{32}$$

When $x(t) \in V \oplus W^\perp$ (W is the sampling space), we have Eq. (33).

$$e(x(t)) = x(t) - \hat{x}(t) = E_{VW^\perp}(x(t)) \tag{33}$$

where, $E_{VW^\perp}(x(t))$ is the oblique projection onto V along $W^\perp$.

Further, the minimum/maximum limit of the sampling error can be obtained by Eq. (34).

$$\|P_{V^\perp}(x(t))\|^2 \Big/ \|\sin(V,W^\perp)\|^2 \leq \|e(x(t))\|^2 \leq \|P_{V^\perp}(x(t))\|^2 \Big/ \|\cos(V,W^\perp)\|^2 \tag{34}$$

5 Experiment and Analysis

In this section, we consider the case s = 10 , $\tau = \frac{T}{17}$, as shown in Figure 1.

We suppose the sparse signal $x(t)$ is defined in Eq. (35) as follows:

$$x(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t) \tag{35}$$

where, f_1, f_2 are carrier frequency. Both of them are selected from (-610 MHz, 610 MHz) at random, indicated by $x(t) \in$ (-610 MHz, 610 MHz).

According to the analysis in Section 2, the above frequency band can be equally divided into m = 61 segments, where there are at most 4 segments that have nonzero elements and the signal sparsity $k \leq 4$. We suppose a time domain expression of generation function $\varphi_p(t)(0 \leq p \leq m-1)$ in Eq.(36) as follows:

$$\varphi_p(t) = \phi(t)\mathrm{e}^{\mathrm{j}2\pi(p-30)t/T} \tag{36}$$

where, $\phi(t) = \mathrm{sinc}(t/T)$, T is the sampling period, $1/T = 20$ MHz .

Given the nature of the sinc function, its space generation function can constitute Riesz basis and can ensure that signal $x(t)$ is expressed by $r_p[n]$ in $V(\varphi)$.

Therefore, according to Eq. (4), the frequency domain of $x(t)$ can be rewritten in Eq. (37) as follows:

$$X(\omega)=\sum_{p=0}^{m-1}R_p(\omega)\theta(\omega-2\pi(p-30)t/T) \tag{37}$$

where, $\theta(\omega)$ is the Fourier transform of $\phi(t)$

It is easy to know that matrix $\boldsymbol{H}(\omega)$'s Kruskal rank. $kr(\boldsymbol{H}(\omega))=s$. According to the above division method of frequency band and Eq. (4), we can have Eq. (38) as follows:

$$x(t)\in\begin{cases}\left\{\sum_{n\in Z}r_0[n]\varphi_0(t-nT):\{r_0[n]\}\in l_2\right\} & f\in(-610MHz,-590MHz]\\ \left\{\sum_{n\in Z}r_1[n]\varphi_1(t-nT):\{r_1[n]\}\in l_2\right\} & f\in[-590MHz,-570MHz]\\ \left\{\sum_{n\in Z}r_2[n]\varphi_2(t-nT):\{r_2[n]\}\in l_2\right\} & f\in[-570MHz,-550MHz]\\ \vdots & \vdots\\ \left\{\sum_{n\in Z}r_{30}[n]\varphi_{30}(t-nT):\{r_{30}[n]\}\in l_2\right\} & f\in[-10MHz,10MHz]\\ \vdots & \vdots\\ \left\{\sum_{n\in Z}r_{58}[n]\varphi_{58}(t-nT):\{r_{58}[n]\}\in l_2\right\} & f\in[550MHz,570MHz]\\ \left\{\sum_{n\in Z}r_{59}[n]\varphi_{59}(t-nT):\{r_{59}[n]\}\in l_2\right\} & f\in[570MHz,590MHz]\\ \left\{\sum_{n\in Z}r_{60}[n]\varphi_{60}(t-nT):\{r_{60}[n]\}\in l_2\right\} & f\in[590MHz,610MHz)\end{cases} \tag{38}$$

5.1 Analysis on Reconstruction Success Rate

We used the minimum L_1 normal algorithm and the traditional method to verify the reconstruction success rate. The proportion of exact recoveries among 100 times of simulation is reported in Figure 3, where the solid line represents the minimum L_1 normal algorithm and the dotted line represents the traditional method. According to the analysis in Section 2 and 3, when L_1 is used in periodic non-uniform sampling to reconstruct a signal, the signal sparsity is

$k < \frac{1}{2}(\mu^{-1} + 1)$. When the sparsity of the input signal is smaller than or equal to 4, the number of sampling channels of the system is only greater than 8 and complete signal reconstruction can be achieved. However, the traditional method needs at least 60 channels.

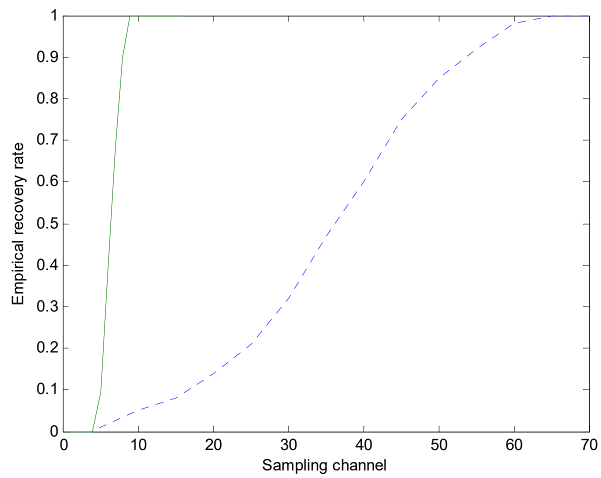

Figure 3 Analysis of reconstruction success rate.

5.2 Comprehensive Test and Analysis of the System

According to the analysis in Section 5.1, we know that the input signal sparsity is 4 at most, so a system with sampling channels $s > 8$ can realize complete reconstruction of the signal. In this section, we set $s = 10$. A 10×61 sensing matrix is formed. The above analysis shows that there are 4 nonzero elements at most in 61 frequency bands. We separately choose a nonzero parameter in the positive and negative frequency bands and R_a and R_b are used to represent nonzero elements in the positive and negative frequency bands.

Figures 4 and 5 separately represent R_a and R_b time domain graphics after interpolation function $\varphi_p(t)$, where the solid line represents the reconstructed signal and the dotted line represents the mirrored part of the reconstructed signal.

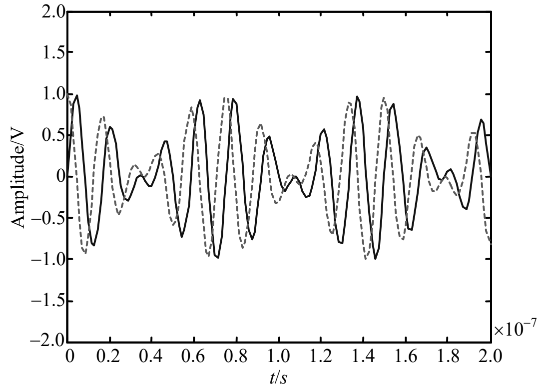

Figure 4 R_a time domain diagram.

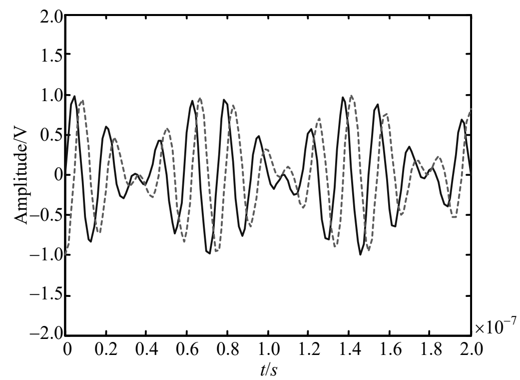

Figure 5 R_b time domain diagram.

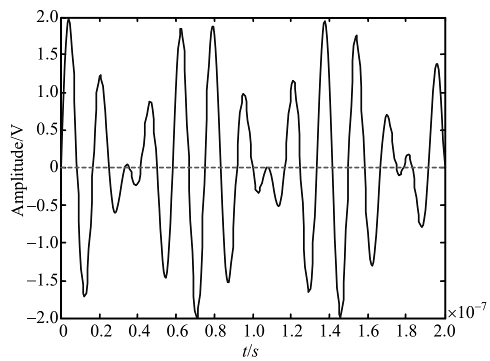

Figure 6 Time domain diagram of the reconstructed signal.

The analysis of Figures 4 and 5 shows time domain graphics of the positive and negative frequency bands corresponding to R_a and R_b. Figure 6 is the overlapping part of bands R_a and R_b, and shows that its amplitude is doubled and the mirrored signal is suppressed. Figure 7 is the time domain diagram of the original signal. Comparison of Figures 7 and 6 shows that the reconstructed signal and the original signal are basically the same.

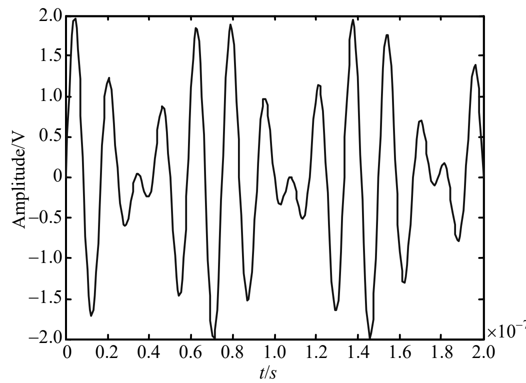

Figure 7 Time domain diagram of the original signal.

6 Conclusion

This paper proposes a periodic non-uniform sampling system that adopts the union of subspaces and uses minimum L1 normal to obtain a unique solution of the undetermined equation. The proposed approach can effectively solve problems in sampling and reconstruction of blind and sparse analog signals. The approach herein, compared with traditional periodic non-uniform sampling system, reduces the number of sampling channels and thus saves system resources. The simulation and test demonstrated that periodic non-uniform sampling can completely reconstruct the original signal far below the Nyquist sampling ratio as long as the number of the system's sampling channels is twice larger than the input RF signal sparsity.

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