New Generalized Algorithm for Developing k-Step Higher Derivative Block Methods for Solving Higher Order Ordinary Differential Equations

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Abstract. This article presents a new generalized algorithm for developing k-step $(m+1)^{th}$ derivative block methods for solving m^{th} order ordinary differential equations. This new algorithm utilizes the concept from the conventional Taylor series approach of developing linear multistep methods. Certain examples are given to show the simplicity involved in the usage of this new generalized algorithm.

Keywords: *block methods; generalized algorithm; higher derivative; higher order; k-step; Taylor series.*

1 Introduction

The main focus of numerical methods for solving differential equations in recent times has been placed on presenting numerical methods with an improved level of accuracy, ranging from first order ordinary differential equations [1-3] to higher order ordinary differential equations [4-6]. Numerical methods adopted to solve such ODEs have been developed using two popular approaches, i.e. numerical integration and collocation. However, there is a need to introduce new approaches to develop these numerical methods so that they become easier and less burdensome compared to the conventional approaches. Referring to the work of [7], three basic methods can be highlighted for developing single linear multistep methods: numerical integration, interpolation and Taylor series expansion. Although the derivations originally involved just single multistep methods for first order ordinary differential equations, over time the numerical methods evolved into a family of block methods and a further improvement with the introduction of higher derivatives, sometimes referred to as Obrechkoff type [8,9].

The aim of this article is to present a novel algorithm that is convenient to adopt for developing new generalized algorithm k-step $(m+1)^{th}$ derivative block methods for solving any m^{th} order ordinary differential equations. This work is

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an extension of [10], which proposed a generalized algorithm for developing block methods without the introduction of higher derivatives.

This article is organized as follows: Section 2 presents the generalized algorithm for developing k-step higher derivative block methods with sample second order and third order block methods developed in Sections 3 and 4 respectively. The basic properties of the block methods developed in Sections 3 and 4 are also highlighted. Certain numerical examples are considered in Section 5 and the article is concluded in Section 6.

2 Generalized Algorithm for k-Step Higher Derivative Block Methods

Considering the general form of m^{th} order ordinary differential equation

$$y^{(m)} = f\left(x, y, y', y'', \cdots, y^{(m-1)}\right),\ x \in [a,b]$$

Extending the generalized algorithm presented by [10] for k-step block methods to accommodate the presence of higher derivatives, the following algorithm is presented for developing k-step $(m+1)^{th}$ derivative block methods for solving m^{th} order ODEs.

$$y_{n+\xi} = \sum_{i=0}^{m-1} \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{k}\left(\phi_{\xi i} f_{n+i} + \tau_{\xi i} g_{n+i}\right),\ \xi = 1,\ 2,\ \cdots,\ k \tag{1}$$

with corresponding derivatives obtained from

$$\begin{aligned} y_{n+\xi}^{(a)} &= \sum_{i=0}^{m-(a+1)} \frac{(\xi h)^i}{i!} y_n^{(i+a)} + \sum_{i=0}^{k}\left(\omega_{\xi i a} f_{n+i} + \varphi_{\xi i a} g_{n+i}\right), \\ a &= 1_{(\xi)},\ 2_{(\xi)},\ \cdots,\ (m-1)_{(\xi)} \end{aligned} \tag{2}$$

where

$$f_{n+i} = f\left(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i}, \cdots, y_{n+i}^{(m-1)}\right),$$

$$g_{n+i} = \frac{df_{n+i} = f\left(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i}, \cdots, y_{n+i}^{(m-1)}\right)}{dx},$$

and the condition $k \geq m$ is set to avoid an underdetermined system. The unknown $\emptyset$, τ, ω and φ coefficients are obtained from $\left(\phi_{\xi 0}, \phi_{\xi 1}, \cdots, \phi_{\xi k}, \tau_{\xi 0}, \tau_{\xi 1}, \cdots, \tau_{\xi k}\right)^T = A^{-1}B$ and

$$\left(\omega_{\xi 0a},\omega_{\xi 1a},\cdots,\omega_{\xi ka},\varphi_{\xi 0a},\varphi_{\xi 1a},\cdots,\varphi_{\xi ka}\right)^T = A^{-1}D;$$

$$A_{(2k+2)\times(2k+2)} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & h & 2h & \cdots & kh & 1 & 1 & 1 & \cdots & 1 \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \cdots & \frac{(kh)^2}{2!} & 0 & h & 2h & \cdots & kh \\ 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \cdots & \frac{(kh)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \cdots & \frac{(kh)^2}{2!} \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ 0 & \frac{(h)^{(2k+1)}}{(2k+1)!} & \frac{(2h)^{(2k+1)}}{(2k+1)!} & \cdots & \frac{(kh)^{(2k+1)}}{(2k+1)!} & 0 & \frac{(h)^{(2k)}}{(2k)!} & \frac{(2h)^{(2k)}}{(2k)!} & \cdots & \frac{(kh)^{(2k)}}{(2k)!} \end{pmatrix}$$

$$B_{(2k+2)\times 1} = \begin{pmatrix} \frac{(\xi h)^m}{m!} \\ \frac{(\xi h)^{(m+1)}}{(m+1)!} \\ \frac{(\xi h)^{(m+2)}}{(m+2)!} \\ . \\ . \\ . \\ \frac{(\xi h)^{((2k+2)+(m-1))}}{((2k+2)+(m-1))!} \end{pmatrix}, \quad D_{(2k+2)\times 1} = \begin{pmatrix} \frac{(\xi h)^{(m-1)}}{(m-1)!} \\ \frac{(\xi h)^m}{m!} \\ \frac{(\xi h)^{(m+1)}}{(m+1)!} \\ . \\ . \\ . \\ \frac{(\xi h)^{((2k+2)+(m-2))}}{((2k+2)+(m-2))!} \end{pmatrix} \tag{3}$$

To ascertain that there are no linearly dependent columns or rows $(\det(A) \neq 0)$, the rank of the matrix is investigated when developing the block methods. Note that the approach in [7] for developing multistep methods using Taylor series expansion is adopted to expand individual terms in Eq. (1) and Eq. (2) to obtain the unknown coefficients. The conventional Taylor expansions concept is defined for $y_{n+a}^{(n)} = y^{(n)}(x_n + ah)$ about x_n as

$$y^{(n)}\left(x_n + ah\right) = y^{(n)}\left(x_n\right) + ahy^{(n+1)}\left(x_n\right) + \frac{(ah)^2}{2!}y^{(n+2)}\left(x_n\right) + \cdots \tag{4}$$

A more detailed explanation is given in the following subsections, where the new generalized algorithm is used to develop k-step $(m+1)^{th}$ derivative block methods for $(m,k) = (2,2)$ and $(m,k) = (3,3)$.

3 Development of Two-Step Third Derivative Block Methods for Second Order ODEs

Consider developing a two-step third derivative block method for second order ODEs, that is $k = 2$ and $m = 2$. Substituting k and m in Eq. (1) and Eq. (2) gives

$$y_{n+\xi} = \sum_{i=0}^{1} \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{2} \left(\phi_{\xi i} f_{n+i} + \tau_{\xi i} g_{n+i} \right), \ \xi = 1, \ 2 \tag{5}$$

and

$$y_{n+\xi}^{(1)} = y_n^{(1)} + \sum_{i=0}^{2} \left(\omega_{\xi i 1} f_{n+i} + \varphi_{\xi i 1} g_{n+i} \right), \ \xi = 1, \ 2 \tag{6}$$

where $a = 1$.

Therefore,

$$\begin{aligned}
y_{n+1} &= y_n + hy_n^{(1)} + \left(\phi_{10} f_n + \phi_{11} f_{n+1} + \phi_{12} f_{n+2} + \tau_{10} g_n + \tau_{11} g_{n+1} + \tau_{12} g_{n+2} \right), \\
y_{n+2} &= y_n + 2hy_n^{(1)} + \left(\phi_{20} f_n + \phi_{21} f_{n+1} + \phi_{22} f_{n+2} + \tau_{20} g_n + \tau_{21} g_{n+1} + \tau_{22} g_{n+2} \right), \\
y_{n+1}^{(1)} &= y_n^{(1)} + \left(\omega_{101} f_n + \omega_{111} f_{n+1} + \omega_{121} f_{n+2} + \varphi_{101} g_n + \varphi_{111} g_{n+1} + \varphi_{121} g_{n+2} \right), \\
y_{n+2}^{(1)} &= y_n^{(1)} + \left(\omega_{201} f_n + \omega_{211} f_{n+1} + \omega_{221} f_{n+2} + \varphi_{201} g_n + \varphi_{211} g_{n+1} + \varphi_{221} g_{n+2} \right)
\end{aligned} \tag{7}$$

Expanding y_{n+1} using Taylor series yields:

$$\begin{aligned}
& y(x_n) + hy^{(1)}(x_n) + \tfrac{(h)^2}{2!} y^{(2)}(x_n) + \tfrac{(h)^3}{3!} y^{(3)}(x_n) + \tfrac{(h)^4}{4!} y^{(4)}(x_n) \\
& + \tfrac{(h)^5}{5!} y^{(5)}(x_n) + \tfrac{(h)^6}{6!} y^{(6)}(x_n) + \tfrac{(h)^7}{7!} y^{(7)}(x_n) + \cdots = y(x_n) \\
& + hy^{(1)}(x_n) + \{\phi_{10}\left(y^{(2)}(x_n) \right) + \phi_{11}(y^{(2)}(x_n) + hy^{(3)}(x_n) \\
& + \tfrac{(h)^2}{2!} y^{(4)}(x_n) + \tfrac{(h)^3}{3!} y^{(5)}(x_n) + \cdots) + \phi_{12}(y^{(2)}(x_n) + 2hy^{(3)}(x_n) \\
& + \tfrac{(2h)^2}{2!} y^{(4)}(x_n) + \tfrac{(2h)^3}{3!} y^{(5)}(x_n) + \cdots) + \tau_{10}\left(y^{(3)}(x_n) \right) + \tau_{11}(y^{(3)}(x_n) \\
& + hy^{(4)}(x_n) + \tfrac{(h)^2}{2!} y^{(5)}(x_n) + \tfrac{(h)^3}{3!} y^{(6)}(x_n) + \cdots) + \tau_{12}(y^{(3)}(x_n) \\
& + 2hy^{(4)}(x_n) + \tfrac{(2h)^2}{2!} y^{(5)}(x_n) + \tfrac{(2h)^3}{3!} y^{(6)}(x_n) + \cdots)\}
\end{aligned}$$

Equating coefficients of $y^{(n)}(x_n)$ on both the left- and right-hand sides of the equation gives the following system of equations:

$$\tfrac{(h)^2}{2!} = \phi_{10} + \phi_{11} + \phi_{12}$$

$$\begin{aligned}
\tfrac{(h)^3}{3!} &= h\phi_{11} + 2h\phi_{12} + \tau_{10} + \tau_{11} + \tau_{12} \\
\tfrac{(h)^4}{4!} &= \tfrac{(h)^2}{2!}\phi_{11} + \tfrac{(2h)^2}{2!}\phi_{12} + h\tau_{11} + 2h\tau_{12} \\
\tfrac{(h)^5}{5!} &= \tfrac{(h)^3}{3!}\phi_{11} + \tfrac{(2h)^3}{3!}\phi_{12} + \tfrac{(h)^2}{2!}\tau_{11} + \tfrac{(2h)^2}{2!}\tau_{12} \\
\tfrac{(h)^6}{6!} &= \tfrac{(h)^4}{4!}\phi_{11} + \tfrac{(2h)^4}{4!}\phi_{12} + \tfrac{(h)^3}{3!}\tau_{11} + \tfrac{(2h)^3}{3!}\tau_{12} \\
\tfrac{(h)^7}{7!} &= \tfrac{(h)^5}{5!}\phi_{11} + \tfrac{(2h)^5}{5!}\phi_{12} + \tfrac{(h)^4}{4!}\tau_{11} + \tfrac{(2h)^4}{4!}\tau_{12}
\end{aligned} \tag{8}$$

where truncation is made such that the number of unknowns equals the number of equations. Rewriting Eq. (8) in matrix form $Ax = B$ gives:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & h & 2h & 1 & 1 & 1 \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & 0 & h & 2h \\ 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} \\ 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} \\ 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} \end{pmatrix} \begin{pmatrix} \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \tau_{10} \\ \tau_{11} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} \frac{(h)^2}{2!} \\ \frac{(h)^3}{3!} \\ \frac{(h)^4}{4!} \\ \frac{(h)^5}{5!} \\ \frac{(h)^6}{6!} \\ \frac{(h)^7}{7!} \end{pmatrix} \tag{9}$$

where $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & h & 2h & 1 & 1 & 1 \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & 0 & h & 2h \\ 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} \\ 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} \\ 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} \end{pmatrix}$ has *rank* = 6, which implies that there are no linearly dependent columns or rows and the inverse exists. These matrices correspond to the definitions assigned in Eq. (3) above and to obtain the unknown coefficients, the matrix inverse approach is adopted to obtain

$$\left(\phi_{10}, \phi_{11}, \phi_{12}, \tau_{10}, \tau_{11}, \tau_{12}\right)^T = \left(\tfrac{13h^2}{42}, \tfrac{h^2}{6}, \tfrac{h^2}{42}, \tfrac{59h^3}{1680}, -\tfrac{8h^3}{105}, -\tfrac{11h^3}{1680}\right).$$

Following the same approach for y_{n+2}, the unknown coefficients are obtained as:

$$\left(\phi_{20}, \phi_{21}, \phi_{22}, \tau_{20}, \tau_{21}, \tau_{22}\right)^T = \left(\tfrac{79h^2}{105}, \tfrac{16h^2}{15}, \tfrac{19h^2}{105}, \tfrac{2h^3}{21}, -\tfrac{16h^3}{105}, -\tfrac{4h^3}{105}\right).$$

Furthermore, in the case of obtaining the coefficients for $y_{n+1}^{(1)}$, the same approach is followed as for y_{n+1} using Taylor series expansion to obtain:

$$
\begin{aligned}
& y^{(1)}(x_n)+hy^{(2)}(x_n)+\tfrac{(h)^2}{2!}y^{(3)}(x_n)+\tfrac{(h)^3}{3!}y^{(4)}(x_n)+\tfrac{(h)^4}{4!}y^{(5)}(x_n) \\
& +\tfrac{(h)^5}{5!}y^{(6)}(x_n)+\tfrac{(h)^6}{6!}y^{(7)}(x_n)+\cdots = y^{(1)}(x_n)+\{\omega_{101}\left(y^{(2)}(x_n)\right) \\
& +\omega_{111}(y^{(2)}(x_n)+hy^{(3)}(x_n)+\tfrac{(h)^2}{2!}y^{(4)}(x_n)+\tfrac{(h)^3}{3!}y^{(5)}(x_n)+\cdots) \\
& +\omega_{121}(y^{(2)}(x_n)+2hy^{(3)}(x_n)+\tfrac{(2h)^2}{2!}y^{(4)}(x_n)+\tfrac{(2h)^3}{3!}y^{(5)}(x_n)+\cdots) \\
& +\varphi_{101}\left(y^{(3)}(x_n)\right)+\varphi_{111}(y^{(3)}(x_n)+hy^{(4)}(x_n)+\tfrac{(h)^2}{2!}y^{(5)}(x_n) \\
& +\tfrac{(h)^3}{3!}y^{(6)}(x_n)+\cdots)+\varphi_{121}(y^{(3)}(x_n)+2hy^{(4)}(x_n) \\
& +\tfrac{(2h)^2}{2!}y^{(5)}(x_n)+\tfrac{(2h)^3}{3!}y^{(6)}(x_n)+\cdots)\}
\end{aligned}
$$

Equating coefficients of $y^{(n)}(x_n)$ on both the left- and right-hand side of the equation gives the following system of equations:

$$
\begin{aligned}
& h=\omega_{101}+\omega_{111}+\omega_{121} \\
& \tfrac{(h)^2}{2!}=h\omega_{111}+2h\omega_{121}+\varphi_{101}+\varphi_{111}+\varphi_{121} \\
& \tfrac{(h)^3}{3!}=\tfrac{(h)^2}{2!}\omega_{111}+\tfrac{(2h)^2}{2!}\omega_{121}+h\varphi_{111}+2h\varphi_{121} \\
& \tfrac{(h)^4}{4!}=\tfrac{(h)^3}{3!}\omega_{111}+\tfrac{(2h)^3}{3!}\omega_{121}+\tfrac{(h)^2}{2!}\varphi_{111}+\tfrac{(2h)^2}{2!}\varphi_{121} \\
& \tfrac{(h)^5}{5!}=\tfrac{(h)^4}{4!}\omega_{111}+\tfrac{(2h)^4}{4!}\omega_{121}+\tfrac{(h)^3}{3!}\varphi_{111}+\tfrac{(2h)^3}{3!}\varphi_{121} \\
& \tfrac{(h)^6}{6!}=\tfrac{(h)^5}{5!}\omega_{111}+\tfrac{(2h)^5}{5!}\omega_{121}+\tfrac{(h)^4}{4!}\varphi_{111}+\tfrac{(2h)^4}{4!}\varphi_{121}
\end{aligned}
\tag{10}
$$

Rewriting Eq. (10) in matrix form $Ax=B$ gives:

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & h & 2h & 1 & 1 & 1 \\
0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & 0 & h & 2h \\
0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} \\
0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} \\
0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!}
\end{pmatrix}
\begin{pmatrix}
\omega_{101} \\ \omega_{111} \\ \omega_{121} \\ \varphi_{101} \\ \varphi_{111} \\ \varphi_{121}
\end{pmatrix}
=
\begin{pmatrix}
h \\ \frac{(h)^2}{2!} \\ \frac{(h)^3}{3!} \\ \frac{(h)^4}{4!} \\ \frac{(h)^5}{5!} \\ \frac{(h)^6}{6!}
\end{pmatrix}
\tag{11}
$$

which also corresponds to the definitions assigned in Eq. (3) above. The matrix inverse approach is adopted to obtain the unknown coefficients as:

$$\left(\omega_{101},\omega_{111},\omega_{121},\varphi_{101},\varphi_{111},\varphi_{121}\right)^T=\left(\tfrac{101h}{240},\tfrac{8h}{15},\tfrac{11h}{240},\tfrac{13h^2}{240},-\tfrac{h^2}{6},-\tfrac{h^2}{80}\right)$$

Similarly, the unknown coefficients for $y_{n+2}^{(1)}$ are obtained as:

$$\left(\omega_{201},\omega_{211},\omega_{221},\varphi_{201},\varphi_{211},\varphi_{221}\right)^T=\left(\tfrac{7h}{15},\tfrac{16h}{15},\tfrac{7h}{15},\tfrac{h^2}{15},0,-\tfrac{h^2}{15}\right)$$

Substituting all obtained coefficients back in Eq. (7) gives the two-step third derivative block method for solving second order ordinary differential equations as:

$$\begin{aligned}
y_{n+1}&=y_n+hy_n^{(1)}+\tfrac{h^2}{42}\left(13f_n+7f_{n+1}+f_{n+2}\right)+\tfrac{h^3}{1680}\left(59g_n-128g_{n+1}-11g_{n+2}\right),\\
y_{n+2}&=y_n+2hy_n^{(1)}+\tfrac{h^2}{105}\left(79f_n+112f_{n+1}+19f_{n+2}\right)+\tfrac{h^3}{105}\left(10g_n-16g_{n+1}-4g_{n+2}\right)\\
y_{n+1}^{(1)}&=y_n^{(1)}+\tfrac{h}{240}\left(101f_n+128f_{n+1}+11f_{n+2}\right)+\tfrac{h^2}{240}\left(13g_n-40g_{n+1}-3g_{n+2}\right),\\
y_{n+2}^{(1)}&=y_n^{(1)}+\tfrac{h}{15}\left(7f_n+16f_{n+1}+7f_{n+2}\right)+\tfrac{h^2}{15}\left(g_n-g_{n+2}\right)
\end{aligned}$$

4 Properties of the Two-Step Third-Derivative Block Method

The following properties of the two-step third-derivative block method are discussed: order, zero-stability, consistency and convergence.

Following the definition of Equation (11), the individual terms of the two-step third derivative block method are expanded using Taylor series expansions about $x=x_n$. The order of the two-step third derivative block method is obtained to be $p=6$.

Secondly, to analyze the two-step third derivative block method for zero-stability, the modulus of the roots of its first characteristic polynomial is expected to be simple or less than one. Thus, the correctors of the two-step third derivative block method are normalized to give the first characteristic polynomial as $\rho(r)=det\left[rI_2-\begin{pmatrix}0&1\\0&1\end{pmatrix}\right]$ with roots satisfying $|r_j|\le 1$.

The two-step third derivative block method is consistent if it has order $p\ge 1$ as satisfied in the previous paragraphs. Therefore, the two-step third derivative block method is convergent [12].

5 Development of Three-Step Fourth Derivative Block Methods for Third Order ODEs

Consider developing a three-step higher derivative block method for third order ODEs, i.e. $k = 3$ and $m = 3$.

Substituting k and m in Eq. (1) and Eq. (2) gives:

$$y_{n+\xi} = \sum_{i=0}^{2} \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{3} \left(\phi_{\xi i} f_{n+i} + \tau_{\xi i} g_{n+i} \right), \; \xi = 1,\; 2,\; 3 \tag{12}$$

and

$$\begin{aligned} y_{n+\xi}^{(1)} &= \sum_{i=0}^{1} \frac{(\xi h)^i}{i!} y_n^{(i+1)} + \sum_{i=0}^{3} \left(\omega_{\xi i1} f_{n+i} + \varphi_{\xi i1} g_{n+i} \right), \; \xi = 1,\; 2,\; 3 \\ y_{n+\xi}^{(2)} &= \sum_{i=0}^{0} \frac{(\xi h)^i}{i!} y_n^{(i+2)} + \sum_{i=0}^{3} \left(\omega_{\xi i2} f_{n+i} + \varphi_{\xi i2} g_{n+i} \right), \; \xi = 1,\; 2,\; 3 \end{aligned} \tag{13}$$

where $a = 1,2$.

$$y_{n+1} = y_n + hy_n^{(1)} + \tfrac{(h)^2}{2!} y_n^{(2)} + \begin{pmatrix} \phi_{10} f_n + \phi_{11} f_{n+1} + \phi_{12} f_{n+2} + \phi_{13} f_{n+3} \\ +\tau_{10} g_n + \tau_{11} g_{n+1} + \tau_{12} g_{n+2} + \tau_{13} g_{n+3} \end{pmatrix},$$

$$y_{n+2} = y_n + 2hy_n^{(1)} + \tfrac{(2h)^2}{2!} y_n^{(2)} + \begin{pmatrix} \phi_{20} f_n + \phi_{21} f_{n+1} + \phi_{22} f_{n+2} + \phi_{23} f_{n+3} \\ +\tau_{20} g_n + \tau_{21} g_{n+1} + \tau_{22} g_{n+2} + \tau_{23} g_{n+3} \end{pmatrix},$$

$$y_{n+3} = y_n + 3hy_n^{(1)} + \tfrac{(3h)^2}{2!} y_n^{(2)} + \begin{pmatrix} \phi_{30} f_n + \phi_{31} f_{n+1} + \phi_{32} f_{n+2} + \phi_{33} f_{n+3} \\ +\tau_{30} g_n + \tau_{31} g_{n+1} + \tau_{32} g_{n+2} + \tau_{33} g_{n+3} \end{pmatrix},$$

$$y_{n+1}^{(1)} = y_n^{(1)} + hy_n^{(2)} + \begin{pmatrix} \omega_{101} f_n + \omega_{111} f_{n+1} + \omega_{121} f_{n+2} + \omega_{131} f_{n+3} \\ +\varphi_{101} g_n + \varphi_{111} g_{n+1} + \varphi_{121} g_{n+2} + \varphi_{131} g_{n+3} \end{pmatrix},$$

$$y_{n+2}^{(1)} = y_n^{(1)} + 2hy_n^{(2)} + \begin{pmatrix} \omega_{201} f_n + \omega_{211} f_{n+1} + \omega_{221} f_{n+2} + \omega_{231} f_{n+2} \\ +\varphi_{201} g_n + \varphi_{211} g_{n+1} + \varphi_{221} g_{n+2} + \varphi_{231} g_{n+3} \end{pmatrix},$$

$$y_{n+3}^{(1)} = y_n^{(1)} + 3hy_n^{(2)} + \begin{pmatrix} \omega_{301} f_n + \omega_{311} f_{n+1} + \omega_{321} f_{n+2} + \omega_{331} f_{n+2} \\ +\varphi_{301} g_n + \varphi_{311} g_{n+1} + \varphi_{321} g_{n+2} + \varphi_{331} g_{n+3} \end{pmatrix},$$

$$y_{n+1}^{(2)} = y_n^{(2)} + \begin{pmatrix} \omega_{102} f_n + \omega_{112} f_{n+1} + \omega_{122} f_{n+2} + \omega_{132} f_{n+3} \\ +\varphi_{102} g_n + \varphi_{112} g_{n+1} + \varphi_{122} g_{n+2} + \varphi_{132} g_{n+3} \end{pmatrix},$$

$$y_{n+2}^{(2)} = y_n^{(2)} + \begin{pmatrix} \omega_{202} f_n + \omega_{212} f_{n+1} + \omega_{222} f_{n+2} + \omega_{232} f_{n+3} \\ +\varphi_{202} g_n + \varphi_{212} g_{n+1} + \varphi_{222} g_{n+2} + \varphi_{232} g_{n+3} \end{pmatrix},$$
$$y_{n+3}^{(2)} = y_n^{(2)} + \begin{pmatrix} \omega_{302} f_n + \omega_{312} f_{n+1} + \omega_{322} f_{n+2} + \omega_{332} f_{n+3} \\ +\varphi_{302} g_n + \varphi_{312} g_{n+1} + \varphi_{322} g_{n+2} + \varphi_{332} g_{n+3} \end{pmatrix} \tag{14}$$

For y_{n+1}:

$$\begin{aligned}
& y(x_n) + hy^{(1)}(x_n) + \tfrac{(h)^2}{2!} y^{(2)}(x_n) + \tfrac{(h)^3}{3!} y^{(3)}(x_n) + \tfrac{(h)^4}{4!} y^{(4)}(x_n) \\
& + \tfrac{(h)^5}{5!} y^{(5)}(x_n) + \tfrac{(h)^6}{6!} y^{(6)}(x_n) + \tfrac{(h)^7}{7!} y^{(7)}(x_n) + \tfrac{(h)^8}{8!} y^{(8)}(x_n) \\
& + \tfrac{(h)^9}{9!} y^{(9)}(x_n) + \cdots = y(x_n) + hy^{(1)}(x_n) + \tfrac{(h)^2}{2!} y^{(2)}(x_n) \\
& + \{\phi_{10}\left(y^{(3)}(x_n)\right) + \phi_{11}(y^{(3)}(x_n) + hy^{(4)}(x_n) + \tfrac{(h)^2}{2!} y^{(5)}(x_n) \\
& + \tfrac{(h)^3}{3!} y^{(6)}(x_n) + \tfrac{(h)^4}{4!} y^{(7)}(x_n) + \cdots) + \phi_{12}(y^{(3)}(x_n) + 2hy^{(4)}(x_n) \\
& + \tfrac{(2h)^2}{2!} y^{(5)}(x_n) + \tfrac{(2h)^3}{3!} y^{(6)}(x_n) + \tfrac{(2h)^4}{4!} y^{(7)}(x_n) + \cdots) + \phi_{13}(y^{(3)}(x_n) \\
& + 3hy^{(4)}(x_n) + \tfrac{(3h)^2}{2!} y^{(5)}(x_n) + \tfrac{(3h)^3}{3!} y^{(6)}(x_n) + \tfrac{(3h)^4}{4!} y^{(7)}(x_n) + \cdots) \\
& + \tau_{10}\left(y^{(4)}(x_n)\right) + \tau_{11}(y^{(4)}(x_n) + hy^{(5)}(x_n) + \tfrac{(h)^2}{2!} y^{(6)}(x_n) \\
& + \tfrac{(h)^3}{3!} y^{(7)}(x_n) + \tfrac{(h)^4}{4!} y^{(8)}(x_n) + \cdots) + \tau_{12}(y^{(4)}(x_n) + 2hy^{(5)}(x_n) \\
& + \tfrac{(2h)^2}{2!} y^{(6)}(x_n) + \tfrac{(2h)^3}{3!} y^{(7)}(x_n) + \tfrac{(2h)^4}{4!} y^{(8)}(x_n) + \cdots) + \tau_{13}(y^{(4)}(x_n) \\
& + 3hy^{(5)}(x_n) + \tfrac{(3h)^2}{2!} y^{(6)}(x_n) + \tfrac{(3h)^3}{3!} y^{(7)}(x_n) + \tfrac{(3h)^4}{4!} y^{(8)}(x_n) + \cdots)\}
\end{aligned}$$

Equating coefficients of $y^{(n)}(x_n)$ on both the left- and right-hand side of the equation gives the following system of equations:

$$\begin{aligned}
& \tfrac{(h)^3}{3!} = \phi_{10} + \phi_{11} + \phi_{12} + \phi_{13} \\
& \tfrac{(h)^4}{4!} = h\phi_{11} + 2h\phi_{12} + 3h\phi_{13} + \tau_{10} + \tau_{11} + \tau_{12} + \tau_{13} \\
& \tfrac{(h)^5}{5!} = \tfrac{(h)^2}{2!}\phi_{11} + \tfrac{(2h)^2}{2!}\phi_{12} + \tfrac{(3h)^2}{2!}\phi_{13} + h\tau_{11} + 2h\tau_{12} + 3h\tau_{13} \\
& \tfrac{(h)^6}{6!} = \tfrac{(h)^3}{3!}\phi_{11} + \tfrac{(2h)^3}{3!}\phi_{12} + \tfrac{(3h)^3}{3!}\phi_{13} + \tfrac{(h)^2}{2!}\tau_{11} + \tfrac{(2h)^2}{2!}\tau_{12} + \tfrac{(3h)^2}{2!}\tau_{13} \\
& \tfrac{(h)^7}{7!} = \tfrac{(h)^4}{4!}\phi_{11} + \tfrac{(2h)^4}{4!}\phi_{12} + \tfrac{(3h)^4}{4!}\phi_{13} + \tfrac{(h)^3}{3!}\tau_{11} + \tfrac{(2h)^3}{3!}\tau_{12} + \tfrac{(3h)^3}{3!}\tau_{13} \\
& \tfrac{(h)^8}{8!} = \tfrac{(h)^5}{5!}\phi_{11} + \tfrac{(2h)^5}{5!}\phi_{12} + \tfrac{(3h)^5}{5!}\phi_{13} + \tfrac{(h)^4}{4!}\tau_{11} + \tfrac{(2h)^4}{4!}\tau_{12} + \tfrac{(3h)^4}{4!}\tau_{13}
\end{aligned}$$

$$\begin{aligned}\frac{(h)^9}{9!} &= \frac{(h)^6}{6!}\phi_{11} + \frac{(2h)^6}{6!}\phi_{12} + \frac{(3h)^6}{6!}\phi_{13} + \frac{(h)^5}{5!}\tau_{11} + \frac{(2h)^5}{5!}\tau_{12} + \frac{(3h)^5}{5!}\tau_{13} \\ \frac{(h)^{10}}{10!} &= \frac{(h)^7}{7!}\phi_{11} + \frac{(2h)^7}{7!}\phi_{12} + \frac{(3h)^7}{7!}\phi_{13} + \frac{(h)^6}{6!}\tau_{11} + \frac{(2h)^6}{6!}\tau_{12} + \frac{(3h)^6}{6!}\tau_{13}\end{aligned} \tag{15}$$

Rewriting Eq. (15) in matrix form $Ax = B$ gives:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & h & 2h & 3h & 1 & 1 & 1 & 1 \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & 0 & h & 2h & 3h \\ 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} \\ 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} \\ 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} \\ 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} & 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} \\ 0 & \frac{(h)^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} & 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} \end{pmatrix} \begin{pmatrix} \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \tau_{10} \\ \tau_{11} \\ \tau_{12} \\ \tau_{13} \end{pmatrix} = \begin{pmatrix} \frac{(h)^3}{3!} \\ \frac{(h)^4}{4!} \\ \frac{(h)^5}{5!} \\ \frac{(h)^6}{6!} \\ \frac{(h)^7}{7!} \\ \frac{(h)^8}{8!} \\ \frac{(h)^9}{9!} \\ \frac{(h)^{10}}{10!} \end{pmatrix} \tag{16}$$

where $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & h & 2h & 3h & 1 & 1 & 1 & 1 \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & 0 & h & 2h & 3h \\ 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} \\ 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} \\ 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} \\ 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} & 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} \\ 0 & \frac{(h)^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} & 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} \end{pmatrix}$ has *rank* = 8, which implies that there are also no linearly dependent columns or rows and the inverse exists. These matrices likewise correspond to the definitions in Eq. (3). The unknown coefficients are obtained below:

$$\begin{aligned}&\left(\phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}\right)^T \\ &= \left(\frac{62387h^3}{544320}, \frac{89h^3}{3360}, \frac{439h^3}{20160}, \frac{1031h^3}{272160}, \frac{1879h^4}{181440}, -\frac{359h^4}{10080}, -\frac{13h^4}{960}, -\frac{17h^4}{18144}\right)\end{aligned}$$

Next the unknown coefficients for y_{n+2} and y_{n+2} are obtained as:

$$\left(\phi_{20},\phi_{21},\phi_{22},\phi_{23},\tau_{20},\tau_{21},\tau_{22},\tau_{23}\right)^{T}$$
$$=\left(\tfrac{5048h^3}{8505},\tfrac{164h^3}{315},\tfrac{4h^3}{21},\tfrac{244h^3}{8505},\tfrac{172h^4}{2835},-\tfrac{4h^4}{15},-\tfrac{34h^4}{315},-\tfrac{4h^4}{567}\right)$$

$$\left(\phi_{30},\phi_{31},\phi_{32},\phi_{33},\tau_{30},\tau_{31},\tau_{32},\tau_{33}\right)^{T}$$
$$=\left(\tfrac{657h^3}{448},\tfrac{2187h^3}{1120},\tfrac{2187h^3}{2240},\tfrac{117h^3}{1120},\tfrac{351h^4}{2240},-\tfrac{729h^4}{1120},-\tfrac{729h^4}{2240},-\tfrac{27h^4}{1120}\right)$$

In obtaining the coefficients for $y_{n+1}^{(1)}$, Taylor series expansion is used, which gives:

$$y^{(1)}(x_n)+hy^{(2)}(x_n)+\tfrac{(h)^2}{2!}y^{(3)}(x_n)+\tfrac{(h)^3}{3!}y^{(4)}(x_n)+\tfrac{(h)^4}{4!}y^{(5)}(x_n)$$
$$+\tfrac{(h)^5}{5!}y^{(6)}(x_n)+\tfrac{(h)^6}{6!}y^{(7)}(x_n)+\tfrac{(h)^7}{7!}y^{(8)}(x_n)+\tfrac{(h)^8}{8!}y^{(9)}(x_n)+\cdots$$
$$=y^{(1)}(x_n)+hy^{(2)}(x_n)+\{\omega_{101}\left(y^{(3)}(x_n)\right)+\omega_{111}(y^{(3)}(x_n)$$
$$+hy^{(4)}(x_n)+\tfrac{(h)^2}{2!}y^{(5)}(x_n)+\tfrac{(h)^3}{3!}y^{(6)}(x_n)+\tfrac{(h)^4}{4!}y^{(7)}(x_n)+\cdots)$$
$$+\omega_{121}(y^{(3)}(x_n)+2hy^{(4)}(x_n)+\tfrac{(2h)^2}{2!}y^{(5)}(x_n)+\tfrac{(2h)^3}{3!}y^{(6)}(x_n)$$
$$+\tfrac{(2h)^4}{4!}y^{(7)}(x_n)+\cdots)+\omega_{131}(y^{(3)}(x_n)+3hy^{(4)}(x_n)+\tfrac{(3h)^2}{2!}y^{(5)}(x_n)$$
$$+\tfrac{(3h)^3}{3!}y^{(6)}(x_n)+\tfrac{(3h)^4}{4!}y^{(7)}(x_n)+\cdots)+\varphi_{101}\left(y^{(4)}(x_n)\right)+\varphi_{111}(y^{(4)}(x_n)$$
$$+hy^{(5)}(x_n)+\tfrac{(h)^2}{2!}y^{(6)}(x_n)+\tfrac{(h)^3}{3!}y^{(7)}(x_n)+\tfrac{(h)^4}{4!}y^{(8)}(x_n)+\cdots)$$
$$+\varphi_{121}(y^{(4)}(x_n)+2hy^{(5)}(x_n)+\tfrac{(2h)^2}{2!}y^{(6)}(x_n)+\tfrac{(2h)^3}{3!}y^{(7)}(x_n)$$
$$+\tfrac{(2h)^4}{4!}y^{(8)}(x_n)+\cdots)+\varphi_{131}(y^{(4)}(x_n)+3hy^{(5)}(x_n)+\tfrac{(3h)^2}{2!}y^{(6)}(x_n)$$
$$+\tfrac{(3h)^3}{3!}y^{(7)}(x_n)+\tfrac{(3h)^4}{4!}y^{(8)}(x_n)+\cdots)\}$$

Equating coefficients of $y^{(n)}(x_n)$ gives:

$$\tfrac{(h)^2}{2!}=\omega_{101}+\omega_{111}+\omega_{121}+\omega_{131}$$
$$\tfrac{(h)^3}{3!}=h\omega_{111}+2h\omega_{121}+3h\omega_{131}+\varphi_{101}+\varphi_{111}+\varphi_{121}+\varphi_{131}$$
$$\tfrac{(h)^4}{4!}=\tfrac{(h)^2}{2!}\omega_{111}+\tfrac{(2h)^2}{2!}\omega_{121}+\tfrac{(3h)^2}{2!}\omega_{131}+h\varphi_{111}+2h\varphi_{121}+3h\varphi_{131}$$
$$\tfrac{(h)^5}{5!}=\tfrac{(h)^3}{3!}\omega_{111}+\tfrac{(2h)^3}{3!}\omega_{121}+\tfrac{(3h)^3}{3!}\omega_{131}+\tfrac{(h)^2}{2!}\varphi_{111}+\tfrac{(2h)^2}{2!}\varphi_{121}+\tfrac{(3h)^2}{2!}\varphi_{131}$$
$$\tfrac{(h)^6}{6!}=\tfrac{(h)^4}{4!}\omega_{111}+\tfrac{(2h)^4}{4!}\omega_{121}+\tfrac{(3h)^4}{4!}\omega_{131}+\tfrac{(h)^3}{3!}\varphi_{111}+\tfrac{(2h)^3}{3!}\varphi_{121}+\tfrac{(3h)^3}{3!}\varphi_{131}$$

$$
\begin{aligned}
\frac{(h)^7}{7!} &= \frac{(h)^5}{5!}\omega_{111} + \frac{(2h)^5}{5!}\omega_{121} + \frac{(3h)^5}{5!}\omega_{131} + \frac{(h)^4}{4!}\varphi_{111} + \frac{(2h)^4}{4!}\varphi_{121} + \frac{(3h)^4}{4!}\varphi_{131} \\
\frac{(h)^8}{8!} &= \frac{(h)^6}{6!}\omega_{111} + \frac{(2h)^6}{6!}\omega_{121} + \frac{(3h)^6}{6!}\omega_{131} + \frac{(h)^5}{5!}\varphi_{111} + \frac{(2h)^5}{5!}\varphi_{121} + \frac{(3h)^5}{5!}\varphi_{131} \\
\frac{(h)^9}{9!} &= \frac{(h)^7}{7!}\omega_{111} + \frac{(2h)^7}{7!}\omega_{121} + \frac{(3h)^7}{7!}\omega_{131} + \frac{(h)^6}{6!}\varphi_{111} + \frac{(2h)^6}{6!}\varphi_{121} + \frac{(3h)^6}{6!}\varphi_{131}
\end{aligned}
\tag{17}
$$

Rewriting Eq. (17) in matrix form $Ax = B$ gives:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & h & 2h & 3h & 1 & 1 & 1 & 1 \\
0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & 0 & h & 2h & 3h \\
0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} \\
0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} \\
0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} & 0 & \frac{(h)^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} \\
0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!} & 0 & \frac{(h)^5}{5!} & \frac{(2h)^5}{5!} & \frac{(3h)^5}{5!} \\
0 & \frac{(h)^7}{7!} & \frac{(2h)^7}{7!} & \frac{(3h)^7}{7!} & 0 & \frac{(h)^6}{6!} & \frac{(2h)^6}{6!} & \frac{(3h)^6}{6!}
\end{pmatrix}
\begin{pmatrix}
\omega_{101} \\ \omega_{111} \\ \omega_{121} \\ \omega_{131} \\ \varphi_{101} \\ \varphi_{111} \\ \varphi_{121} \\ \varphi_{131}
\end{pmatrix}
=
\begin{pmatrix}
\frac{(h)^2}{2!} \\ \frac{(h)^3}{3!} \\ \frac{(h)^4}{4!} \\ \frac{(h)^5}{5!} \\ \frac{(h)^6}{6!} \\ \frac{(h)^7}{7!} \\ \frac{(h)^8}{8!} \\ \frac{(h)^9}{9!}
\end{pmatrix}
\tag{18}
$$

corresponding to the definitions in Eq. (3). The values of the unknown coefficients are given below:

$$
\begin{aligned}
&\left(\omega_{101}, \omega_{111}, \omega_{121}, \omega_{131}, \varphi_{101}, \varphi_{111}, \varphi_{121}, \varphi_{131}\right)^T \\
&= \left(\frac{19519h^2}{68040}, \frac{1301h^2}{10080}, \frac{181h^2}{2520}, \frac{3329h^2}{272160}, \frac{371h^3}{12960}, -\frac{313h^3}{2520}, -\frac{89h^3}{2016}, -\frac{137h^3}{45360}\right)
\end{aligned}
$$

Similarly, the unknown coefficients for $y_{n+2}^{(1)}$ and $y_{n+3}^{(1)}$ are obtained as:

$$
\begin{aligned}
&\left(\omega_{201}, \omega_{211}, \omega_{221}, \omega_{231}, \varphi_{201}, \varphi_{211}, \varphi_{221}, \varphi_{231}\right)^T \\
&= \left(\frac{5731h^2}{8505}, \frac{296h^2}{315}, \frac{109h^2}{315}, \frac{344h^2}{8505}, \frac{2606h^3}{2835}, -\frac{20h^3}{63}, -\frac{52h^3}{315}, -\frac{4h^3}{405}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\omega_{301}, \omega_{311}, \omega_{321}, \omega_{331}, \varphi_{301}, \varphi_{311}, \varphi_{321}, \varphi_{331}\right)^T \\
&= \left(\frac{603h^2}{560}, \frac{2187h^2}{1120}, \frac{729h^2}{560}, \frac{27h^2}{160}, \frac{27h^2}{224}, -\frac{243h^3}{560}, -\frac{243h^3}{1120}, -\frac{9h^3}{280}\right)
\end{aligned}
$$

while the coefficients for $y_{n+1}^{(2)}$, $y_{n+2}^{(2)}$ and $y_{n+3}^{(2)}$ are obtained as:

$$
\begin{aligned}
&\left(\omega_{102}, \omega_{112}, \omega_{122}, \omega_{132}, \varphi_{102}, \varphi_{112}, \varphi_{122}, \varphi_{132}\right)^T \\
&= \left(\frac{6893h}{18144}, \frac{313h}{672}, \frac{89h}{672}, \frac{397h}{18144}, \frac{1283h^2}{30240}, -\frac{851h^2}{3360}, -\frac{269h^2}{3360}, -\frac{163h^2}{30240}\right)
\end{aligned}
$$

$$\left(\omega_{202},\omega_{212},\omega_{222},\omega_{232},\varphi_{202},\varphi_{212},\varphi_{222},\varphi_{232}\right)^T$$
$$=\left(\tfrac{223h}{567},\tfrac{20h}{21},\tfrac{13h}{21},\tfrac{20h}{567},\tfrac{43h^2}{945},-\tfrac{16h^2}{105},-\tfrac{19h^2}{105},-\tfrac{8h^2}{945}\right)$$

$$\left(\omega_{302},\omega_{312},\omega_{322},\omega_{332},\varphi_{302},\varphi_{312},\varphi_{322},\varphi_{332}\right)^T$$
$$=\left(\tfrac{93h}{224},\tfrac{243h}{224},\tfrac{243h}{224},\tfrac{93h}{224},\tfrac{57h^2}{1120},-\tfrac{81h^2}{1120},\tfrac{81h^2}{1120},-\tfrac{57h^2}{1120}\right)$$

Substituting all obtained coefficients back in Eq. (14) gives the three-step fourth derivative block method for solving third order ordinary differential equations as:

$$y_{n+1}=y_n+hy_n^{(1)}+\tfrac{(h)^2}{2!}y_n^{(2)}+\tfrac{h^3}{544630}\left(62387f_n+14418f_{n+1}+11853f_{n+2}+2062f_{n+3}\right)$$
$$+\tfrac{h^4}{544320}\left(5637g_n-19386g_{n+1}-7371g_{n+2}-510g_{n+3}\right),$$
$$y_{n+2}=y_n+2hy_n^{(1)}+\tfrac{(2h)^2}{2!}y_n^{(2)}+\tfrac{h^3}{8505}\left(5048f_n+4428f_{n+1}+1620f_{n+2}+244f_{n+3}\right)$$
$$+\tfrac{h^4}{544320}\left(516g_n-2268g_{n+1}-918g_{n+2}-60g_{n+3}\right),$$
$$y_{n+3}=y_n+3hy_n^{(1)}+\tfrac{(3h)^2}{2!}y_n^{(2)}+\tfrac{h^3}{2240}\left(3285f_n+4374f_{n+1}+2187f_{n+2}+234f_{n+3}\right)$$
$$+\tfrac{h^4}{544320}\left(351g_n-1458g_{n+1}-729g_{n+2}-54g_{n+3}\right),$$
$$y_{n+1}^{(1)}=y_n^{(1)}+hy_n^{(2)}+\tfrac{h^2}{272160}\left(78076f_n+35127f_{n+1}+19548f_{n+2}+3329f_{n+3}\right)$$
$$+\tfrac{h^3}{272160}\left(7791g_n-33804g_{n+1}-12015g_{n+2}-822g_{n+3}\right),$$
$$y_{n+2}^{(1)}=y_n^{(1)}+2hy_n^{(2)}+\tfrac{h^2}{8505}\left(5731f_n+7992f_{n+1}+2943f_{n+2}+344f_{n+3}\right)$$
$$+\tfrac{h^3}{272160}\left(618g_n-2700g_{n+1}-1404g_{n+2}-84g_{n+3}\right),$$
$$y_{n+3}^{(1)}=y_n^{(1)}+3hy_n^{(2)}+\tfrac{h^2}{1120}\left(1206f_n+2187f_{n+1}+1458f_{n+2}+189f_{n+3}\right)$$
$$+\tfrac{h^3}{272160}\left(135g_n-486g_{n+1}-243g_{n+2}-36g_{n+3}\right)$$
$$y_{n+1}^{(2)}=y_n^{(2)}+\tfrac{h}{90720}\left(34465f_n+422f_{n+1}+19548f_{n+2}+3329f_{n+3}\right)$$
$$+\tfrac{h^2}{272160}\left(7791g_n-33804g_{n+1}-12015g_{n+2}-822g_{n+3}\right),$$
$$y_{n+2}^{(2)}=y_n^{(2)}+\tfrac{h}{8505}\left(5731f_n+7992f_{n+1}+2943f_{n+2}+344f_{n+3}\right)$$
$$+\tfrac{h^2}{272160}\left(618g_n-2700g_{n+1}-1404g_{n+2}-84g_{n+3}\right),$$
$$y_{n+3}^{(2)}=y_n^{(2)}+\tfrac{h}{1120}\left(1206f_n+2187f_{n+1}+1458f_{n+2}+189f_{n+3}\right)$$
$$+\tfrac{h^2}{272160}\left(135g_n-486g_{n+1}-243g_{n+2}-36g_{n+3}\right)$$

6 Properties of the Three-Step Fourth-Derivative Block Method

The following properties of the three-step fourth derivative block method are discussed: order, zero-stability, consistency and convergence.

Following the same approach as the two-step third derivative block method, the correctors of the three-step fourth derivative block method are also expanded using Taylor series about $= x_n$. The order of the three-step fourth derivative block method is obtained to be $p = 8$.

Secondly, to analyze the three-step fourth derivative block method for zero-stability, the modulus of the roots of its first characteristic polynomial is expected to be simple or less than one. Thus, the correctors of the three-step fourth derivative block method are normalized to give the first characteristic polynomial as $\rho(r) = det\left[rI_3 - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$ with roots satisfying $|r_j| \leq 1$.

The three-step fourth derivative block method is consistent if it has order $p \geq 1$ as satisfied in the previous paragraphs. Therefore, the three-step fourth derivative block method is convergent [12].

7 Numerical Examples

This section considers certain linear and non-linear second and third order ordinary differential equations. The developed methods in Sections 3 and 4 above are adopted to solve these ODEs (encompassing both initial and boundary value problems) and a comparison is made with previous works.

Problem 1. Consider the non-linear second order initial value problem

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \tfrac{1}{2},\ x \in [0,1]$$

with exact solution: $y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$.

This initial value problem was compared with [9], where a hybrid order six block method was presented with $h = \frac{1}{100}$ as presented in Table 1. The common ground for comparison is in terms of the order of the block methods since the two-step third-derivative block method and the hybrid block method presented by [9] are both of order six.

Problem 2. Consider the linear second order initial value problem

$$y''+\tfrac{6}{x}y'+\tfrac{6}{x^2}y=0, y(1)=1, y'(1)=1,\ x\in[0,hN]$$

with exact solution: $y=\frac{5}{3x}-\frac{2}{3x^4}$

The number of steps (N) considered for this initial value problem is $N=10$ with respect to the selected step-size $h=\frac{0.1}{32}$. The results as displayed in Table 2 were compared to the solution obtained by order six hybrid block method presented by [13]. The block method compared to [13] is the order six two-step third-derivative block method.

Problem 3. Consider the non-linear second order boundary value problem

$$y''=\tfrac{1}{8}\left(32+2x^3-yy'\right), y(1)=17, y(3)=\tfrac{43}{3},\ x\in[1,3]$$

with exact solution: $y=x^2+\frac{16}{x}$

The boundary value problem defined here was solved by [14] using an order six self-starting block method and $h=0.1$. Thus, since the basis for comparison is the equal order of the methods, the suitable block method is the two-step third-derivative block method. The details of the results obtained can be seen in Table 3.

Problem 4. Consider the special third order initial value problem

$$y'''=-y, y(0)=1, y'(0)=-1, y''(0)=1,\ x\in[0,1]$$

with exact solution: $y=e^{-x}$.

This third order IVP was compared with [4] in which the authors present an order eight block method with $h=0.1$. Table 4 shows the comparison of the method in [9] with the order eight three-step fourth-derivative block method.

Problem 5. Consider the non-linear third order boundary value problem

$$y'''=-2e^{-3y}+4(1+x)^{-3}, y(0)=0, y'(0)=1, y(1)=\ln 2,\ x\in[0,1]$$

with exact solution: $y=\ln(1-x)$.

The third order BVP defined here was solved by [15] using an order eight block method and compared with the three-step fourth-derivative method also of order eight. The step-size h for this problem varied with respect to the number of steps (N) as shown in Table 5. The maximum error (MAXE) at the boundary $x=1$ was recorded for $N=7,14,28,56$.

Table 1 Comparison of the Two-Step Third Derivative Block Method with [9] for Solving Problem 1.

x	Exact Solution	Computed Solution	Error [9]	Error (TSTD)
0.1	1.0500417292784914	1.0550555592082120	6.661338E-16	2.220446E-16
0.2	1.1003353477310756	1.1003353477310753	1.332268E-15	2.220446E-16
0.3	1.1511404359364668	1.1511404359364668	4.440892E-16	0.000000E+00
0.4	1.2027325540540823	1.2027325540540821	1.332268E-15	2.220446E-16
0.5	1.2554128118829952	1.2554128118829952	3.774758E-15	0.000000E+00
0.6	1.3095196042031119	1.3095196042031116	1.065814E-14	2.220446E-16
0.7	1.3654437542713964	1.3654437542713957	2.642331E-14	6.661338E-16
0.8	1.4236489301936019	1.4236489301936006	5.861978E-14	1.332268E-15
0.9	1.4847002785940520	1.4847002785940489	1.265654E-13	3.108624E-15
1.0	1.5493061443340550	1.5493061443340488	2.711165E-13	6.217249E-15
CPU Time			N/A	≈ 0.59 secs

Note: TSTD: Two-step third derivative block method.

Table 2 Comparison of the Two-Step Third Derivative Block Method with [13] for Solving Problem 2.

x	Exact Solution	Computed Solution	Error [13]	Error (TSTD)
1.003125	1.0030765258576961	1.0030765258576961	N/A	0.000000E+00
1.006250	1.0060575030835164	1.0060575030835164	N/A	0.000000E+00
1.009375	1.0089449950888376	1.0089449950888376	9.661126E-08	0.000000E+00
1.012500	1.0117410181679887	1.0117410181679887	9.425732E-08	0.000000E+00
1.015625	1.0144475426864139	1.0144475426864139	9.197108E-08	0.000000E+00
1.018750	1.0170664942356729	1.0170664942356726	8.975049E-08	2.220446E-16
1.021875	1.0195997547562881	1.0195997547562876	8.759359E-08	4.440892E-16
1.025000	1.0220491636294322	1.0220491636294318	8.549846E-08	4.440892E-16
1.028125	1.0244165187384029	1.0244165187384029	8.346327E-08	0.000000E+00
1.031250	1.0267035775008062	1.0267035775008062	8.148622E-08	0.000000E+00
CPU Time			N/A	≈ 0.06 secs

Note: TSTD: Two-step third derivative block method.

Table 3 Comparison of the Two-Step Third Derivative Block Method with [14] for Solving Problem 3.

x	Exact Solution	Computed Solution	Error [14]	Error (TSTD)
1.1	15.755454545454546	15.755455083760621	1.50E-06	5.383061E-07
1.2	14.773333333333333	14.773334073676603	3.97E-07	7.403433E-07
1.3	13.997692307692306	13.997693050796402	8.12E-08	7.431041E-07
1.4	13.388571428571428	13.388572125044842	1.57E-06	6.964734E-07
1.5	12.916666666666666	12.916667287517622	1.83E-06	6.208510E-07
1.6	12.559999999999999	12.560000544356562	1.79E-06	5.443566E-07
1.7	12.301764705882352	12.301765175115765	1.58E-06	4.692334E-07
1.8	12.128888888888888	12.128889289986821	7.93E-07	4.010979E-07
1.9	12.031052631578946	12.031052970831665	1.07E-06	3.392527E-07
2.0	12.000000000000000	12.000000284503839	1.07E-06	2.845038E-07
2.1	12.029047619047619	12.029047854947255	6.42E-07	2.358996E-07
2.2	12.112727272727273	12.112727465928481	1.27E-06	1.932012E-07
2.3	12.246521739130435	12.246521894771508	2.18E-07	1.556411E-07
2.4	12.426666666666666	12.426666789477615	1.84E-06	1.228109E-07
2.5	12.650000000000000	12.650000094140001	3.82E-07	9.414000E-08

x	Exact Solution	Computed Solution	Error [14]	Error (TSTD)
2.6	12.913846153846155	12.913846223094138	1.86E-06	6.924798E-08
2.7	13.215925925925927	13.215925973633468	1.35E-06	4.770754E-08
2.8	13.554285714285715	13.554285743482653	1.22E-06	2.919694E-08
2.9	13.927241379310345	13.927241392694905	2.75E-06	1.338456E-08
3.0	14.333333333333336	14.333333333333336	0.000000	0.000000E+00
CPU Time			N/A	≈ 0.12 secs

Note: TSTD: Two-step third derivative block method.

Table 4 Comparison of the Three-Step Fourth Derivative Block Method with [4] for Solving Problem 4.

x	Exact Solution	Computed Solution	Error [4]	Error (TSFD)
0.1	0.90483741803595952	0.90483741803595963	2.138401E-12	1.110223E-16
0.2	0.81873075307798182	0.81873075307798204	6.055156E-13	2.220446E-16
0.3	0.74081822068171777	0.74081822068171832	7.395751E-12	5.551115E-16
0.4	0.67032004603563933	0.67032004603564022	2.158163E-12	8.881784E-16
0.5	0.60653065971263342	0.60653065971263520	1.484579E-11	1.776357E-15
0.6	0.54881163609402639	0.54881163609402939	1.098521E-11	2.997602E-15
0.7	0.49658530379140947	0.49658530379141408	3.142886E-11	4.607426E-15
0.8	0.44932896411722156	0.44932896411722822	2.309530E-11	6.661338E-15
0.9	0.40656965974059905	0.40656965974060832	5.154149E-11	9.270362E-15
1.0	0.36787944117144222	0.36787944117145466	8.200535E-11	1.243450E-14
CPU Time			N/A	≈ 0.06 secs

TSFD: Three-step fourth derivative block method.

Table 5 Comparison of the MAXE for the Three-Step Fourth Derivative Block Method with [15] for Solving Problem 5.

N	Exact Solution	Computed Solution (TSFD)	CPU Time	MAXE (TSFD)	MAXE [15]
7	0.69314718055994529	0.69314718055994529	≈ 0.10 secs	0.000000E+00	5.24E-09
14	0.69314718055994529	0.69314718055994529	≈ 0.15 secs	0.000000E+00	2.39E-11
28	0.69314718055994529	0.69314718055994529	≈ 0.30 secs	0.000000E+00	9.50E-14
56	0.69314718055994529	0.69314718055994529	≈ 0.61 secs	0.000000E+00	3.62E-16

Note: TSTD: Two-step third derivative block method, TSFD: Three-step fourth derivative block method, N: Number of steps, MAXE: Maximum error, N/A: Not available.

8 Conclusion

This article presented a new approach to developing block methods for solving m^{th} order ordinary differential equations with the presence of $(m+1)^{th}$. The generalized approach is seen to be quite flexible as the algorithm can simultaneously produce block methods of step length k for solving any order m of ordinary differential equations. The sample methods developed using this new approach were seen to satisfy the basic properties to ensure convergence and their accuracy is also displayed (see Tables 1-5). Thus, this new generalized approach is quite suitable for developing block methods for solving higher order ODEs.

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