Continuous-Like Linear Operators on Bilinear Spaces

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Abstract. This paper introduces continuous-like linear operators on bilinear spaces as a generalization of continuous linear operators on Hilbert spaces. It is well known that the existence of the adjoint of a linear operator on a Hilbert space is equivalent to the operator being continuous. In this paper, this result is extended to the class of linear operators on bilinear spaces. It is shown that the existence of the adjoint of a linear operator on a bilinear space is guaranteed if and only if the operator is continuous-like.

Keywords: *adjoint operators; bilinear spaces; closed subspaces; continuous; continuous-like; Hilbert spaces.*

1 Introduction

Let F be a field and V be a vector space over the field F. A bilinear form on V is a map that assigns any pair elements of V, (v,w) to an element of F, denoted by $[v,w]$, which is linear with respect to each variable, v and w. In this article, a bilinear space is a vector space equipped with a non-degenerate bilinear form. An example of bilinear spaces that are massively utilized in the study and development of algebraic system theory are the spaces of truncated Laurent series (see for example [1,2]).

A subclass of the class of bilinear spaces that is greatly developed is the subclass of inner product spaces over the real field R. Considering the number of results in the area of inner product spaces, it is of interest to be able to generalize those results to bilinear spaces. In previous studies, we have been able to generalize the Riesz representation theorem to the class of linear functionals of bilinear spaces [3]. This triggered the question whether the result on inner product spaces regarding adjoint operators on inner product spaces could also be extended to the class of linear operators on bilinear spaces, which was the goal of this study.

In the class of finite dimensional spaces, the existence of the adjoint of any linear operator is guaranteed without any additional condition. In contrast, for the underlying spaces being infinite dimensional, particularly on infinite

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dimensional Hilbert spaces, the existence of the adjoint of any linear operator is guaranteed if and only if the linear operator is continuous or if and only if it is bounded. Meanwhile, the concepts of norm and topology are not relevant to bilinear spaces. How can this be generalized to bilinear spaces?

This article identifies a necessary and sufficient condition in terms of closed subspaces for a linear operator on a Hilbert space being continuous. The finding leads to the introduction of the continuous-like concept for linear operators on a bilinear space as the generalization of the continuous notion of linear operators on a Hilbert space. Finally, it is shown that a necessary and sufficient condition for the existence of the adjoint of a linear operator on a bilinear space is that the linear operator is continuous-like.

2 Continuous-like Operators

In this section we introduce the continuous-like notion of a linear operator on a bilinear space as a generalization of the continuous notion of linear operators on a Hilbert space. For that, we will first investigate an equivalent condition for the continuity of any linear operator on a Hilbert space that enables us to do the generalization. From now on, what we mean with an inner product space or a Hilbert space is always over the real field R.

Let H be a Hilbert space. We have the norm and topology on H induced by the inner product on H. A subset $S \subseteq H$ is called closed if its complement $H \backslash S$ is an open set. Further, a closed subspace can be identified using its double orthogonal complement. Let S be a subspace of H. The orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vectors in H that are orthogonal to S,

$$S^{\perp} = \{v \in H : \langle v, s \rangle = 0 \text{ for all } s \in S\}$$

This set forms a subspace. The double orthogonal complement of S, denoted by $S^{\perp\perp}$, contains the original set S. It is a well-known fact that the closeness property of the subspace S is equivalent to $S^{\perp\perp} = S$ [4, p. 335]. This fact inspired the formulation of closed subspaces in bilinear spaces (see e.g. [1,2]).

Let V be a bilinear space over a field F with the associated bilinear form denoted by [-,-]. Two vectors $v, w \in V$ are called orthogonal, written as $v \perp w$, if $[v, w] = 0$ holds. Two subsets of V, say X and Y, are called orthogonal, denoted by $X \perp Y$, if any element of X is orthogonal to any element of Y. The orthogonal complement of a subset $S \in V$ is the set of all vectors that are orthogonal to the set S, denoted by $S^{\perp}$. A subspace $S \in V$ is called closed if it is equal to its double complement, that is if $S^{\perp\perp} = S$ holds.

Continuity is a fundamental concepts that provides significant advantages in the development of various areas of mathematics. In the field of inner product spaces, the continuity property has led to the development of the class of bounded operators on Hilbert spaces that forms a subalgebra. The results related to bounded operators on Hilbert spaces are huge, including their applications to various areas of mathematics. One of them is the connection among three notions: continuity, bounded operators, and adjoint operators. Let *f* be a linear operator on a Hilbert space *H*. A linear operator on *H* is called the adjoint *f*, denoted by f^*, if the following condition holds:

$$for\ every\quad v,w\ \in H\ \langle f(v),w\rangle = \langle v,f^*(w)\rangle \tag{1}$$

Theorem 2.1[4] Let *f* be a linear operator on a Hilbert space *H*. The following statements are equivalent.

The linear operator *f* is continuous.

The linear operator *f* is bounded.

There exists $f*$ a unique linear operator on *H* such that the condition in Eq. (1) holds.

Bilinear spaces can be considered as a generalization of inner product spaces. From this perspective, we would like to obtain a result which can be considered as an extension of Theorem 2.1 to the class of bilinear spaces. The question that immediately rises is: What do we mean with a linear operator on a bilinear space being continuous? Investigation to answer that question leads to the following equivalent condition for continuity of a linear operator on a Hilbert space associated with closed subspaces.

Theorem 2.2 Let f be a linear operator on a Hilbert space *H*. Then f is continuous if and only if for every $S\ \subseteq H$, closed subspaces of *H*, the subspace $f^{-1}(S) = \{v\ \in H : f(v)\ \in S\}$ is also closed.

Proof. It is obvious that if *f* is continuous then for any *S* closed subspace of *H*, the subspace $f^{-1}(S)$ is also closed. Hence, we only need to prove the sufficient condition for *f* being continuous. Let the following condition hold:

$f^{-1}(S)$ is closed for any closed subspace S.

We will show f being continuous. Referring to Theorem 2.1 it is enough to show that the adjoint operator of f is existence. That is we will show that for every $w\ \in H$ there exists a unique vector $z\ \in H$ such that for any $v\ \in H$ the following equation holds:

$$\langle f(v),w\rangle = \langle v,\mathrm{z}\rangle \tag{2}$$

Let $w \in H$. If $\langle f(v), w\rangle = 0$ for all $v \in H$ we have the unique vector $z = 0$. Suppose the functional $\lambda_w(v) = \langle f(v), w\rangle$ for all $v \in H$ is not the zero functional. It is linear with the kernel

$$\begin{aligned} Ker(\lambda_w) &= \{ v \in H : \langle f(v), w\rangle = 0\} \\ &= \{ v \in H : \langle f(v), y\rangle = 0 \; for\ all\ \ y \in Span\{w\}\} \\ &= \{ v \in H : f(v) \in Span\{w\}^{\perp}\} \\ &= f^{-1}(Span\{w\}^{\perp}) \end{aligned}$$

Since $Span\{w\}^{\perp}$ is a closed subspace, we have $Ker(\lambda_w)$ is closed. As a result $Ker(\lambda_w)^{\perp} = Span\{u\}$ for some $u \in H$ with $u \neq 0$. Let $a = \lambda_w(u) = \langle f(u), w\rangle \neq 0$, $b=\langle u, u\rangle \neq 0$ and denote $z = \left(\frac{a}{b}\right) u$. Then, for every $v \in H$ we have $v = v_1 + \alpha u$ for some $v_1 \in Ker(\lambda_w)$ and $\alpha \in R$. Then Eq. (2) holds since

$$\langle f(v), w\rangle = \langle f(v_1), w\rangle + \langle f(\alpha u), w\rangle = \alpha\langle f(u), w\rangle = \alpha a$$

$$\langle v, z\rangle = \langle v_1, z\rangle + \langle \alpha u, z\rangle = \alpha\langle u, \left(\frac{a}{b}\right) u\rangle = \alpha \left(\frac{a}{b}\right)\langle u, u\rangle = \alpha a.$$

Thus, we have obtained that there exists a $z \in H$ such that for all $v \in H$ Eq. (2) holds. Then, applying the property that the inner product is nondegenerate, we obtain that z that satisfies Eq. (2) for all $v \in H$ is unique.

The sufficient and necessary condition in Theorem 2.2 is weaker than the original condition for continuity. Particularly it only concerns the collection of all closed subspaces compared to the collection of all closed subsets.

It is understood that Theorem 2.2 is a small result and the proof is also elementary and applies a number of well-known properties. Nonetheless, the proposed novelty of Theorem 2.2 is that it gives a new perspective on the continuity in inner product spaces, providing a proper means to generalize the concept of continuity in inner product spaces to bilinear spaces, as stated in the following definition. Further, considering the impact of continuity in the development of mathematics, it is natural to expect that the following continuous-like concept will influence the development of the study of linear operators in bilinear spaces.

Definition 2.3 Let V and W be two bilinear spaces over a field F. A linear map f from V to W is called continuous-like if for any closed subspace $S \subseteq W$, the subspace $f^{-1}(S) = \{ v \in V \mid f(v) \in S \}$ is also closed.

Remarks:

1. It is obvious that any continuous linear mapping in Hilbert spaces is continuous-like.

2. Let V be a bilinear space over a field F. It is easy to show that the zero operator and the identity operator in V are continuous-like.
3. Let V and W be two finitely dimensional bilinear spaces over a field F. Then any linear mapping from V into W is continuous-like since any subspace on a finitely dimensional bilinear space is closed.
4. As a result of the definition, the closeness of the kernel is a necessary condition for a continuous-like linear mapping in a bilinear space. That means a linear mapping in bilinear spaces is not continuous-like if its kernel is not closed.

Example 2.4 Let F be a field, n be positive integer, and F^n be the vector space over F consisting of n-column vectors with components in F. On F^n we can define a non-degenerate bilinear form, defined as $[u, v] = v^t u$ for all $u, v \in F^n$. A truncated Laurent series space $F^n((z^{-1}))$ is the vector space over F consists of all truncated Laurent series of the form $f = \sum_{j=k}^{\infty} f_j z^{-j}$ with $f_j \in F^n$ and for some k integer. The space $F^n((z^{-1}))$ can be formed as a bilinear space with bilinear form is defined as the following:

$$[f, g] = \sum_{j=-\infty}^{\infty} [f_{-j-1}, g_j] \quad for\ all\ f, g \in F^n((z^{-1})).$$

Note that the space $F((z^{-1}))$ actually can be considered as a field with the product operation as an extension of the product between two polynomials. From this perspective we can consider the space $F^n((z^{-1}))$ equal to $F((z^{-1}))^n$ and it is an n-dimensional vector space over the field $F((z^{-1}))$. Further, we obtain $[f, g] = (g^t f)_{-1}$ for all $f, g \in F((z^{-1}))^n$.

1. Let A be an $n \times n$ matrix with components in $F((z^{-1}))$. The linear operator on $F((z^{-1}))^n$, say ϕ, that is defined as $\phi(f) = Af$ for any $f \in F((z^{-1}))^n$, is a F-linear mapping. ϕ is continuous-like since it is also $F((z^{-1}))$-linear mapping.
2. Now we will construct a linear operator on $F^n((z^{-1}))$ that is not continuous-like. Let $f = \sum_{j=k}^{\infty} f_j z^{-j} \in F^n((z^{-1}))$ for some integer k. The element f is said to have finite support if the number of non-zero coefficient $f_j \neq 0$ is finite. Let U be the set of all finite support elements of $F^n((z^{-1}))$. It is easy to show that U is a proper subspace of $F^n((z^{-1}))$. Hence there exists a nonzero subspace, say V, such that $F^n((z^{-1})) = U \oplus V$. Let us define T the projection operator on the subspace V along the subspace U, that is $T(u + v) = v\ for\ all\ u \in U\ and\ v \in V$.

We will show that the operator projection T is not continuous-like. Consider that the set $S = \{0\}$ is a closed subspace. If we can show that $T^{-1}(S)$ is not closed, then T is not continuous-like. The set $T^{-1}(S)$ nonetheless is Ker(T) and according to the definition of T we have $T^{-1}(S) = U$. Let $T = \{ e_1, \cdots, e_n \} \subseteq F^n$ be the standard basis of F^n. We have that $e_i z^{-j} \in U$ for all $i = 1, \cdots, n$ and $j \in Z$, where Z is the set of integers. Consider that for any $f \in T^{-1}(S)^{\perp}$ we have $[f, e_i z^{-j}] = 0$ for all $i = 1, \cdots, n$ and $j \in Z$. As a result, $f = 0$. Therefore, we obtain $T^{-1}(S)^{\perp} = 0$ which implies $T^{-1}(S)^{\perp\perp} = F^n((z^{-1}))$. Meanwhile $T^{-1}(S) = U \neq F^n((z^{-1}))$ since it is a proper subspace. Thus $T^{-1}(S)$ is not closed. As a consequence, T is not continuous-like.

3 Adjoint Operators

In this section we will investigate a characterization of an adjoint operator on a bilinear space in terms of continuous-like as an extension of Theorem 2.1. For that, first we will review the term adjoint mapping/operator in a bilinear space.

Let $f: V \to W$ be a linear mapping from bilinear space V to bilinear space W over a field F with each bilinear form is $[-,-]_V$ and $[-,-]_W$ respectively. A mapping from W to V is called an adjoint of f, denoted by $f^*: W \to V$, if the following condition holds: for every $v \in V$ and $w \in W$

$$[f(v), w]_W = [v, f^*(w)]_V \tag{3}$$

By using the nondegenerate property of the corresponding bilinear forms, it can be shown that if an adjoint mapping of a linear mapping exists, then it is linear and unique. Therefore, a more crucial problem is: When will an adjoint mapping of a linear mapping f be existing?

The following theorem is one of the main results of this article and it can be thought of as a generalization of Theorem 2.1 to the class of linear maps on bilinear spaces.

Theorem 3.1 Let V and W be two bilinear spaces, and f be a linear mapping from V to W. Then the existence and uniqueness of the adjoint of f are guaranteed if and only if the linear mapping f is continuous-like.

Proof. Let there exist a unique linear mapping, f^*, the adjoint of f, such that Eq. (3) holds for every $v, w \in W$. We will show that f is continuous-like. Let $S \subseteq W$ be a closed subspace, i.e. $S^{\perp\perp} = S$. We obtain

$$f^{-1}(S) = \{ v \in V \mid f(v) \in S\}$$

$$= \{ v \in V \mid f(v) \; S^{\perp\perp} \}$$
$$= \{ v \in V \mid [f(v), y]_W = 0 \; for \; all \; y \in S^{\perp} \}$$
$$= \{ v \in V \mid [v, f^*(y)]_V = 0 \; for \; all \; y \in S^{\perp} \}$$
$$= f^*(S^{\perp})^{\perp}$$

Since the orthogonal complement of any subspace is closed, we conclude that $f^{-1}(S)$ is closed. Thus f is continuous-like.

Conversely, suppose f is continuous-like. Let $w \in W$. We will identify a unique vector as the map of w by f^*, denoted by $f^*(w)$, such that Eq. (3) holds for all $v \in V$. If $w \in f(V)^{\perp}$ we define $f^*(w) = 0$ then Eq. (3) holds for all $v \in V$. Suppose $w \notin f(V)^{\perp}$. In this case the mapping

$$\lambda_w(v) = [f(v), w]_W \quad for \; all \quad v \in V$$

is a non-zero linear functional on V with $Ker(\lambda_w) = f^{-1}(Span\{w\}^{\perp})$, hence it is closed. According to Riesz representation theorem [3, Theorem 2.3 p. 35] there exists a unique vector $z \in V$ such that the following equation holds:

$$\lambda_w(v) = [f(v), z]_W \quad for \; all \quad v \in V.$$

Define $f^*(w) = z$. Then Eq. (3) holds for all $v \in V$. As a result we obtain a mapping f^* that satisfies Eq. (3) for all $v \in V$ and $w \in W$. It is routine to show that such mapping is linear and unique.

Example 3.2 We continue our discussion of two linear operators in Example 2.4. The first linear operator is ϕ, which is defined as $\phi(u) = Au \; for \; all \; u \in F((z^{-1}))^n$ for some A, an $n \times n$ matrix with components in $F\big((z^{-1})\big)$. We already obtained that ϕ is continuous-like. Hence, according to Theorem 3.1 the adjoint of ϕ, denoted by ϕ^*, is guaranteed and is unique. To obtain the mapping of any element by ϕ^*, let us consider the relation between ϕ and ϕ^*. For any $u, v \in F((z^{-1}))^n$ we obtain

$$[u, \phi^*(v)] = [\phi(u), v] = [Au, v] = (v^t(Au))_{-1} = ((v^tA)u)_{-1}$$
$$= ((A^tv)^tu)_{-1} = [u, A^tv].$$

Since the above equation holds for all u in $F((z^{-1}))^n$, then according to the nondegenerate property of the bilinear of $F((z^{-1}))^n$we obtain $\phi^*(v) = A^tv$ for all $v \in F((z^{-1}))^n$. The second operator is the projection operator T. We already obtained that T is not continuous-like. Hence, according to Theorem 3.1 the adjoint of T does not exist.

Finally, we closed this section with the following corollary.

Corollary 3.3 Let V be a bilinear space over a field F and
$B(V) = \{ f\colon V \to V \,|f \text{ linear and continuous} - \text{like} \}$.
Then B (V) is a subalgebra of the algebra consisting of all linear operators on V.

Proof. Let V be a bilinear space over a field F. According to Theorem 3.1 the set $B(V)$ defined in the corollary is equal to the set of all linear operators on V whose adjoints exist. Since for any $f, g \in B(V)$ and $a \in F,$ by using the nondegenerate property of the bilinear form we can show that

$$(af)^* = a f^*, \quad (f+g)^* = f^* + g^*, \quad \text{and} \quad (fg)^* = g^* f^*$$

We obtain that $B(V)$ is closed under addition, product, and scalar product. Hence, $B(V)$ is a subalgebra of the algebra of linear operator on V.

Note that from Corollary 3.3 we can conclude that sums, compositions, and scalar products of continuous-like linear operators are continuous-like.

4 Concluding Remarks

In this study we were able to identify the necessary and sufficient condition for linear operators on Hilbert spaces being continuous in terms of closed subspaces. This fact gave us the opportunity to introduce the continuous-like notion of linear mappings on bilinear spaces. We obtained that the class of continuous-like linear mappings on bilinear spaces is nonetheless the class of linear mappings that have adjoint mappings. As a consequence, the class of continuous-like linear operators on a bilinear space forms a subalgebra. It is interesting to see in how far the structure of the subalgebra of bounded linear operators on a Hilbert space can be extended to the subalgebra of continuous-like operators on a bilinear space.

As a last note, let us consider the result in [5] by Wójcik concerning equivalent conditions for orthogonal preserving operators on norm spaces equipped with sesquilinear forms. One of the conditions is the operators being bounded, which is equivalent to being continuous. That result and the current development of continuous-like notion in bilinear spaces trigger the question if that characterization of orthogonal preserving operators can be extended to bilinear spaces.

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