A New Family of Exponential Type Estimators in the Presence of Non-Response

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Abstract. We propose families of estimators for the population mean using an exponential function in case of non-response. This situation is examined under two cases, Case I and II. The bias, MSE and minimum MSE are separately obtained for both cases. We compare the proposed estimators theoretically with the main estimators from the literature, such as Hansen and Hurwitz (1946), ratio, regression and exponential estimators. The conditions for which the proposed estimators are most efficient are obtained. Moreover, different empirical studies are conducted to support the theoretical results for both cases.

Keywords: auxiliary variable; efficiency; exponential type estimators; family of estimators; non-response; population mean.

1 Introduction

Several authors have introduced different types of estimators to estimate unknown population parameters. When estimating population parameters, the information of an auxiliary variable (x) is generally used for enhancing the efficiency. For instance, Yadav and Mishra [1] and Yadav et al. [2] have proposed an estimator for population mean using auxiliary information. For this reason, ratio, product, regression, etc. type estimators have been proposed using auxiliary information to introduce more efficient estimators than others in the literature.

Some of the main estimators to estimate the population mean under the SRSWOR scheme are the following.

The ratio type estimator was proposed by Cochran [3]:

\[ t_R = \frac{\bar{y}}{\bar{x}} \hat{X} \] (1)

In Eq. (1), \( \hat{X} \) and \( \hat{x} \) refer to the population and sample mean of \( x \), respectively, and the sample mean of \( y \) is defined as \( \bar{y} \). The MSE of the \( t_R \) is given by:

\[ MSE(t_R) = \lambda \bar{y}^2 \left( C_y^2 + C_x^2 - 2C_{xy} \right), \] (2)

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\[ \lambda = \frac{1-f}{n}, C_y^2 = \frac{s_y^2}{\bar{y}^2}, C_x^2 = \frac{s_x^2}{\bar{x}^2}, C_{xy} = \rho_{xy}C_xC_y. \]

Besides, \( \bar{Y} \) is the population mean of \( y \). Here, \( f = \frac{n}{N} \) and \( \rho_{xy} \) is the population correlation coefficient between \( x \) and \( y \).

Cochran [4] proposed the classical regression estimator and obtained its MSE as follows:

\[
\begin{align*}
t_{\text{reg}} & = \bar{y} + b(\bar{X} - \bar{x}), \\
MSE(t_{\text{reg}}) & = \bar{Y}^2 \lambda C_y^2 (1 - \rho_{xy}^2),
\end{align*}
\]

respectively, where \( b \) is the regression coefficient.

The product, ratio and regression type estimators have equal efficiency when the relation between \( x \) and \( y \) is a straight line passing through the origin. However, this situation may not occur most of the time [5]. Recently, estimators have been proposed to take advantage of an exponential function.

Bahl and Tuteja [6] were the first to introduce an exponential type estimator:

\[
\begin{align*}
t_{BT} & = \bar{y} \exp \left( \frac{\bar{X} - \bar{x}}{\bar{x} + \bar{X}} \right), \\
MSE(t_{BT}) & = \bar{Y}^2 \lambda C_y^2 \left( 1 - \rho_{xy}^2 \right),
\end{align*}
\]

Following Bahl and Tuteja [6], Yadav and Kadilar [7] proposed as estimator:

\[
\begin{align*}
t_{YK} & = k\bar{y} \exp \exp \left( \frac{(c\bar{X} - c\bar{x})}{(c\bar{X} + c\bar{x}) + 2d} \right), \\
MSE_{\text{min}}(t_{YK}) & = \bar{Y}^2 \left( 1 - \frac{(\lambda(2\xi^2c_y^2 - \xi c_{xy}) + 1)^2}{\lambda(c_y^2 + 5\xi^2c_x^2 - 4\xi c_{xy}) + 1} \right)
\end{align*}
\]

where \( \xi = \frac{c\bar{X}}{2(c\bar{X} + d)} \).

After the significant contributions of these studies, Singh et al. [8], Kumar and Saini [9], and Singh et al. [10] proposed exponential type estimators for the population mean.

In real life, all information on various variables may not be available. Hansen and Hurwitz [11] introduced a new sub-sampling method to deal with non-response situations. In this method a population consist of \( N \) units, \( S \), and \( n \) units are drawn from the population under SRSWOR. \( N \) is composed of \( N_1 \) and
$N_2$ for responding and non-responding units, respectively. Furthermore, $n$ is divided into two units, responding ($n_1$) and non-responding ($n_2$). From $n_2$ a sub-sample of size $r = \frac{n_2}{z} (z > 1)\bar{X}$ is randomly drawn. Here, $z$ means the inverse sampling rate at the second phase sample. In addition to this technique, the methods of Srinath [12] and Bouza [13] can be used as alternatives to Hansen Hurwitz’s method for determining the subsample size [14].

Hansen and Hurwitz [11] were the first to introduce an unbiased estimator for non-response situations for the population mean as follows:

$$t_{HH} = w_1\bar{y}_1 + w_2\bar{y}_2(r),$$

where $w_1 = \frac{n_1}{n}$ is the response proportion in the sample. Similarly, $w_2 = \frac{n_2}{n}$ refers to the non-response proportion. In addition, $\bar{y}_2(r)$ and $\bar{y}_1$ indicate the sample means of the $y$ contingent on $r$ and $n_1$ units, respectively. The variance of $t_{HH}$ is given by:

$$V(t_{HH}) = \bar{Y}^2 \left( \lambda C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2 \right),$$

where $C_y^2 = \frac{s_y^2}{\bar{Y}^2}$ is the coefficient of variation of $Y$ for $N_2$ units. $W_2 = \frac{n_2}{N}$ is the non-response proportion of the population.

The non-response situation will be examined under Case I and Case II. In Case I, non-response is known and exists only in $y$.

For this case, Rao [15] adapted the ratio estimator and the regression estimator, $t^*_R$ and $t^*_reg$, in Eq. (1) and Eq. (7) respectively, using the Hansen and Hurwitz [11] technique:

$$t^*_R = \frac{\bar{X}}{\bar{x}} \bar{y}^*,$$

$$t^*_reg = \bar{y}^* + b^*(\bar{X} - \bar{x}),$$

where $b^* = \frac{s_y}{\bar{X} - \bar{x}}$ and $\bar{y}^*$ is the sample mean of $y$ under the non-response scheme.

The $MSE(t^*_R)$ and $MSE(t^*_reg)$ are given, respectively, as

$$MSE(t^*_R) = \bar{Y}^2 \left( \lambda C_x^2 - 2C_{yx} + C_y^2 \right) + \frac{W_2(z-1)}{n} C_{y(2)}^2,$$

$$MSE(t^*_reg) = \bar{Y}^2 \left( \lambda C_y^2 \left(1 - \rho_{x,y}^2\right) + \frac{W_2(z-1)}{n} C_{y(2)}^2 \right).$$

Singh et al. [16] proposed an exponential type estimator using a similar technique by adapting the $t_{BT}$ estimator as:

$$t^*_BT = \bar{y}^* \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right).$$
The MSE for the $t_{BT}^*$ estimator is given by:

$$MSE(t_{BT}^*) = \bar{Y}^2 \left( \frac{C_y^2}{4} + C_{yx}^2 - \lambda + \frac{W_2(z-1)}{n} C_{y(z)}^2 \right).$$  \hspace{1cm} (16)$$

After the significant contributions of these studies, Yunusa and Kumar [17], Dansawad [18], Singh and Usman [19], Pal and Singh [20,21], Yadav et al. [22], Sinha and Kumar [23], Pal and Singh [24], Kumar and Kumar [25], Sanaullah et al. [26] and Javaid et al. [27] have proposed exponential type estimators for the population mean for Case I.

For Case II, non-response exists in both $x$ and $y$ and $\bar{X}$ is known. Cochran [4] adapted the estimator in Eq. (1) for the Case II as follows:

$$t_{R}^{**} = \bar{y}^* \overline{X},$$ \hspace{1cm} (17)

where $\bar{x}^*$ denotes the sample mean of $x$ in case of non-response. The MSE of the $t_{R}^{**}$ is:

$$MSE(t_{R}^{**}) = \lambda \bar{Y}^2 \left( C_x^2 - 2C_{yx} + C_y^2 \right) + \bar{Y}^2 \left( \frac{W_2(z-1)}{n} \left( C_y^2 + C_{x(z)}^2 - 2C_{yx(z)} \right) \right).$$ \hspace{1cm} (18)

where $C_{x(z)}^2 = \frac{s_x^2}{\bar{x}^2}$ and $C_{yx(z)} = C_x(z)C_y(z)\rho_{yx(z)}$. Here, $\rho_{yx(z)}$ is the population correlation coefficient of the non-response group between $y$ and $x$.

Singh et al. [16] adapted the exponential type estimator in Eq. (3) using the Hansen and Hurwitz [11] technique for Case II:

$$t_{BT}^{**} = \bar{y}^* \exp \left( \frac{\bar{x} - \bar{x}^*}{\bar{x} + \bar{x}^*} \right).$$ \hspace{1cm} (19)

$MSE(t_{BT}^{**})$ is given as:

$$MSE(t_{BT}^{**}) = \bar{Y}^2 \left( \lambda \frac{C_x^2}{4} - \lambda C_{yx} + \lambda C_y^2 + \frac{W_2(z-1)}{n} \left( C_y^2 + C_{x(z)}^2 - \right) \right) \left( C_{yx(z)} \right).$$ \hspace{1cm} (20)

Cochran [4] proposed a classical regression estimator under non-response by adapting the estimator in Eq. (7) as follows:

$$t_{reg}^{**} = \bar{y}^* + b^* (\bar{X} - \bar{x}^*),$$ \hspace{1cm} (21)

whose MSE is given as:
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\[ MSE(t_{\text{reg}}^{**}) = \lambda C^2_y \bar{Y}^2 (1 - \rho_{xy}^2) + \bar{Y}^2 \left( \frac{W_2(x-1)}{n} \left( c_y^2 \right) - 2\rho_{xy} \frac{c_y}{c_x} c_{yx(2)} + \rho_{xy}^2 \frac{c_y^2}{c_x} c_{x(2)}^2 \right) \] (22)

After Singh et al. [16], Kumar and Bhougal [26], Kumar [29], Yadav et al. [22], Pal and Singh [20], Singh and Usman [19], Ünal and Kadilar [30, 31], Muneer et al. [32], Sanaullah et al. [26], Riaz et al. [33], Sinha and Kumar [23], Pal and Singh [24], Kumar and Kumar [25] and Sanaullah et al. [26] also proposed new estimators, taking advantage of an exponential function for Case II.

In this study, families of estimators taking advantage of an exponential function to estimate the population mean by adapting the estimator in Eq. (7) are proposed for non-response situations. The properties will be examined in Section 2 and comparisons between the proposed estimator and existing estimators from the literature will be made in Section 3 and Section 4, respectively.

2 The Proposed Families of Estimators

Based on Yadav and Kadilar [7], we adapt the exponential type estimators in Eq. (5) to a family of estimators, taking advantage of the exponential function for the population mean for Case I and Case II.

2.1 Case I:

We propose the following family of estimators for the first case:

\[ t_{CC1,i} = k \hat{y}^* \exp \left[ \frac{(a_i \hat{x} + b_i) - (a_i \hat{x} + b_i)}{(a_i \hat{x} + b_i) + (a_i \hat{x} + b_i)} \right], \quad i = 1, \ldots, 10. \] (23)

Here, \( k \) is a chosen constant to make \( MSE(t_{CC1,i}) \), \( i = 1, \ldots, 10 \) min and \( a_i, b_i \) either functions of known parameters of \( x \), such as, \( \beta_2(x), C_x \) etc. or real numbers.

To obtain expressions for \( B(t_{CC1,i}) \) and \( MSE(t_{CC1,i}) \), \( i = 1, \ldots, 10 \), we consider:

\[ \hat{y}^* = (\bar{Y}e_y^* + \bar{Y}), \hat{x} = (\bar{x}e_x + \bar{x}), \]

Then, \( E(e_x) = 0 \), \( E(e_y^*e_x) = \lambda C_{xy}, E(e_y^{**}) = 0 \), \( E(e_y^{*2}) = \lambda C_y^2 + \frac{W_2(x-1)}{n} C_{y(2)}^2 \), \( E(e_x^2) = \lambda C_x^2 \).

Now, expressing \( t_{CC1,i} \), \( i = 1, \ldots, 10 \) in terms of \( e_x \) and \( e_y^* \), we have:

\[ t_{CC1,i} = \bar{Y} (k + ke_y^*) \exp \left( \frac{a_i \hat{x} + b_i - a_i \hat{x} e_x - b_i}{a_i \hat{x} + b_i + a_i \hat{x} e_x + b_i} \right), \] (24)
\[ k\hat{Y}(1 + e_y^*) \exp \left[ \frac{-\theta e_x}{2} \left( 1 + \frac{\theta e_x}{2} \right)^{-1} \right] \] (25)

\[ \hat{Y} \left( k + ke_y^* - \frac{k\theta_i}{2} e_x + \frac{3k\theta_i^2}{8} e_x^2 - \frac{k\theta_i}{2} e_y^* e_x \right), \quad i = 1, \ldots, 10 \] (26)

where \( \theta_i = \frac{a_i \lambda}{a_i \lambda + b_i} \).

Expanding the right-hand side of Eq. (26), we have:

\[ (t_{CC1,i} - \hat{Y}) = \hat{Y} \left( k + \theta_i e_x + \frac{3k\theta_i^2}{8} e_x^2 - \frac{k\theta_i}{2} e_y^* e_x \right) + k(1) \] (27)

We take the expectation on both sides of Eq. (27) as the bias and we get:

\[ B(t_{CC1,i}) = \hat{Y} \left( k - 1 + \frac{3k\theta_i^2}{8} \lambda C^2_x - \frac{k\theta_i}{2} \lambda \rho_{xy} C_x C_y \right), \quad i = 1, \ldots, 10. \] (28)

We take the square of both sides of \((t_{CC1,i} - \hat{Y})\) and then we take the expectation, so we obtain \(MSE(t_{CC1,i}), i = 1, \ldots, 10\) as:

\[ MSE(t_{CC1,i}) = \hat{Y}^2 \left( k^2 \left( \frac{3k\theta_i^2}{8} \lambda C^2_x + \frac{W_x(z-1)}{n} C^2_y \right) + (k-1)^2 + \lambda C^2_x \left( k^2 \theta_i^2 - \frac{3k\theta_i^2}{4} \right) - \lambda \rho_{xy} C_x C_y (2k^2 \theta_i - k \theta_i) \right), \] (29)

After obtaining the optimal \(k\) as:

\[ k^* = \frac{A_1}{A_2}, \] (30)

here,

\[ A_1 = \lambda \left( \frac{3}{4} \theta_i^2 C^2_x - \theta_i C_{xy} \right) + 2 \]

and

\[ A_2 = 2 \left( \lambda C^2_y + \frac{W_x(z-1)}{n} C^2_y \right) + \lambda \theta_i^2 C^2_x - 2 \lambda \theta_i C_{xy} + 1 \right). \]

we have the min \(MSE(t_{CC1,i}), i = 1, \ldots, 10\) estimators as follows:

\[ MSE_{min}(t_{CC1,i})^\sim \left( 1 - \frac{A_1^2}{2A_2} \right), \quad i = 1, \ldots, 10. \] (31)

Some members of the estimators in Eq. (23) are given in Table 1.

### 2.2 Case II:

We propose the family of estimators for the second case as follows:
\[ t_{CC2,i} = k \bar{y}^* \exp \left( \left[ \frac{(a_i \bar{x} + b_i) - (a_i \bar{x}^* + b_i)}{(a_i \bar{x} + b_i) + (a_i \bar{x}^* + b_i)} \right] , i = 1, \ldots, 10. \] 

(32)

**Table 1** Some members of the family of estimators.

<table>
<thead>
<tr>
<th>Values</th>
<th>Estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>$a_1 = 1$, $b_1 = 1$</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>$\beta_2(x)$</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>$c_x$</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>$\beta_2(x)$, $c_x$</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>$c_x$</td>
</tr>
<tr>
<td>$\theta_7$</td>
<td>$c_x$</td>
</tr>
<tr>
<td>$\theta_8$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$\theta_9$</td>
<td>$\beta_2(x)$</td>
</tr>
<tr>
<td>$\theta_{10}$</td>
<td>$\rho$</td>
</tr>
</tbody>
</table>

Using Table 1, we can write some members of $t_{CC2,i}$, $i = 1, \ldots, 10$ for Case II.

To obtain $B(t_{CC2,i})$ and $MSE(t_{CC2,i})$ we consider:

\[ \bar{y}^* = (\bar{Y} + \bar{e}_y^*), \bar{x}^* = (\bar{X} + \bar{e}_x^*) \]

Then, we have:

\[ E(e_x^*) = 0, \quad E(e_y^*) = \lambda C_x^2 + \frac{\lambda C_y^2}{n} C_{xy}^2, \quad E(e_x^* e_y^*) = 0, \]

\[ E(e_y^* e_y^*) = \lambda C_y^2 + \frac{\lambda C_y^2}{n} C_{xy}^2, \quad E(e_x^* e_y^*) = \lambda C_{xy}^2 + \frac{\lambda C_y^2}{n} C_{xy}^2. \]

Now, expressing $t_{CC2,i}$, $i = 1, \ldots, 10$ estimators in Eq. (32), we get:

\[ t_{CC2,i} = \bar{Y} (k + l e_y^*) \exp \left( \frac{(a \bar{x} + b) - (a \bar{x}^* + b)}{a \bar{x} + b + a \bar{x}^* + b} \right), \]  

(33)

\[ = k \bar{Y} (1 + e_0^*) \exp \left( -\theta e_1^* \left( 1 + \frac{\theta e_1^*}{2} \right)^{-1} \right) \]  

(34)
\[
\tilde{Y} \left( k + ke^*_Y \left( -\frac{k\theta_1^2e_{x^2}^2}{8} - \frac{k\theta_1^2e_{x^2}^2}{8} \right) \right), \quad i = 1, \ldots, 10. 
\] (35)

Using the same procedure as was used for the first proposed family, we obtain \( B(t_{CC,1}) \) and \( MSE(t_{CC,1}) \) as follows:

\[
B(t_{CC,1}) = \tilde{Y} \left( \frac{3k\theta_1^2}{8} \left( \lambda^2 C_x^2 + \frac{W_2(z-1)}{n} C_{x(2)}^2 \right) + (k - 1) - \frac{k\theta_1^2}{2} \left( \lambda^2 \rho_{yx} C_y C_x + \frac{W_2(z-1)}{n} C_{yx(2)} \right) \right), \quad i = 1, \ldots, 10, 
\] (36)

\[
MSE(t_{CC,1}) = \tilde{Y}^2 \left( k^2 \left( \frac{3k\theta_1^2}{8} \left( \lambda^2 C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2 \right) + (k - 1)^2 + \frac{k\theta_1^2}{2} \left( \lambda^2 C_y^2 + \frac{W_2(z-1)}{n} C_{y(2)}^2 \right) - k\theta_1(2k - 1) \left( \lambda C_y + \frac{W_2(z-1)}{n} C_{yx(2)} \right) \right) \right), \quad i = 1, \ldots, 10. 
\] (37)

Minimization of \( MSE(t_{CC,1}) \) with respect to \( k \), the optimal \( k \) is obtained as:

\[
k^{**} = \frac{B_1}{B_2}, \quad (38)
\]

where

\[
B_1 = \left( \lambda \left( \frac{3}{4} \theta_1^2 C_x^2 - \theta_1 C_y \right) + \frac{W_2(z-1)}{n} \left( \frac{3}{4} \theta_1^2 C_{x(2)}^2 - \theta_1 C_{yx(2)} \right) + 2 \right)
\]

and

\[
B_2 = 2 \left( \lambda \left( C_y^2 - 2 \theta_1 C_y C_x + \theta_1^2 C_x^2 \right) + \frac{W_2(z-1)}{n} \left( C_{y(2)}^2 + \theta_1^2 C_{x(2)}^2 - 2\theta_1 C_{yx(2)} \right) + 1 \right).
\]

Using \( k^{**} \) in \( MSE(t_{CC,1}) \), we have \( \min \) \( MSE(t_{CC,1}) \), \( i = 1, \ldots, 10 \) estimators as follows:

\[
MSE_{\min}(t_{CC,1})^2 = \tilde{Y}^2 \left( 1 - \frac{B_1^2}{2B_2} \right), \quad i = 1, \ldots, 10 \quad (39)
\]

3 \hspace{1em} \textbf{Efficiency Comparisons}

Now we will investigate the efficiencies of \( t_{CC,1} \) and \( t_{CC,2}, i = 1, \ldots, 10 \) given in Eq. (23) and Eq. (32) with various estimators from the literature for Case I and Case II.

3.1 \hspace{1em} \textbf{Efficiency Comparisons for Case I}

Using Eqs. (10), (13), (14), (16) and (31) we find the efficiency conditions of \( t_{CC,1}, i = 1, \ldots, 10 \) as follows:

i) \( V(t_{HH}) - MSE(t_{CC,1})_{\min} > 0 \)
\[(1 - \frac{A_i}{2A_z}) < (\lambda C_y^2 + \frac{W_z(x-1)}{n} C_{y(2)}) \] (40)

ii) \(MSE(t^*_R) - MSE(t_{CC1,i})_{\text{min}} > 0\)
\[(1 - \frac{A_i}{2A_z}) < (\lambda(-2C_{yx} + C_x^2 + C_y^2) + \frac{W_z(x-1)}{n} C_{y(2)}) \] (41)

iii) \(MSE(t^*_BT) - MSE(t_{CC1,i})_{\text{min}} > 0\)
\[(1 - \frac{A_i}{2A_z}) < (\lambda C_y^2 + C_x^2 - C_{yx}) + \frac{W_z(x-1)}{n} C_{y(2)} \] (42)

iv) \(MSE(t^*_reg) - MSE(t_{CC1,i})_{\text{min}} > 0\)
\[(1 - \frac{A_i}{2A_z}) < (\lambda C_y^2 (1 - \rho_{xy}^2) + \frac{W_z(x-1)}{n} C_{y(2)}) \] (43)

3.2 Efficiency Comparisons for Case II

Using Eqs. (10), (18), (20), (22) and (39) for \(t_{CCZ,i} i = 1, \ldots, 10\) we have:

i) \(V(t_{HH}) - MSE(t_{CC2,i})_{\text{min}} > 0\)
\[(1 - \frac{B_i^2}{2B_2}) < (\lambda C_y^2 + \frac{W_z(x-1)}{n} C_{y(2)}) \] (44)

ii) \(MSE(t^*_{R'''}) - MSE(t_{CC2,i})_{\text{min}} > 0\)
\[(1 - \frac{B_i^2}{2B_2}) < (\lambda(C_y^2 + C_x^2 - 2C_{yx}) + \frac{W_z(x-1)}{n}(C_y^2 + C_x^2 - 2C_{yx(2)})) \] (45)

iii) \(MSE(t^*_{BT''}) - MSE(t_{CC2,i})_{\text{min}} > 0\)
\[(1 - \frac{B_i^2}{2B_2}) < (\lambda(C_y^2 + C_x^2 - 2C_{yx}) + \frac{W_z(x-1)}{n}(C_y^2 + C_x^2 - 2C_{yx(2)})) \] (46)

iv) \(MSE(t^*_{reg''}) - MSE(t_{CC2,i})_{\text{min}} > 0\)
\[(1 - \frac{B_i^2}{2B_2}) < (\lambda C_y^2 (1 - \rho_{xy}^2) + \frac{W_z(x-1)}{n}(C_y^2 + \rho_{xy}^2 C_x^2 C_{x(2)} - 2\rho_{yx} C_y C_{xy(2)})) \] (47)

When the conditions Eq. (40-43) and Eq. (44-47) are satisfied, we infer that \(t_{CC1,i}\) and \(t_{CCZ,i}\) are more efficient than the other estimators for both \(i\) values, respectively.
4 Numerical Illustrations

We used different data sets considered by Khare and Sinha [34] and Khare and Srivastava [35] for Case I and II, respectively, to examine the performances of $t_{CC1,i}$ and $t_{CC2,i}, i = 1, \ldots, 10$ in practice compared to other estimators from the literature.

The percent relative efficiencies (PREs) were also computed using various values of $z$ for both cases separately by using the following formula:

$$PRE(\ast, t_{HH}) = \frac{V(t_{HH})}{MSE(\ast)} \times 100.$$ 

4.1 Numerical Illustration of Case I

We used the data set from Khare and Sinha [34] for Case I. The descriptive statistics are given in Table 2. Note that for Population 1:

<table>
<thead>
<tr>
<th>Parameter values for Population I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 96, n = 40$</td>
</tr>
<tr>
<td>$\bar{Y} = 137.92$</td>
</tr>
<tr>
<td>$\bar{X} = 1.32$</td>
</tr>
<tr>
<td>$\rho_{yx(2)} = 0.72$</td>
</tr>
<tr>
<td>$\lambda = 0.4167$</td>
</tr>
<tr>
<td>$\beta(x) = 1.2$</td>
</tr>
<tr>
<td>$f = 0.1448$</td>
</tr>
<tr>
<td>$C_y = 0.81$</td>
</tr>
<tr>
<td>$\rho_{xy} = 0.77$</td>
</tr>
<tr>
<td>$C_x = 0.823$</td>
</tr>
<tr>
<td>$W_2 = 0.25$</td>
</tr>
<tr>
<td>$C_{x(2)} = 0.94$</td>
</tr>
<tr>
<td>$C_{y(2)} = 2.08$</td>
</tr>
</tbody>
</table>

The MSE values of $t_{CC1,i}, i = 1, \ldots, 10$ and $t_{HH}, t^*_R, t^*_BT, t^*_reg$ for various values of $z$ for Case I are given in Table 3.

<table>
<thead>
<tr>
<th>Table 3 MSE Values of $(t_{CC1,i}, i = 1, \ldots, 10)$ and other estimators for Population I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

Based on these results we conclude that all estimators in $(t_{CC1,i}, i = 1, \ldots, 10)$ are more efficient than the other estimators in Case I.
According to the Table 4, the PRE values of \( t_{CC1,i}, i = 1, \ldots, 10 \), especially \( t_{CC1,9} \), were better compared to those of the other estimators. We also found that the PRE values of \( t_{CC1,i}, i = 1, \ldots, 10 \) increased with increasing values of \( z \).

### 4.2 Numerical Illustration for Case II

We used the data set from Khare and Srivastava [35] for Case II and the descriptive statistics are given in Table 5. Note that for Population 2:

<table>
<thead>
<tr>
<th>( N = 70, n = 35 )</th>
<th>( \bar{X} = 1755.53 )</th>
<th>( \lambda = 0.014 )</th>
<th>( \rho_{yx} = 0.778 )</th>
<th>( C_{yx} = 0.39 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f = 0.50 )</td>
<td>( \bar{Y} = 981.29 )</td>
<td>( C_x = 0.801 )</td>
<td>( \rho_{yx(2)} = 0.445 )</td>
<td>( C_{yx(2)} = 0.104 )</td>
</tr>
<tr>
<td>( W_2 = 0.2 )</td>
<td>( C_y = 0.625 )</td>
<td>( C_{x(2)} = 0.5739 )</td>
<td>( C_{y(2)} = 0.409 )</td>
<td>( \beta_2(x) = 0.34 )</td>
</tr>
</tbody>
</table>

The MSE values of the \( t_{CC2,i}, i = 1, \ldots, 10 \) and \( t_{HH}, t_{BT}^{**}, t_{reg}^{**} \) estimators for various values of \( z \) in Case II are given in Table 6.

According to the MSE values, the \( t_{CC2,i}, i = 1, \ldots, 10 \) estimators are more efficient than the other estimators in a non-response situation both in \( y \) and \( x \). The PRE values which are given in Table 7, were also better compared to those of the other estimators. Furthermore, we also found that the PRE values of \( (t_{CC2,i}, i = 1, \ldots, 10) \) decreased with increasing values of \( z \).
Table 6  MSE values of \( (t_{CC2,i}; i = 1, \ldots, 10) \) and other estimators for Population II.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( t_{HH} )</th>
<th>( t_{R} )</th>
<th>( t_{RT} )</th>
<th>( t_{EG} )</th>
<th>( t_{CC2,1} )</th>
<th>( t_{CC2,2} )</th>
<th>( t_{CC2,3} )</th>
<th>( t_{CC2,4} )</th>
<th>( t_{CC2,5} )</th>
<th>( t_{CC2,6} )</th>
<th>( t_{CC2,7} )</th>
<th>( t_{CC2,8} )</th>
<th>( t_{CC2,9} )</th>
<th>( t_{CC2,10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8137.694</td>
<td>9056.8011</td>
<td>9975.908</td>
<td>10895.0147</td>
<td>11814.12</td>
<td>12397.94638</td>
<td>13456.222</td>
<td>14562.222</td>
<td>15684.842</td>
<td>16823.081</td>
<td>17862.488</td>
<td>18824.204</td>
<td>19822.218</td>
<td>20821.251</td>
</tr>
<tr>
<td>5</td>
<td>8231.209</td>
<td>9883.9618</td>
<td>11396.72</td>
<td>12979.4683</td>
<td>14562.222</td>
<td>15684.842</td>
<td>16823.081</td>
<td>17862.488</td>
<td>18824.204</td>
<td>19822.218</td>
<td>20821.251</td>
<td>21822.284</td>
<td>22822.318</td>
<td>23821.351</td>
</tr>
<tr>
<td>6</td>
<td>4619.291</td>
<td>5417.1474</td>
<td>6215.003</td>
<td>7012.8594</td>
<td>7810.715</td>
<td>8682.436</td>
<td>9682.204</td>
<td>10682.063</td>
<td>11682.292</td>
<td>12682.428</td>
<td>13682.567</td>
<td>14682.706</td>
<td>15682.845</td>
<td>16682.984</td>
</tr>
<tr>
<td>7</td>
<td>4794.201</td>
<td>5684.361</td>
<td>6574.521</td>
<td>7464.6817</td>
<td>8354.842</td>
<td>9244.982</td>
<td>10244.121</td>
<td>11244.260</td>
<td>12244.409</td>
<td>13244.558</td>
<td>14244.707</td>
<td>15244.856</td>
<td>16244.995</td>
<td>17245.134</td>
</tr>
<tr>
<td>8</td>
<td>4576.81</td>
<td>5357.387</td>
<td>6135.116</td>
<td>6910.0099</td>
<td>7682.081</td>
<td>8557.387</td>
<td>9457.387</td>
<td>10457.387</td>
<td>11457.387</td>
<td>12457.387</td>
<td>13457.387</td>
<td>14457.387</td>
<td>15457.387</td>
<td>16457.387</td>
</tr>
</tbody>
</table>

Table 7  PREs of \( (t_{CC2,i}; i = 1, \ldots, 10) \) and Others for Population II.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( t_{HH} )</th>
<th>( t_{R} )</th>
<th>( t_{RT} )</th>
<th>( t_{EG} )</th>
<th>( t_{CC2,1} )</th>
<th>( t_{CC2,2} )</th>
<th>( t_{CC2,3} )</th>
<th>( t_{CC2,4} )</th>
<th>( t_{CC2,5} )</th>
<th>( t_{CC2,6} )</th>
<th>( t_{CC2,7} )</th>
<th>( t_{CC2,8} )</th>
<th>( t_{CC2,9} )</th>
<th>( t_{CC2,10} )</th>
</tr>
</thead>
</table>

5 Conclusion

We proposed families of estimators, \( t_{CC1,i}, t_{CC2,i}; i = 1, \ldots, 10 \), taking advantage of an exponential function for estimating the population mean under non-response for two cases. Equations for the bias and minimum MSE of \( t_{CC1,i}, t_{CC2,i}; i = 1, \ldots, 10 \) were also obtained for both cases. In this way, the \( t_{CC1,i}, t_{CC2,i}; i = 1, \ldots, 10 \) estimators were found to be more efficient in theory than the other estimators under the obtained conditions. Using the data sets from Khare and Sinha [34] and Khare and Srivastava [35], we concluded that \( t_{CC1,i} \) and \( t_{CC2,i} \) are more efficient than the other estimators in Case I and II.
respectively. Therefore, the proposed $t_{Cc1,i}, t_{Cc2,i}; i = 1, \ldots, 10$ estimators are recommended for both cases of non-response situations based on the obtained results.

References


A New Family of Exponential Type Estimators in The Presence of Nonresponse


