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Inclusion Properties of Orlicz and Weak Orlicz Spaces

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Abstract. In this paper we discuss the structure of Orlicz spaces and weak Orlicz spaces on $\mathbb{R}^n$. We obtain some necessary and sufficient conditions for the inclusion property of these spaces. One of the keys is to compute the norm of the characteristic functions of the balls in $\mathbb{R}^n$.

Keywords: *Inclusion property; Lebesgue spaces; Orlicz spaces; Young function; weak Orlicz spaces.*

1 Introduction

Orlicz spaces were introduced by Birnbaum and Orlicz in 1931 [1]. Let $\Phi: [0,\infty) \to [0,\infty)$ be a Young function, that is, Φ is convex, left-continuous, $\Phi(0) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Given a measure space (X, dx), we define the Orlicz space $L_\Phi(X)$ to be the set of measurable functions $f: X \to \mathbb{R}$ such that

$$\int_X \Phi(a\,|\,f(x)\,|)dx < \infty$$

for some $a > 0$. The space $L_\Phi(X)$ is a Banach space equipped with the norm

$$\|\, f \,\|_{L_\Phi(X)} := \inf\left\{ b > 0 : \int_X \Phi\left(\frac{|\,f(x\,|)}{b}\right) dx \le 1 \right\}$$

(see [2,3]). Note that, if $\Phi(t) := t^p$ for some $p \ge 1$ and $X := \mathbb{R}^n$, then $L_\Phi(X) = L_p(\mathbb{R}^n)$, the Lebesgue space of p-th integrable functions on $\mathbb{R}^n$ [4]. Thus, Orlicz spaces can be viewed as a generalization of Lebesgue spaces.

Several authors have made important observations about Orlicz spaces (see [2-8], etc.). Here we are interested in the inclusion property of these spaces. In [8], Welland proved the following inclusion property: Let X be of finite measure, and Φ, Ψ be two Young functions. If there is $C > 0$ such that $\Phi(t) \le \Psi(Ct)$ for every $t > 0$, then $L_\Phi(X) \subseteq L_\Psi(X)$. Accordingly, if X is of finite measure, Φ, Ψ

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are two Young functions, and there is $C > 0$ such that $\Psi(\frac{t}{C}) \leq \Phi(t) \leq \Psi(Ct)$ for every $t > 0$, then we have $L_\Phi(X) = L_\Psi(X)$. A refinement of this result may be found in [5], which states that $L_\Phi(X) \subseteq L_\Psi(X)$ if only if there are $C > 0$ and $T > 0$ such that $\Phi(t) \leq \Psi(Ct)$ for every $t \geq T$. Related results can be found in [6,9,10]. Motivated by these results, the purpose of this work is to get the inclusion property of Orlicz spaces $L_\Phi(\mathbb{R}^n)$ and extend the results to weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ (see [11-14]). (Here $X := \mathbb{R}^n$ has an infinite measure.)

The rest of this paper is organized as follows. The main results are presented in Sections 2 and 3. In Section 2, we state the inclusion property of Orlicz spaces $L_\Phi(\mathbb{R}^n)$ as Theorem 2.5, which contains a necessary and sufficient condition for the inclusion property to hold. An analogous result for the weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ is stated as Theorem 3.3.

To prove the results, we pay attention to the characteristic functions of balls in $\mathbb{R}^n$ and use the inverse function of Φ, which is given by $\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}$. The reader will find the following lemma useful.

Lemma 1.1 *Suppose that* Φ *is a Young function and* $\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}$. *We have*

(1) $\Phi^{-1}(0) = 0$.
(2) $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ *for* $s_1 \leq s_2$.
(3) $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ *for* $0 \leq s < \infty$.
(4) *Let* $C > 0$. *Then* $\Phi_1(t) \leq \Phi_2(Ct)$ *if only if* $C\Phi_1^{-1}(t) \geq \Phi_2^{-1}(t)$, *for every* $t \geq 0$.
(5) *Let* $C > 0$. *Then* $\Phi_1(t) \leq C\Phi_2(t)$ *if only if* $\Phi_1^{-1}(Ct) \geq \Phi_2^{-1}(t)$, *for every* $t \geq 0$.

Proof. The proof of parts (1)-(3) can be found in [15]. Now we will prove (4) and (5).

(4) Let $C > 0$. We will prove $\Phi_1(t) \leq \Phi_2(Ct)$ if only if $C\Phi_1^{-1}(t) \geq \Phi_2^{-1}(t)$, for every $t \geq 0$.

Take an arbitrary $C > 0$ such that $\Phi_1(t) \leq \Phi_2(Ct)$ for every $t \geq 0$. Let $\Phi_1^{-1}(t) = \inf A_1$ where $A_1 = \inf\{r \geq 0 : \Phi_1(r) > t\}$, and write $B_1 = \{r \geq 0 : \Phi_2(Cr) > t\}$. Observe that

$$\inf B_1 = \inf\{r \geq 0 : \Phi_2(Cr) > t\}$$

$$= \inf\{\frac{x}{C} \geq 0 : \Phi_2(x) > t\}$$

$$= \frac{1}{C}\inf\{x \geq 0 : \Phi_2(x) > t\}$$

$$= \frac{1}{C}\Phi_2^{-1}(t)$$

for $x = Cr$. For an arbitrary $r \in A_1$, we have $\Phi_2(Cr) \geq \Phi_1(r) > t$, and thus it follows that $r \in B_1$. Hence we conclude that $A_1 \subseteq B_1$. Accordingly, we obtain $\frac{1}{C}\Phi_2^{-1}(t) = \inf B_1 \leq \inf A_1 = \Phi_1^{-1}(t)$.

Now, suppose that $C\Phi_1^{-1}(t) \geq \Phi_2^{-1}(t)$, for $C > 0$ and every $t \geq 0$. Observe that, by Lemma 1.1 (3) we have

$$\Phi_1(\frac{t}{C}) \leq \Phi_1(\frac{\Phi_2^{-1}(\Phi_2(t))}{C})$$

$$\leq \Phi_1(C\frac{\Phi_1^{-1}(\Phi_2(t))}{C})$$

$$= \Phi_1(\Phi_1^{-1}(\Phi_2(t)))$$

$$\leq \Phi_2(t).$$

As a result, we have $\Phi_1(\frac{t}{C}) \leq \Phi_2(t)$ or $\Phi_1(t) \leq \Phi_2(Ct)$.

(5) Let $C > 0$. We will prove $\Phi_1(t) \leq C\Phi_2(t)$ if only if $\Phi_1^{-1}(Ct) \geq \Phi_2^{-1}(t)$, for every $t \geq 0$.

Take an arbitrary $C > 0$ such that $\Phi_1(t) \leq C\Phi_2(t)$ for every $t \geq 0$. Let $\Phi_1^{-1}(Ct) = \inf A_2$ where $A_2 = \inf\{r \geq 0: \Phi_1(r) > Ct\}$, and write $B_2 = \{r \geq 0: \Phi_2(r) > t\}$. Observe that, for $r \in A_2$ we have $C\Phi_2(t) \geq \Phi_1(r) > Ct$, so that $r \in B_2$. Hence we conclude that $A_2 \subseteq B_2$. Accordingly, we obtain $\Phi_2^{-1}(t) = \inf B_2 \leq \inf A_2 = \Phi_1^{-1}(Ct)$.

Now, suppose that $\Phi_1^{-1}(Ct) \geq \Phi_2^{-1}(t)$, for $C > 0$ and every $t \geq 0$. Observe that, by Lemma 1.1 (3) we have

$$\Phi_1(t) \leq \Phi_1(\Phi_2^{-1}(\Phi_2(t)))$$

$$\leq \Phi_1(\Phi_1^{-1}(C\Phi_2(t)))$$

$$\leq C\Phi_2(t).$$

As a result, we have $\Phi_1(t) \leq C\Phi_2(t)$.

More lemmas (and their proofs) will be presented in the next sections.

2 Inclusion Property of Orlicz Spaces

Let us first recall several Lemmas below.

Lemma 2.1 *If Φ is a Young function, then $\Phi(\alpha t) \leq \alpha\Phi(t)$ for $t > 0$ and $0 \leq \alpha \leq 1$.*

Lemma 2.2 [2] *Let Φ be a Young function and $f \in L_{\Phi(\mathbb{R}^n)}$. If $0 < \| f \|_{L_\Phi(\mathbb{R}^n)} < \infty$, then*

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\| f \|_{L_\Phi(\mathbb{R}^n)}}\right) dx \leq 1.$$

Furthermore, $\| f \|_{L_\Phi(\mathbb{R}^n)} \leq 1$ if only if $\int_{\mathbb{R}^n} \Phi(|f(x)|)dx \leq 1$.

Corollary 2.3 *Let Φ, Ψ be Young functions. If there exists $C > 0$ such that $\Phi(t) \leq \Psi(Ct)$ for $t > 0$, then $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$ with $\| f \|_{L_\Phi(\mathbb{R}^n)} \leq C \| f \|_{L_\Psi(\mathbb{R}^n)}$ for every $f \in L_\Psi(\mathbb{R}^n)$.*

Proof. Suppose that $f \in L_\Psi(\mathbb{R}^n)$. Observe that

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{C\| f \|_{L_\Psi(\mathbb{R}^n)}}\right) dx \leq \int_{\mathbb{R}^n} \Psi\left(\frac{C|f(x)|}{C\| f \|_{L_\Psi(\mathbb{R}^n)}}\right) dx = \int_{\mathbb{R}^n} \Psi\left(\frac{|f(x)|}{\| f \|_{L_\Psi(\mathbb{R}^n)}}\right) dx \leq 1.$$

By the definition of $\| f \|_{L_\Phi(\mathbb{R}^n)}$, we have $\| f \|_{L_\Phi(\mathbb{R}^n)} \leq C \| f \|_{L_\Psi(\mathbb{R}^n)}$. This proves that $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$, as desired.

Remark. From Corollary 2.3, we note that if $\Phi \leq \Psi$, then $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$ with $\| f \|_{L_\Phi(\mathbb{R}^n)} \leq \| f \|_{L_\Psi(\mathbb{R}^n)}$ for every $f \in L_\Psi(\mathbb{R}^n)$. As we shall see below, the converse of this statement also holds. We need the following lemma.

Lemma 2.4 [7] *Let Φ be a Young function, $a \in \mathbb{R}^n$, and $r > 0$. Then $\| \chi_{B(a,r)} \|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)}$, where $|B(a,r)|$ denotes the volume of $B(a,r)$.*

Theorem 2.5 *Let Φ, Ψ be Young functions. Then the following statements are equivalent:*

(1) $\Phi(t) \leq \Psi(Ct)$ *for every* $t > 0$.
(2) $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$.
(3) *For every* $f \in L_\Psi(\mathbb{R}^n)$, *we have* $\| f \|_{L_\Phi(\mathbb{R}^n)} \leq C \| f \|_{L_\Psi(\mathbb{R}^n)}$..

Proof. We have seen that (1) implies (2). Next, since $(L_\Psi(\mathbb{R}^n), L_\Phi(\mathbb{R}^n))$ is a Banach pair, it follows from [16, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1). Now assume that (3) holds. By Lemma 2.4, we have

$$\frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)} = \| \chi_{B(a,r)} \|_{L_\Phi(\mathbb{R}^n)} \leq C \| \chi_{B(a,r)} \|_{L_\Psi(\mathbb{R}^n)} = \frac{C}{\Psi^{-1}\left(\frac{1}{|B(a,r)|}\right)},$$

or $C\Phi^{-1}(\frac{1}{|B(a,r)|}) \geq \Psi^{-1}(\frac{1}{|B(a,r)|})$, for every $a \in \mathbb{R}^n, r > 0$. By Lemma 1.1 (4), we obtain $\Phi(\frac{1}{|B(a,r)|}) \leq \Psi(\frac{C}{|B(a,r)|})$. Since $r > 0$ is arbitrary, we conclude that $\Phi(t) \leq \Psi(Ct)$ for every $t > 0$.

2.1 A Special Case

One may ask whether from inclusion relations between Orlicz spaces we may deduce some known fact of those in Lebesgue spaces. The answer is affirmative; we need the following lemma for this purpose.

Lemma 2.6 *Let Φ_1, Φ_2, and Φ_3 be Young functions such that $\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$ for every $t \geq 0$. If $f \in L_{\Phi_1}(\mathbb{R}^n)$ and $g \in L_{\Phi_2}(\mathbb{R}^n)$, then $fg \in L_{\Phi_3}(\mathbb{R}^n)$ with*

$$\| fg \|_{L_{\Phi_3}(\mathbb{R}^n)} \leq 2 \| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)} .$$

Proof. Let $s, t \geq 0$. Without loss of generality, suppose that $\Phi_1(s) \leq \Phi_2(t)$. By Lemma 1.1 (3), we obtain

$$st \leq \Phi_1^{-1}\left(\Phi_1(s)\right)\Phi_2^{-1}\left(\Phi_2(t)\right) \leq \Phi_1^{-1}\left(\Phi_2(t)\right)\Phi_2^{-1}\left(\Phi_2(t)\right) \leq \Phi_3^{-1}\left(\Phi_2(t)\right).$$

Hence $\Phi_3(st) \leq \Phi_3(\Phi_3^{-1}(\Phi_2(t))) \leq \Phi_2(t) \leq \Phi_2(t) + \Phi_1(s)$. From Lemma 2.1 we have

$$\int_{\mathbb{R}^n} \Phi_3\left(\frac{|f(x)g(x)|}{2\| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx \le \frac{1}{2}\int_{\mathbb{R}^n} \Phi_3\left(\frac{|f(x)g(x)|}{\| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx.$$

On the other hand, by Lemma 2.2 we obtain

$$\int_{\mathbb{R}^n} \Phi_3\left(\frac{|f(x)g(x)|}{\| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx \le \int_{\mathbb{R}^n} \Phi_1\left(\frac{|f(x)|}{\| f \|_{L_{\Phi_1}(\mathbb{R}^n)}}\right) dx + \int_{\mathbb{R}^n} \Phi_2\left(\frac{|g(x)|}{\| g \|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx \le 2$$

whenever $f \in L_{\Phi_1}(\mathbb{R}^n)$ and $g \in L_{\Phi_2}(\mathbb{R}^n)$. By using the definition of $\| fg \|_{L_{\Phi_3}(\mathbb{R}^n)}$, we have $\| fg \|_{L_{\Phi_3}(\mathbb{R}^n)} \le 2 \| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}$, as desired.

Corollary 2.7 *Let* $X := B(a, r_0) \subset \mathbb{R}^n$ *for some* $a \in \mathbb{R}^n$ *and* $r_0 > 0$. *If* Φ_1, Φ_2 *are two Young functions and there is a Young function* Φ *such that*

$$\Phi_1^{-1}(t)\Phi^{-1}(t) \le \Phi_2^{-1}(t)$$

for every $t \ge 0$, *then* $L_{\Phi_1}(X) \subseteq L_{\Phi_2}(X)$ *with*

$$\| f \|_{L_{\Phi_2}(X)} \le \frac{2}{\Phi^{-1}\left(\frac{1}{|B(a,r_0)|}\right)} \| f \|_{L_{\Phi_1}(X)}$$

for $f \in L_{\Phi_1}(X)$.

Proof. Let $f \in L_{\Phi_1}(X)$. By Lemma 2.4 and choosing $g := \chi_{B(a,r_0)}$, we obtain

$$\| f\chi_{B(a,r_0)} \|_{L_{\Phi_2}(X)} \le 2 \| \chi_{B(a,r_0)} \|_{L_{\Phi}(X)} \| f \|_{L_{\Phi_1}(X)} = \frac{2}{\Phi^{-1}\left(\frac{1}{|B(a,r_0)|}\right)} \| f \|_{L_{\Phi_1}(X)}.$$

This shows that $L_{\Phi_1}(X) \subseteq L_{\Phi_2}(X)$.

Corollary 2.8 *Let* $X := B(a, r_0)$ *for some* $a \in \mathbb{R}^n$ *and* $r_0 > 0$. *If* $1 \le p_2 < p_1 < \infty$, *then* $L_{p_1}(X) \subseteq L_{p_2}(X)$.

Proof. Let $\Phi_1(t) := t^{p_1}, \Phi_2(t) := t^{p_2}$, and $\Phi(t) := t^{\frac{p_1p_2}{p_1-p_2}}$ $(t \ge 0)$. Since $1 \le p_2 < p_1 < \infty$, we have $\frac{p_1p_2}{p_1-p_2} > 1$. Thus, Φ_1, Φ_2, and Φ are three Young functions. Observe that, using the definition of Φ^{-1} and Lemma 1.1, we have

$$\Phi_1^{-1}(t) = t^{\frac{1}{p_1}}, \Phi_2^{-1}(t) = t^{\frac{1}{p_2}}, \text{and } \Phi^{-1}(t) = t^{\frac{p_1-p_2}{p_1p_2}}.$$

Moreover, $\Phi_1^{-1}(t)\Phi^{-1}(t) = t^{\frac{1}{p_1}} t^{\frac{p_1-p_2}{p_1p_2}} = t^{\frac{1}{p_2}} = \Phi_2^{-1}(t)$, and so it follows from Corollary 2.7 that $\| f \|_{L_{p_2}(X)} \leq \frac{2}{\Phi^{-1}(\frac{1}{|B(a,r_0)|})} \| f \|_{L_{p_1}(X)}$, and therefore $L_{p_1}(X) \subseteq L_{p_2}(X)$.

Remark. Of course we can prove the inclusion of property of Lebesgue spaces on a finite measure space directly via Hölder's inequality. What we showed here is that we can obtain the result through the lens of Orlicz spaces.

3 Inclusion Property of Weak Orlicz Spaces

First, we recall the definition of weak Orlicz spaces [14]. Let Φ be a Young function. We define the weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ to be the set of measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that $\| f \|_{wL_\Phi(\mathbb{R}^n)} < \infty$, where

$$\| f \|_{wL_\Phi(\mathbb{R}^n)} := \inf\left\{b > 0: \sup_{t>0} \Phi(t)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq 1\right\}.$$

Remark. Note that $\|\cdot\|_{wL_\Phi(\mathbb{R}^n)}$ defines a quasi-norm in $wL_\Phi(\mathbb{R}^n)$, and that $(wL_\Phi(\mathbb{R}^n), \|\cdot\|_{wL_\Phi(\mathbb{R}^n)})$ forms a quasi-Banach space (see [11,12]).

The relation between weak Orlicz spaces and (strong) Orlicz spaces is clear, as presented in the following theorem.

Theorem 3.1 [12,17] *Let Φ be a Young function. Then $L_\Phi(\mathbb{R}^n) \subset wL_\Phi(\mathbb{R}^n)$ with $\| f \|_{wL_\Phi(\mathbb{R}^n)} \leq \| f \|_{L_\Phi(\mathbb{R}^n)}$ for every $f \in L_\Phi(\mathbb{R}^n)$.*

Proof. The proof of this theorem can be found in [12,17]. We rewrite the proof here for convenience.

Given $f \in L_\Phi(\mathbb{R}^n)$, let $A_{\Phi,w} := \{b > 0: \sup_{t>0} \Phi(t)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq 1\}$ and $B_{\Phi,w} := \left\{b > 0: \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x|)}{b}\right) dx \leq 1\right\}$. Then $\| f \|_{wL_\Phi(\mathbb{R}^n)} = \inf A_{\Phi,w}$ and $\| f \|_{L_\Phi(\mathbb{R}^n)} = \inf B_{\Phi,w}$. Observe that, for arbitrary $b \in B_{\Phi,w}$ and $t > 0$, we have

$$\Phi(t)\left|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}\right| \leq \int_{\{x\in\mathbb{R}^n: \frac{|f(x)|}{b} > t\}} \Phi\left(\frac{|f(x)|}{b}\right) dx$$

$$\leq \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \leq 1.$$

Since $t > 0$ is arbitrary, we have $\sup_{t>0} \Phi(t)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq 1$, and $B_{\Phi,w} \subseteq A_{\Phi,w}$. Hence, $f \in wL_\Phi$ with $\| f \|_{wL_\Phi(\mathbb{R}^n)} \leq \| f \|_{L_\Phi(\mathbb{R}^n)}$.

Remark. As the strong and weak Orlicz spaces contain the strong and weak Lebesgue spaces respectively, the inclusion in the above theorem is proper. See [18] for a counterexample.

In addition to Lemma 2.4, we have the following lemma for the characteristic functions of balls in weak Orlicz spaces.

Lemma 3.2 [13] *Let Φ be a Young function, $a \in \mathbb{R}^n$, and $r > 0$ be arbitrary. Then we have* $\| \chi_{B(a,r)} \|_{wL_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)}$.

Now we come to the inclusion property of weak Orlicz spaces.

Theorem 3.3 *Let Φ, Ψ be Young functions. Then the following statements are equivalent:*

(1) $\Phi(t) \leq \Psi(Ct)$ *for every* $t > 0$.
(2) $wL_\Psi(\mathbb{R}^n) \subseteq wL_\Phi(\mathbb{R}^n)$.
(3) *For every* $f \in wL_\Psi(\mathbb{R}^n)$, *we have* $\| f \|_{wL_\Phi(\mathbb{R}^n)} \leq C \| f \|_{wL_\Psi(\mathbb{R}^n)}$.

Proof. Assume that (1) holds, and let $f \in wL_\Psi(\mathbb{R}^n)$. Put

$$A_{\Phi,w} = \{b > 0: \sup_{t>0} \Phi(t)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq 1\}$$

and

$$A_{\Psi,w} = \{b > 0: \sup_{t>0} \Psi(Ct)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq 1\}$$
$$= \{b > 0: \sup_{s>0} \Psi(s)|\{x \in \mathbb{R}^n: \frac{C|f(x)|}{b} > s\}| \leq 1\},$$

for $s = Ct$. Then $\| f \|_{wL_\Phi(\mathbb{R}^n)} = \inf A_{\Phi,w}$ and $\| Cf \|_{wL_\Psi(\mathbb{R}^n)} = \inf A_{\Psi,w}$. Observe that, for arbitrary $b \in A_{\Psi,w}$ and $t > 0$, we have

$$\Phi(t)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq \Psi(Ct)|\{x \in \mathbb{R}^n: \frac{|f(x)|}{b} > t\}| \leq 1.$$

Thus, $\sup_{t>0} \Phi(t)|\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}| \leq 1$. Hence it follows that $b \in A_{\Phi,w}$, and so we conclude that $A_{\Psi,w} \subseteq A_{\Phi,w}$. Accordingly, we obtain

$$\| f \|_{wL_\Phi(\mathbb{R}^n)} = \inf A_{\Phi,w} \leq \inf A_{\Psi,w} = C \| f \|_{wL_\Psi(\mathbb{R}^n)},$$

which also proves that $wL_\Psi(\mathbb{R}^n) \subset wL_\Phi(\mathbb{R}^n)$.
As mentioned in [10, Appendix G], we are aware that [16, Lemma 3.3] still holds for quasi-Banach spaces, and so (2) and (3) are equivalent.

Assume now that (3) holds. By Lemma 3.2, we have

$$\frac{1}{\Phi^{-1}\left(\frac{1}{|B(a,r_0)|}\right)} = \| \chi_{B(a,r_0)} \|_{wL_\Phi(\mathbb{R}^n)} \leq C \| \chi_{B(a,r_0)} \|_{wL_\Psi(\mathbb{R}^n)} = \frac{C}{\Psi^{-1}\left(\frac{1}{|B(a,r_0)|}\right)},$$

or

$$C\Phi^{-1}\left(\frac{1}{|B(a,r_0)|}\right) \geq \Psi^{-1}\left(\frac{1}{|B(a,r_0)|}\right),$$

for arbitrary $a \in \mathbb{R}^n$ and $r_0 > 0$. By Lemma 1.1, we have

$$\Phi\left(\frac{1}{|B(a,r_0)|}\right) \leq \Psi\left(\frac{C}{|B(a,r_0)|}\right).$$

Since $a \in \mathbb{R}^n$ and $r_0 > 0$ are arbitrary, we conclude that $\Phi(t) \leq \Psi(Ct)$ for every $t > 0$.

4 Concluding Remarks

We have proved the inclusion property of (strong) Orlicz spaces and of weak Orlicz spaces. Both proofs use the norm of the characteristic functions of the balls in $\mathbb{R}^n$. As our final conclusion, we have the following corollary which states that the inclusion property of (strong) Orlicz spaces are equivalent to that of weak Orlicz spaces, and both can be observed just by comparing the associated Young functions. To be precise, if Φ, Ψ are two Young functions, then the following statements are equivalent:

(1) $\Phi(t) \leq \Psi(Ct)$ for every $t > 0$.
(2) $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$.
(3) For every $f \in L_\Psi(\mathbb{R}^n)$, we have $\| f \|_{L_\Phi(\mathbb{R}^n)} \leq C \| f \|_{L_\Psi(\mathbb{R}^n)}$.
(4) $wL_\Psi(\mathbb{R}^n) \subseteq wL_\Phi(\mathbb{R}^n)$.
(5) For every $f \in wL_\Psi(\mathbb{R}^n)$, we have $\| f \|_{wL_\Phi(\mathbb{R}^n)} \leq \| f \|_{wL_\Psi(\mathbb{R}^n)}$.

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