Inclusion Properties for a Class of Meromorphic Functions Defined by a Linear Operator

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Abstract. This study targets a specific class of meromorphic univalent functions $f(z)$ defined by the linear operator $L(a,b)f(z)$. This paper aims to demonstrate some properties for the class $\Sigma_{a,b}^{k,\lambda}(h)$ to satisfy a certain subordination.

Keywords: *meromorphic functions; hypergeometric functions; subordination; linear operator; Hadamard product (convolution).*

AMS Subject Classification: 30C45.

1 Introduction

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n, \tag{1}$$

which are analytic in the punctured unit disk

$$\Delta^*=\{z:z\in\mathbb{C} \quad \text{and} \quad 0<|z|<1\}=\Delta\setminus\{0\}.$$

For functions $f_k(z)\ (k=1,2)$ given by

$$f_k(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,k}z^n \qquad (k=1,2), \tag{2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1*f_2)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,1}a_{n,2}z^n. \tag{3}$$

Received December 1st, 2015, Revision December 22nd, 2016; Accepted December 30th, 2016.
 ISSN: 2337-5760, DOI: 10.5614/j.math.fund.sci.2016.48.3.8

Let the function $\phi(a,b;z)$ be defined by

$$\phi(a,b;z)=\frac{1}{z}+\sum_{n=0}^{\infty}\left|\frac{(a)_{n+1}}{(b)_{n+1}}\right|z^{n}, \tag{4}$$

for $b\neq 0,-1,-2,...,$ and $a\in\mathbb{C}\setminus\{0\}$.

Here, and in the remainder of this paper, $(\lambda)_{\kappa}\,(\lambda,\kappa\in\mathbb{C})$ denotes the general Pochhammer symbol defined, in terms of the gamma function, by

$$(\lambda)_{\kappa}:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}=\begin{cases}1 & (\kappa=0;\lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)...(\lambda+n-1) & (\kappa=n\in\mathbb{N};\lambda\in\mathbb{C})\end{cases}. \tag{5}$$

Corresponding to the function $\phi(a,b;z)$, using the Hadamard product for $f(z)\in\Sigma,$ we define a new linear operator $L(a,b)$ on Σ by

$$L(a,b)f(z)=\phi(a,b;z)*f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left|\frac{(a)_{n+1}}{(b)_{n+1}}\right|a_{n}z^{n}. \tag{6}$$

The generalized and Gaussian hypergeometric functions together with the meromorphic functions were studied recently by several authors [1-9].

We define the following operator for the function $f\in L(a,b)f(z)$ by

$$D^{0}\left(L(a,b)f(z)\right)=L(a,b)f(z)$$

and for *k = 1,2,3,...,*

$$D^{k}\left(L(a,b)f(z)\right)=z\left(D^{k-1}L(a,b)f(z)\right)'+\frac{2}{z}=\frac{1}{z}+\sum_{n=1}^{\infty}n^{k}\left|\frac{(a)_{n+1}}{(b)_{n+1}}\right|a_{n}z^{n}. \tag{7}$$

The above differential operator D^{k} was studied by Ghanim and Darus [10-12].

In addition, we derive from the Eq. (6) and Eq. (7)

$$z\left(L(a,b)f(z)\right)'=aL(a+1,b)f(z)-(a+1)L(a,b)f(z). \tag{8}$$

and

$$z\left(D^{k}L(a,b)f(z)\right)'=aD^{k}L(a+1,b)f(z)-(a+1)D^{k}L(a,b)f(z). \tag{9}$$

respectively.

Let Ω be the class of all analytic, convex and univalent functions in the open unit disk and let $h(z)\in\Omega$ satisfy $h(0)=1$, with

$$\Re\{h(z)\}>0,\ |z|<1. \tag{10}$$

For two functions $f,g\in\Omega$, we say that f is *subordinate* to g or g is *superordinate* to f in Δ and write $f\prec g,\ z\in\Delta,$ if there exists a Schwarz function ω, analytic in Δ with $\omega(0)=0$ and $|\omega(z)|\leq 1$ when $z\in\Delta$ such that $f(z)=g(\omega(z)),z\in\Delta.$ Furthermore, if function g is univalent in Δ, then we have the following equivalence:

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\ \text{ and }\ f(\Delta)\subset g(\Delta),\qquad (z\in\Delta).$$

Definition. If a function $f\in\Sigma$ satisfies the following subordination condition

$$(1+\lambda)z\left(D^kL(a,b)f(z)\right)+\lambda z^2\left(D^kL(a,b)f(z)\right)'\prec h(z) \tag{11}$$

then f is in the class $\Sigma_{a,b}^{k,\lambda}(h)$, where λ is a complex number and $h(z)\in\Omega$. Let A be a class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{12}$$

which are analytic in Δ.

A function $f(z)\in A$ is in the class of starlike functions $S^*(\alpha)$ of order α in Δ, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\qquad (z\in\Delta),$$

for some $\alpha,\ 0<\alpha<1.$

A function $f(z)\in A$ is in the class of prestarlike function $R(\alpha)$ of order α in Δ, if

$$\frac{z}{(1-z)^{2(1-\alpha)}}*f(z)\in S^*(\alpha)\qquad (\alpha<1)$$

(see for example [13-15]). $f(z)$ is convex univalent in Δ and $R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$ if and only if $f(z) \in R(0)$.

2 Preliminary Results

Lemma 1. [16] Let $g(z)$ and $h(z)$ are two analytic functions in Δ. $h(z)$ is convex univalent with $h(0)=g(0)$. If

$$g(z)+\frac{1}{\mu}zg'(z) \prec h(z) \tag{13}$$

where $\Re\mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu}\int_0^z t^{\mu-1}h(t)\,dt \prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of Eq. (13).

Lemma 2. [13] *If* $\Re a \geq 0$ *and* $a \neq 0$, then,

$$\Sigma_{a,b}^{k,\lambda}\left(\tilde{h}\right) \subset \Sigma_{a,b}^{k,\lambda}(h),$$

where

$$\tilde{h}(z) = az^{-a}\int_0^z t^{a-1}h(t)\,dt \prec h(z).$$

Lemma 3. [13] *If* $f(z) \in \Sigma_{a,b}^{k,\lambda}(h)$, $g(z) \in \Sigma$ and $\Re\left(zg(z)\right) > \frac{1}{2} \qquad (z \in \Delta)$,

then,

$$(f * g)(z) \in \Sigma_{a,b}^{k,\lambda}(h).$$

3 Main Results

Theorem 1. Let $f(z) \in \Sigma_{a,b}^{k,\lambda}(h)$. Then $F(z)$ is the function defined by

$$F(z) = \frac{\mu-1}{z^{\mu}}\int_0^z t^{\mu-1}f(t)\,dt \qquad (\Re\mu > 1) \tag{14}$$

and in the class $\Sigma_{a,b}^{k,\lambda}\left(\tilde{h}\right)$, where

$$\tilde{h}(z) = (\mu - 1) z^{1-\mu} \int_0^z t^{\mu-2} h(t)\, dt \prec h(z).$$

Proof. For $f(z) \in \Sigma$ and $\Re\mu > 1$, we can obtain from (14) that $F(z) \in \Sigma$ and

$$(\mu - 1) f(z) = \mu F(z) + zF'(z), \qquad F(z) \in \Sigma. \tag{15}$$

Define $H(z)$ by

$$H(z) = (1+\lambda) z \left(D^k L(a,b) F(z)\right) + \lambda z^2 \left(D^k L(a,b) F(z)\right)'. \tag{16}$$

From Eq. (15) and Eq. (16) it follows that:

$$(1+\lambda) z \left(D^k L(a,b) f(z)\right) + \lambda z^2 \left(D^k L(a,b) f(z)\right)'$$

$$= (1+\lambda) z \left(D^k L(a,b) \left(\frac{\mu F(z) + zF'(z)}{\mu - 1} \right) \right) + \lambda z^2 \left(D^k L(a,b) \left(\frac{\mu F(z) + zF'(z)}{\mu - 1} \right) \right)'$$

$$= \frac{\mu}{\mu - 1} H(z) + \frac{1}{\mu - 1} \left(zH'(z) - H(z) \right) = H(z) + \frac{zH'(z)}{\mu - 1}. \tag{17}$$

Let $f(z) \in \Sigma_{a,b}^{k,\lambda}(h)$. Then, by Eq. (17)

$$H(z) + \frac{zH'(z)}{\mu - 1} \prec h(z) \qquad (\Re\mu > 1),$$

and hence we obtain from Lemma 1:

$$H(z) \prec \tilde{h}(z) = (\mu - 1) z^{1-\mu} \int_0^z t^{\mu-2} h(t)\, dt \prec h(z).$$

Thus, Lemma 2 contributes to

$$F(z) \in \Sigma_{a,b}^{k,\lambda}(\tilde{h}) \subset \Sigma_{a,b}^{k,\lambda}(h).$$

Theorem 2. Let $F(z)$ be defined as in Eq. (14) and $f(z) \in \Sigma$. If

$$(1+\alpha) z \left(D^k L(a,b) F(z)\right) + \alpha z \left(D^k L(a,b) f(z)\right) \prec h(z) \quad (\alpha > 0), \tag{18}$$

then $F(z) \in \Sigma_{a,b}^{k}(\tilde{h}) = \Sigma_{a,b}^{k,0}(\tilde{h})$, where $\Re\mu > 1$ and

$$\tilde{h}(z) = \frac{(\mu - 1)}{\alpha} z^{\frac{1-\mu}{\alpha}} \int_0^z t^{\frac{\mu-1}{\alpha} - 1} h(t)\, dt \prec h(z).$$

Proof. Let us define the analytic function $H(z)$ in Δ as follows:

$$H(z)=z\left(D^{k}L(a,b)F(z)\right) \tag{19}$$

with $H(0)=1$, and

$$zH'(z)=H(z)+z^{2}\left(D^{k}L(a,b)F(z)\right)'. \tag{20}$$

By using Eq. (15), Eq. (18), Eq. (19) and Eq. (20), we conclude that:

$$(1-\alpha)z\left(D^{k}L(a,b)F(z)\right)+\alpha z\left(D^{k}L(a,b)f(z)\right)$$

$$=(1-\alpha)z\left(D^{k}L(a,b)F(z)\right)+\frac{\alpha}{\mu-1}\left(\mu zD^{k}L(a,b)F(z)\right)+z^{2}\left(D^{k}L(a,b)F(z)\right)'$$

$$=H(z)+\frac{\alpha}{\mu-1}zH'(z)\prec h(z)$$

for $\Re\mu>1$ and $\alpha>0$.

Therefore, an application of Lemma 1 asserts Theorem 2.

Theorem 3. Let $f(z)\in\Sigma_{a,b}^{k,\lambda}(h)$. If $F(z)$ is the function given by

$$F(z)=\frac{\mu-1}{z^{\mu}}\int_{0}^{z}t^{\mu-1}f(t)\,dt \qquad (\mu>1) \tag{21}$$

then,

$$\sigma f(\sigma z)\in\Sigma_{a,b}^{k,\lambda}(h)$$

where

$$\sigma=\sigma(\mu)=\frac{\sqrt{\mu^{2}-2(\mu-1)}-1}{(\mu-1)}\in(0,1). \tag{22}$$

When

$$h(z)=\delta+(1-\delta)\frac{1+z}{1-z} \qquad (\delta\neq1) \tag{23}$$

consequently, bound σ is sharp.

Proof. For $F(z)\in\Sigma_{a,b}^{k,\lambda}(h)$, we could verify that: $F(z)=F(z)*\dfrac{z^{-1}}{1-z}$ and $zF'(z)=F(z)*\left(\dfrac{1}{(1-z)^2}-\dfrac{1}{z(1-z)}\right)$.

Then, using Eq. (21), we obtain:

$$f(z)=\frac{\mu F(z)+zF'(z)}{\mu-1}=(F*g)(z) \qquad \left(z\in\Delta^*,\ \mu>1\right), \tag{24}$$

where

$$g(z)=\frac{1}{\mu-1}\left(\frac{1}{z(1-z)^2}-(\mu-1)\frac{1}{z(1-z)}\right)\in\Sigma. \tag{25}$$

Now, we prove that:

$$\Re\left(zg(z)\right)>\frac{1}{2} \qquad \left(|z|<\sigma\right), \tag{26}$$

where $\sigma=\sigma(\mu)$ is given by Eq. (22). Setting

$$\frac{1}{1-z}=\mathrm{Re}^{i\theta} \qquad \left(R>0,|z|=r<1\right)$$

we have:

$$\cos\theta=\frac{1+R^2\left(1-r^2\right)}{2R} \text{ and } R\geq\frac{1}{1+r}. \tag{27}$$

By Eq. (25) and Eq. (27) with $\mu>1$, we have:

$$2\Re\left\{zg(z)\right\}=\frac{2}{\mu-1}\left[(\mu-1)R\cos\theta+R^2\left(2\cos^2\theta-1\right)\right]$$

$$=\frac{1}{\mu-1}\left[(\mu-1)\left(1+R^2\left(1-r^2\right)\right)+\left(1+R^2\left(1-r^2\right)\right)^2-R^2\right]$$

$$=\frac{R^2}{\mu-1}\left[R^2\left(1-r^2\right)^2+\mu\left(1-r^2\right)-1\right]+1\geq\frac{R^2}{\mu-1}\left[\left(1-r^2\right)^2+\mu\left(1-r^2\right)-1\right]+1$$

$$=\frac{R^2}{\mu-1}\left[(1-\mu)r^2+\mu-2r\right]+1.$$

This would eventually give Eq. (26) and hence

$$\Re\left(z\sigma g\left(\sigma z\right)\right) > \frac{1}{2} \qquad \left(z \in \Delta\right). \tag{28}$$

Let $F\left(z\right) \in \Sigma_{a,b}^{k,\lambda}\left(h\right)$. Using Eq. (24) and Eq. (28) with Lemma 3, we have:

$$\sigma f\left(\sigma z\right) = F\left(z\right) * \sigma g\left(\sigma z\right) \in \Sigma_{a,b}^{k,\lambda}\left(h\right).$$

For $h\left(z\right)$ defined by Eq. (23), function $F\left(z\right) \in \Sigma$ is given by:

$$\left(1+\lambda\right)z\left(D^k L\left(a,b\right)F\left(z\right)\right) + \lambda z^2\left(D^k L\left(a,b\right)F\left(z\right)\right)' = \delta + \left(1-\delta\right)\frac{1+z}{1-z}. \tag{29}$$

$\left(\delta \neq 1\right)$. By using Eq. (29), Eq. (16) and Eq. (17), we obtain the following:

$$\left(1+\lambda\right)z\left(D^k L\left(a,b\right)f\left(z\right)\right) + \lambda z^2\left(D^k L\left(a,b\right)f\left(z\right)\right)'$$

$$= \delta + \left(1-\delta\right)\frac{1+z}{1-z} + \frac{z}{\mu-1}\left(\delta + \left(1-\delta\right)\frac{1+z}{1-z}\right)'$$

$$= \delta + \frac{\left(1-\delta\right)\left(\mu+2z-1+\left(1-\mu\right)z^2\right)}{\left(\mu-1\right)\left(1-z\right)^2} = \delta \qquad \left(\sigma = -z\right).$$

Hence, for each $\mu\left(\mu > 1\right)$ the bound $\sigma = \sigma\left(\mu\right)$ cannot be increased.

Acknowledgements

The authors would like to thank the referees for giving suggestions for improving this work. The work was supported by FRGS/1/2016/STG06/UKM/01/1.

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