On Subspace-ergodic Operators

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Abstract. In this paper, we define subspace-ergodic operators and give examples of these operators. We show that by any given separable infinite dimensional Banach space, subspace-ergodic operators can be constructed. We demonstrate that an invertible operator $T$ is subspace-ergodic if and only if $T^{-1}$ is subspace-ergodic. We prove that the direct sum of two subspace-ergodic operators is subspace-ergodic and if the direct sum of two operators is subspace-ergodic, then each of them is subspace-ergodic. Also, we investigate relations between subspace-ergodic and subspace-mixing operators. For example, we show that if $T$ is subspace-mixing and invertible, then $T^n$ and $T^{-n}$ are subspace-ergodic for any $n \in \mathbb{N}$.

Keywords: ergodic operators; mixing operators; subspace-ergodic operators; subspace-mixing operators.

1 Introduction

Let $X$ be a complex and separable Banach space and $B(X)$ be the set of all bounded linear operators on $X$. Let $\mathbb{N}_0$ be the set of non-negative integers and let $\mathbb{N}$ be the set of natural numbers. We say that $T$ is topologically transitive if for any non-empty open sets $U \subseteq X$ and $V \subseteq X$, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V \neq \emptyset$. One can read more information about these operators in [1-3]. In the statement of [3] and [4], an operator $T \in B(X)$ is called mixing, if for any two non-empty open sets $U \subseteq X$ and $V \subseteq X$, there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V$ non-empty for every $n \geq N$.

Costakis and Sambarino showed in [4] that $T = \lambda B$ is mixing, where $\lambda$ is a scalar with $|\lambda| > 1$ and $B$ is the backward shift on $l^2$. It is interesting that one can construct mixing operators on every infinite-dimensional separable Banach space [5].

Theorem 1.1. If $X$ is any infinite-dimensional separable Banach space, then $X$ supports a mixing operator [5].

Let $T: X \to X$ be an operator. Then for any sets $A \subseteq X$ and $B \subseteq X$, the return set from $A$ to $B$ is defined as:

$$N_T(A, B) = \{n \in \mathbb{N}_0; T^n(A) \cap B \neq \emptyset\}.$$
So, if an operator $T$ is topologically transitive, then $N_T(U, V)$ is non-empty for any open sets $U \subseteq X$ and $V \subseteq X$. As it mentioned in [1], if $T$ is topologically transitive, then $N_T(U, V)$ is infinite. Note that if $T$ is mixing, then $N_T(U, V)$ is cofinite. Remember that we say a set $S$ is cofinite if $\mathbb{N} \setminus S$ is finite.

We call a strictly increasing sequence $(n_k)_k$ of positive integers syndetic if

$$\sup_{k \geq 1} (n_{k+1} - n_k) < \infty.$$ 

We say a subset $A$ of $\mathbb{N}_0$ is syndetic if the strictly increasing sequence of positive integers forming $A$ is syndetic [1]. The complement of syndetic sets does not contain arbitrary long intervals.

We say an operator $T \in B(X)$ is topologically ergodic if for any pair of non-empty open sets $U \subseteq X$ and $V \subseteq X$, $N_T(U, V)$ is syndetic [1]. It is not hard to see that mixing operators are topologically ergodic and topologically ergodic operators are topologically transitive. Grosse-Erdmann and Peris proved in [6] that ergodic operators are weakly-mixing. An operator $T$ is called weakly-mixing if $T \otimes T$ is topologically transitive. Remember that if $X$ and $Y$ are two Banach spaces, then $X \oplus Y = \{(x, y); x \in X, y \in Y\}$ and if $S \in B(X)$ and $T \in B(Y)$, then the operator $S \otimes T: X \oplus Y \to X \oplus Y$ is defined by $(S \otimes T)(x, y) = (Sx, Ty)$.

Madore and Martinez-Avendano introduced subspace-transitive operators in [7]. An operator $T$ is called subspace-transitive with respect to a closed and non-trivial subspace $M$ of $X$ if for any non-empty relatively open sets $U \subseteq M$ and $V \subseteq M$, $T^{-n}(U) \cap V$ contains a non-empty open subset of $M$ for some $n \in \mathbb{N}_0$. They also defined subspace-hypercyclic operators. For more information, see [8-10]. Also, in [11], one can read interesting properties of subspace-supercyclic operators.

Talebi and Moosapoor defined subspace-mixing operators in [12]. Let $M$ be a closed and non-empty subspace of $X$. We say an operator $T \in B(X)$ is $M$-mixing if for any relatively open sets $U \subseteq M$ and $V \subseteq M$ there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $T^n(U) \cap V \neq \emptyset$.

In this paper, we define subspace-ergodic operators and give examples of these operators. We show that by any given separable infinite-dimensional Banach space, we can construct subspace-ergodic operators. We demonstrate that an invertible operator $T$ is subspace-ergodic if and only if $T^{-1}$ is subspace-ergodic. We prove that the direct sum of two subspace-ergodic operators is subspace-ergodic. Also, we prove that if the direct sum of two operators is subspace-ergodic, then each of them is subspace-ergodic. Moreover, we investigate relations between subspace-ergodic operators and subspace-mixing operators. For example, we show that if $T$ is subspace-mixing and invertible, then $T^n$ and $T^{-n}$ are subspace-ergodic for any $n \in \mathbb{N}$. 

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2 Definitions and Some Results

First, we define the return set with respect to a subspace. As usual, when we talk about a subspace, it is considered a closed subspace. Also, the idea of this paper is given from subspace-hypercyclic and subspace-mixing operators. So, in the following definitions we will assume that $M$ is a closed subspace.

**Definition 2.1.** Let $T \in B(X)$ and let $M$ be a closed and non-empty subspace of $X$. For $A \subseteq M$ and $B \subseteq M$, we define the return set from $A$ to $B$ with respect to $M$ as follows:

$$N_T(A, B)_M = \{ n \in \mathbb{N}_0 : T^n(A) \cap B \neq \emptyset \}.$$  

By Definition 2.1, if $T$ is $M$-transitive, then $N_T(U, V)_M$ is non-empty for any non-empty relatively open sets $U \subseteq M$ and $V \subseteq M$. Also, if $T$ is an $M$-mixing operator, then $N_T(U, V)_M$ is cofinite.

Now we define subspace-ergodic operators as follows:

**Definition 2.2.** Let $T \in B(X)$ and let $M$ be a closed and non-empty subspace of $X$. We say that $T$ is $M$-ergodic or subspace-ergodic with respect to $M$ if for any non-empty relatively open sets $U \subseteq M$ and $V \subseteq M$, the set $N_T(U, V)_M$ is syndetic.

By definition, it is clear that any ergodic operator is subspace-ergodic since it is sufficient to consider $M = X$.

**Example 2.3.** Let $T \in B(X)$ be an ergodic operator and let $I$ be the identity operator on $X$. Then $T^p \oplus \alpha I$ is $M$-ergodic with respect to $M := X \oplus \{0\}$ for any $p \in \mathbb{N}$ and for any scalar $\alpha$.

**Proof.** Let $\alpha$ be a scalar. First, we show that $T \oplus \alpha I$ is $M$-ergodic. Let $U_1$ and $V_1$ be non-empty relatively open subsets of $M$. Thus, there exist non-empty open subsets $U$ and $V$ of $X$ such that $U_1 = U \oplus \{0\}$ and $V_1 = V \oplus \{0\}$. By hypothesis, $T$ is an ergodic operator. Thus, the set $\{ n \in \mathbb{N}_0 : T^n(U) \cap (V) \neq \emptyset \}$ is syndetic. Note that we have

$$(T \oplus \alpha I)^n(U \oplus \{0\}) \cap (V \oplus \{0\}) = (T^n(U) \oplus \{0\}) \cap (V \oplus \{0\})$$

$$= (T^n(U) \cap V) \oplus ([0] \cap \{0\}) \quad (1)$$

$$= (T^n(U) \cap V) \oplus \{0\}.$$ 

By Eq. (1), we deduce that $\{ n \in \mathbb{N}_0 : (T \oplus \alpha I)^n(U_1) \cap (V_1) \neq \emptyset \}$ is syndetic too. Therefore, $T \oplus \alpha I$ is $M$-ergodic. Now if $T$ is an ergodic operator, then $T^p$ is ergodic for any $p \in \mathbb{N}$ [1, p.62]. Hence, similar to what was shown, $T^p \oplus \alpha I$ is $M$-ergodic for any $p \in \mathbb{N}$. On the other hand, $\alpha$ is an arbitrary scalar. So, for any scalar $\alpha$, $T^p \oplus \alpha I$ is $M$-ergodic.

Similarly, $\alpha I \oplus T^p$ is $N$-ergodic with respect to $N := \{0\} \oplus X$ for any $p \in \mathbb{N}$ and for any scalar $\alpha$. 

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By the above examples, we can gain more subspace-ergodic operators from known ergodic operators. For instance, note to the following example:

**Example 2.4.** Let $T$ be a weighted shift on $l^2$ given by

$$T(x_1, x_2, x_3, x_4, \ldots) = \left(2x_2, \frac{3}{2}x_3, \frac{4}{3}x_4, \ldots \right).$$

As was proved in [1, Example 2.39], for any two non-empty open subsets $U$ and $V$ of $l^2$, we have $N_T(U, V)$ is syndetic. Hence, $T$ is an ergodic operator and by Example 2.3, $T^p \oplus \alpha I$ is $M$-ergodic with respect to $M = l^2 \oplus \{0\}$ for any $p \in \mathbb{N}$ and for any scalar $\alpha$.

As mentioned before, mixing operators are ergodic. Also, it is not hard to see that if $T$ is a mixing operator, then $T^p$ is mixing for any $p \in \mathbb{N}$. So, by Example 2.3, we can create the following example:

**Example 2.5.** Let $T \in B(X)$ be a mixing operator and let $I$ be the identity operator on $X$. Then $T^p \oplus \alpha I$ is $M$-ergodic with respect to $M = X \oplus \{0\}$ for any $p \in \mathbb{N}$ and for any scalar $\alpha$. For example, let $T = \lambda B$ be the Rolewicz’s operator on $l^2$, where $B$ is the backward shift on $l^2$ and $\lambda$ be a scalar with $|\lambda| > 1$. As it said in the introduction, $T = \lambda B$ is mixing. So, for any scalar $\alpha$, the operator $(\lambda B)^p \oplus \alpha I$ is $M$-ergodic with respect to $M = l^2 \oplus \{0\}$. Also, it is easy to see that $\alpha I \oplus (\lambda B)^p$ is $N$-ergodic with respect to $N = \{0\} \oplus l^2$.

**Corollary 2.6.** Let $X$ be a separable and infinite-dimensional Banach space. Then there exists a subspace-ergodic operator on $X \oplus X$.

**Proof.** By Theorem 1.1, there exists a mixing operator $T$ on $X$. Then $T \oplus I$ is the desired operator by Example 2.5.

Now like subspace-hypercyclicity and subspace-supercyclicity, some questions arise for subspace-ergodicity as follows:

**Question 1.** Let $T$ be an invertible $M$-ergodic operator. Can we conclude that $T^{-1}$ is also $M$-ergodic?

**Question 2.** Let $T$ be an $M$-ergodic operator. Is $T^n$ is $M$-ergodic for any $n \in \mathbb{N}$?

**Question 3.** Let $\lambda$ be a scalar with $|\lambda| = 1$. Does the $M$-ergodicity of $T$ imply $M$-ergodicity of $\lambda T$?

**Question 4.** Let $T$ be an ergodic operator on $X$. Is there a closed and non-trivial subspace $M$ of $X$ such that $T$ is $M$-ergodic?

**Question 5.** Is the direct sum of two subspace-ergodic operators also subspace-ergodic?

In this section, we answer Question 1 affirmatively. Also, we partially answer Question 2 and Question 3. In the next section, we answer Question 5.

In the next theorem, we show that the answer to Question 1 is positive.
**Theorem 2.7.** Let $T \in B(X)$ be an invertible operator and let $M$ be a closed and non-empty subspace of $X$. Then $T$ is $M$-ergodic if and only if $T^{-1}$ is $M$-ergodic.

**Proof.** Let $U \subseteq M$ and $V \subseteq M$ be two non-empty relatively open sets. Let $n \in N_T(U,V)_M$. So, $T^n(U) \cap V \neq \emptyset$. $T$ is invertible and

$$T^{-n}(T^n(U) \cap V) \neq \emptyset.$$  

Hence, $T^{-n}(V) \cap U \neq \emptyset$. This means that $n \in N_{T^{-1}}(V,U)_M$. So,

$$N_T(U,V)_M \subseteq N_{T^{-1}}(V,U)_M.$$  

Similarly, we have

$$N_{T^{-1}}(V,U)_M \subseteq N_T(U,V)_M.$$  

Therefore, $N_T(U,V)_M = N_{T^{-1}}(V,U)_M$.

Hence, $N_T(U,V)_M$ is syndetic if and only if $N_{T^{-1}}(V,U)_M$ is syndetic. Therefore, $T$ is $M$-ergodic if and only if $T^{-1}$ is $M$-ergodic.

Now it is natural to note the powers of a subspace-ergodic operator. In the next theorem, we prove that if $T^n$ is subspace-ergodic for some $n \in \mathbb{N}$, then $T$ is subspace-ergodic too.

**Theorem 2.8.** Let $T \in B(X)$ and let $M$ be a closed and non-empty subspace of $X$. If $T^n$ is $M$-ergodic for some $n \in \mathbb{N}$, then $T$ is also $M$-ergodic.

**Proof.** Let $U \subseteq M$ and $V \subseteq M$ be two non-empty relatively open sets. By hypothesis, $T^n$ is $M$-ergodic. So, $N_{T^n}(U,V)_M$ is syndetic. But if $(T^n)^k(U) \cap V \neq \emptyset$, then $T^{nk}(U) \cap V \neq \emptyset$. Hence, if $k \in N_{T^n}(U,V)_M$, then $k \in N_T(U,V)_M$. So,

$$N_{T^n}(U,V)_M \subseteq N_T(U,V)_M.$$  

Let $(m_k)_k$ be the elements of $N_{T^n}(U,V)_M$ and let $(t_k)_k$ be the elements of $N_T(U,V)_M$. Hence,

$$\sup_{1 \leq k < \infty} (m_{k+1} - m_k) \geq \sup_{1 \leq k < \infty} (t_{k+1} - t_k).$$  

By definition, $\sup_{1 \leq k < \infty} (m_{k+1} - m_k) < \infty$ and so $\sup_{1 \leq k < \infty} (t_{k+1} - t_k)$ is less than infinity too. That means $N_T(U,V)_M$ is syndetic and hence, $T$ is an $M$-ergodic operator.

In the next theorem, we show that if $T$ is an $M$-mixing operator, then $T^n$ is $M$-ergodic for any $n \in \mathbb{N}$ and therefore, we have a partial answer to Question 2.

**Theorem 2.9.** Let $T \in B(X)$ and let $M$ be a closed and non-empty subspace of $X$. If $T$ is an $M$-mixing operator, then $T^n$ is $M$-ergodic operator for any $n \in \mathbb{N}$.

**Proof.** It is immediately obtained by Definition 2.2 that $T$ is $M$-ergodic. Now, let $n > 1$ be an arbitrary natural number. First, we show that $T^n$ is $M$-mixing.
Suppose that $U \subseteq M$ and $V \subseteq M$ are non-empty relatively open sets. By hypothesis, there exists $N \in \mathbb{N}$ such that $T^k(U) \cap V$ is non-empty for any $k \geq N$. On the other hand, for every $k \in \mathbb{N}$ we have $kn \geq k$. So, $T^{kn}(U) \cap V \neq \emptyset$ for any $k \geq N$. Hence,

$$(T^n)^k(U) \cap V \neq \emptyset, \quad \text{for any } k \geq N.$$ 

Therefore, $T^n$ is M-mixing and hence, $T^n$ is $M$-ergodic. But $n > 1$ is an arbitrary natural number. So, $T^n$ is $M$-ergodic for any $n \in \mathbb{N}$.

**Lemma 2.10.** Let $T \in B(X)$ be an invertible operator and let $M$ be a closed and non-empty subspace of $X$. Then $T$ is $M$-mixing if and only if $T^{-1}$ is $M$-mixing.

**Proof.** Let $U \subseteq M$ and $V \subseteq M$ be two non-empty relatively open sets. By hypothesis, $T$ is an $M$-mixing operator. So, $N_T(U,V)_M$ is cofinite. But

$$N_T(U,V)_M = N_{T^{-1}}(V,U)_M.$$ 

Hence, $N_T(U,V)_M$ is cofinite if and only if $N_{T^{-1}}(V,U)_M$ is cofinite. This means that $T$ is $M$-mixing if and only if $T^{-1}$ is $M$-mixing.

The next corollary is a direct result of Theorem 2.9 and Lemma 2.10.

**Corollary 2.11.** Let $T$ be an invertible and $M$-mixing operator. Then for any $n \in \mathbb{N}$, $T^n$ and $T^{-n}$ are $M$-ergodic.

Now we mention a theorem from [13] and by this we have a partial answer to Question 3.

**Theorem 2.12.** Let $T \in B(X)$. Then $T$ is $M$-mixing with respect to a non-empty and closed subspace $M$ of $X$ if and only if for any non-empty relatively open set $U \subseteq M$ and any 0-neighborhood $W$ in $M$ there exists a positive integer $N$ such that for any $n \geq N$, $T^n(U) \cap W \neq \emptyset$ and $T^n(W) \cap U \neq \emptyset$ [13].

In other words, $T$ is $M$-mixing if and only if $N_T(U,W)_M$ and $N_T(W,U)_M$ are cofinite, for any non-empty relatively open set $U \subseteq M$ and any 0-neighborhood $W$ in $M$.

**Theorem 2.13.** Let $T \in B(X)$ be an $M$-mixing operator. Let $\lambda$ be a scalar with $|\lambda| = 1$. Then $\lambda T$ is $M$-ergodic.

**Proof.** Let $T$ be a subspace-mixing operator with respect to a closed and non-empty subspace $M$ of $X$. We show that $\lambda T$ is an $M$-mixing operator and hence it is $M$-ergodic. Let $U \subseteq M$ be a relatively open set and let $W$ be a 0-neighborhood in $M$. By hypothesis, $N_T(U,W)_M$ and $N_T(W,U)_M$ are cofinite. We can find a balanced set $W_1$, a neighborhood of zero in $M$ such that $W_1 \subseteq W$.

Again by hypothesis, $N_T(U,W_1)_M$ is cofinite. So, there exists $N \in \mathbb{N}$ such that

$$T^n(U) \cap W_1 \neq \emptyset \quad (n \geq N).$$

(2)
Let $n \geq N$ be an arbitrary natural number. By Eq. (2), $T^n(U) \cap W_1$ is non-empty. So, there exists $x \in U$ such that $T^n x \in W_1$. Since $|\lambda^n| = 1$ and $W_1$ is a balanced set,

$$\lambda^n T^n x \in \lambda^n W_1 \subseteq W_1.$$ 

Hence, $\lambda^n T^n x \in W_1$ and therefore, $(\lambda^n T^n)(U) \cap W_1 \neq \emptyset$. Since $W_1 \subseteq W$,

$$(\lambda^n T^n)(U) \cap W \neq \emptyset.$$ 

Therefore, $N_{2T}(U, W)_M$ is cofinite and similarly, $N_{2T}(W, U)_M$ is cofinite. So, by Theorem 2.12, $\lambda T$ is $M$-mixing which completes the proof.

3 On the Direct Sum of Two Subspace-ergodic Operators

First, we show that the direct sum of two subspace-ergodic operators is subspace-ergodic. In fact, we show that the answer to Question 5 is positive. In this section, $M$ and $N$ always indicate closed and non-zero subspaces of $X$ and $Y$ respectively.

**Theorem 3.1.** Let $S \in B(X)$ be an $M$-ergodic operator and let $T \in B(Y)$ be an $N$-ergodic operator. Then, $S \oplus T$ is $M \oplus \{0\}$-ergodic and $\{0\} \oplus N$-ergodic.

Especially, $T \oplus T$ is $N \oplus \{0\}$-ergodic and $\{0\} \oplus N$-ergodic.

**Proof.** Let $U \subseteq M$ and $V \subseteq M$ be non-empty relatively open sets. Then,

$$(S \oplus T)^n(U \oplus \{0\}) \cap (V \oplus \{0\}) = (S^n \oplus T^n)(U \oplus \{0\}) \cap (V \oplus \{0\})$$

$$= (S^n(U) \cap V) \oplus (T^n(\{0\}) \cap \{0\})$$

$$= (S^n(U) \cap V) \oplus \{0\}.$$ 

So,

$$N_{S \oplus T}(U \oplus \{0\}, V \oplus \{0\})_{M \oplus \{0\}} = N_S(U, V)_M. \quad (3)$$

By hypothesis, $N_S(U, V)_M$ is syndetic. So, by Eq. (3), $N_{S \oplus T}(U \oplus \{0\}, V \oplus \{0\})_{M \oplus \{0\}}$ is syndetic. This means that $S \oplus T$ is $M \oplus \{0\}$-ergodic. Similarly, $S \oplus T$ is $\{0\} \oplus N$-ergodic.

By Theorem 3.1 and Theorem 2.7 we can conclude the following corollary:

**Corollary 3.2.** Let $S \in B(X)$ be an $M$-ergodic operator and let $T \in B(Y)$ be an $N$-ergodic operator. If $S$ and $T$ are invertible operators, then $(S \oplus T)^{-1}$ is $M \oplus \{0\}$-ergodic and $\{0\} \oplus N$-ergodic.

If $S \in B(X)$ and $T \in B(Y)$ are topologically ergodic operators, then $S^p \oplus T^q$ is topologically ergodic on $X \oplus Y$ for any $p, q \in \mathbb{N}$ [1, p. 173].

Now the question arises if this is also true for subspace-ergodic operators? We partially answer this question in the next theorem.
Theorem 3.3. Let $S \in B(X)$ be an $M$-ergodic operator and let $T \in B(Y)$ be an $N$-ergodic operator. Then

(i) if $S$ is an $M$-mixing operator, then $S^p \oplus T$ is $M \oplus N$-ergodic for any $p \in \mathbb{N}$,

(ii) if $T$ is an $N$-mixing operator, then $S \oplus T^q$ is $M \oplus N$-ergodic for any $q \in \mathbb{N}$.

Proof. We prove part (i) and the proof of part (ii) is similar. First, we prove that $S \oplus T$ is a $M \oplus N$-ergodic. Let $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq N$ be non-empty relatively open sets. By hypothesis, $S$ is an $M$-mixing operator. So, there exists a natural number $p$ such that for any $n \geq p$,

$$S^n(U_1) \cap (U_2) \neq \emptyset.$$ 

So,

$$\{n \in \mathbb{N}; n \geq p\} \subseteq N_S(U_1, U_2)_M.$$ 

By hypothesis, $T$ is an $N$-ergodic operator and so $N_T(V_1, V_2)_N$ is syndetic. On the other hand,

$$(S \oplus T)^n(U_1 \oplus V_1) \cap (U_2 \oplus V_2) = (S^n(U_1) \cap (U_2)) \oplus (T^n(V_1) \cap (V_2)).$$

Hence,

$$N_{S \oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M \oplus N} = N_S(U_1, U_2)_M \cap N_T(V_1, V_2)_N \\
\supseteq \{n \in \mathbb{N}; n \geq p\} \cap N_T(V_1, V_2)_N \\
= \{n \in \mathbb{N}; n \geq p \text{ and } n \in N_T(V_1, V_2)_N\}.$$ 

Since $N_T(V_1, V_2)_N$ is syndetic, we can deduce that $N_{S \oplus T}((U_1 \oplus V_1), (U_2 \oplus V_2))_{M \oplus N}$ is syndetic. Now note that if $S$ is $M$-mixing, then $S^p$ is $M$-mixing for any $p \in \mathbb{N}$ which completes the proof.

Bamerni and Kilicman showed in [14] that the direct sum of two subspace-mixing operators is also subspace-mixing. Now we extend their statement as follows:

Theorem 3.4. Let $S \in B(X)$ be an $M$-mixing operator and let $T \in B(Y)$ be an $N$-mixing operator. Then, $(S \oplus T)^n$ is $M \oplus N$-mixing for any $n \in \mathbb{N}$.

Moreover, if $S$ and $T$ are invertible, then $(S \oplus T)^{-n}$ is also $M \oplus N$-mixing for any $n \in \mathbb{N}$. Especially, $(S \oplus T)^n$ and $(S \oplus T)^{-n}$ are $M \oplus N$-ergodic for any $n \in \mathbb{N}$.

Proof. Let $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq N$ be non-empty relatively open sets. $S$ is an $M$-mixing operator. So, there exists a natural number $N_1$ such that for any $n \geq N_1$,

$$S^n(U_1) \cap (U_2) \neq \emptyset.$$  

(4)
On the other hand, $T$ is an $N$-mixing operator. So, there exists a natural number $N_2$ such that for any $n \geq N_2$,
\[
T^n(V_1) \cap (V_2) \neq \emptyset. \tag{5}
\]
Let $p = \max\{N_1, N_2\}$. So, by Eq. (4) and Eq. (5), for any $n \geq p$ we have,
\[
S^n(U_1) \cap (U_2) \neq \emptyset \quad \text{and} \quad T^n(V_1) \cap (V_2) \neq \emptyset.
\]
Hence, for any $n \geq p$,
\[
(S \oplus T)^n(U_1 \oplus V_1) \cap (U_2 \oplus V_2) = (S^n(U_1) \cap U_2) \oplus (T^n(V_1) \cap V_2) \neq \emptyset.
\]
So, $N_{S \oplus T}(U_1 \oplus V_1, (U_2 \oplus V_2))_{M \oplus N}$ is cofinite and hence, $(S \oplus T)^n$ is $M \oplus N$-mixing.

The proof of the rest of the theorem is an easy consequence of Lemma 2.10 and Theorem 2.9.

Finally, we prove that subspace-ergodicity of the direct sum of two operators, indicates subspace-ergodicity of each of them.

**Theorem 3.5.** Let $S \in B(X)$ and $T \in B(Y)$. If $S \oplus T$ is an $M \oplus N$-ergodic operator, then $S$ is an $M$-ergodic operator and $T$ is an $N$-ergodic operator.

Especially, if $T \oplus T$ is $N \oplus N$ ergodic, then $T$ is $N$-ergodic.

**Proof.** Let $U_1 \subseteq M$ and $U_2 \subseteq M$ be non-empty relatively open sets. We prove that $N_S(U_1, U_2)_M$ is syndetic.

Suppose that $V_1 \subseteq N$ and $V_2 \subseteq N$ be non-empty relatively open sets. By hypothesis, $S \oplus T$ is $M \oplus N$-ergodic. So, $N_{S \oplus T}(U_1 \oplus V_1, (U_2 \oplus V_2))_{M \oplus N}$ is syndetic. Let $n \in N_{S \oplus T}(U_1 \oplus V_1, (U_2 \oplus V_2))_{M \oplus N}$. So,
\[
(S \oplus T)^n(U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \emptyset.
\]
And hence,
\[
(S^n(U_1) \cap (U_2)) \oplus (T^n(V_1) \cap (V_2)) \neq \emptyset.
\]
Therefore, $S^n(U_1) \cap U_2$ must be non-empty and hence, $n \in N_S(U_1, U_2)_M$. This means
\[
N_S(U_1 \oplus V_1, (U_2 \oplus V_2))_{M \oplus N} \subseteq N_S(U_1, U_2)_M.
\]
Since $N_{S \oplus T}(U_1 \oplus V_1, (U_2 \oplus V_2))_{M \oplus N}$ is syndetic, $N_S(U_1, U_2)_M$ is syndetic. Similarly, $T$ is also $N$-ergodic.

**References**

On Subspace-ergodic Operators


