Generalization of Slightly Compressible Modules

Samruam Baupradist[1,3], Phatsarapa Janmuang[1,3] & Suphawat Asawasamrit[2,3]

[1]Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand
[2]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
[3]Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Road, Bangkok 10400, Thailand
E-mail: samruam.b@chula.ac.th

Abstract. In this paper, we give a generalization of slightly compressible modules. We introduce the notion of M-slightly compressible modules, i.e. a right R module N is called *M-slightly compressible* if for every nonzero submodule A of N there exists a nonzero R-homomorphism s from M to N such that $s(M) \hookrightarrow A$. Many examples of M-slightly compressible modules are provided. Some results on M-slightly compressible modules are obtained, which are interesting and important.

Keywords: *compressible modules; M-sightly compressible modules; slightly compressible modules.*

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1 Introduction and Preliminaries

Throughout this paper, R is an associative ring with identity and Mod-R is the category of unitary right R-modules. For a right R-module M, let $S = EndR(M)$ be the endomorphism ring of M. A right R-module N is called M-generated if there exists an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called finitely M-generated. In particular, N is called M-cyclic if it is isomorphic to M / L for some submodule L of M or equivalent to saying that any M-cyclic submodule X of M can be considered the image of an endomorphism of M. Following Wisbauer [1], $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are submodules of M-generated modules. A right R-module M is called a self-generator if it generates all of its submodules. A right R-module M is called a subgenerator if it is a generator of $\sigma[M]$. For undefined notation, terminology and all the basic results on rings and modules see [1-3].

In 1976, Zelmanowitz [4] introduced the notion of compressible modules. A right R-module M is called *compressible* if for each nonzero submodule N of M

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there exists an R-module monomorphism from M to N. For example, if R is a domain, then the right R-module R is compressible. Generalizations of compressible modules have been studied in several papers (see [5-7]). Recently, P.F. Smith [8] introduced the concept of a slightly compressible module, which is a generalization of the compressible module. According to P.F. Smith, a right R-module M is called *slightly compressible* if for any nonzero submodule N of M there exists a nonzero R-module homomorphism from M to N. See for example [8], Example 1.2: if S is a nonzero ring and R is the ring of $2x2$ upper triangular matrices over S, then the right S-module R is slightly compressible.

In this paper, the notion of M-slightly compressible modules where M is a right R-module is introduced and studied, which is a general form of slightly compressible modules. Moreover we provide conditions for any right R-module to be an M-slightly compressible module and an example of M-slightly compressible modules. Some results on slightly compressible modules [8] are extended to M-slightly compressible modules.

2 M-slightly Compressible Modules

In this section, we introduce the concept of M-slightly compressible modules. We investigate the basic properties of M-slightly compressible modules. Some of these properties are analogous to the properties of slightly compressible modules. First, we give the following definition:

Definition 2.1 Let M and N be right R-modules. N is called M-slightly compressible if for every nonzero submodule A of N there exists a nonzero R-homomorphism s from M to N such that $s(M) \hookrightarrow A$. In the case that $M = N$, N is called a *slightly compressible module*, referring to [8].

Example 2.2

(1) This example is taken from [9]. A right R-module N is called *fully-M-cyclic* if for every submodule A of N there exists $s \in Hom_R(M, N)$ such that $A = s(M)$. A right R-module M is called *quasi-fully-cyclic* if it is a fully-M-cyclic module. It is clear that every fully-M-cyclic module is an M-slightly compressible module.

(2) Let M and N be right R-modules. If N is an M-generated module, then N is an M slightly compressible module (see, [[3], Exercise 2(b) and (d), pp. 361-362]).

(3) Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ the ring of all matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $a, b, c \in F$, $M_R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $N_R = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then, M_R and N_R are R_R-slightly compressible modules.

Theorem 2.3 Let M be a Noetherian right R-module. If N is an M-slightly compressible module, then $Soc(M) \cong Soc(N)$.

Proof. Assume that N is an M-slightly compressible module. Let A be a simple submodule of N. There exists $0 \neq s \in Hom_R(M, N)$ such that $0 \neq s(M) \hookrightarrow A$. But A is a simple submodule of N, $A = s(M)$. Let $0 \neq a \in A$. Then, $aR = A = s(M)$, so $a = s(b)$ for some $b \in M$. In Noetherian module bR, there exists a simple submodule B containing b such that $A \cong B$. Therefore $Soc(M) \cong Soc(N)$.

Theorem 2.4 Let M, $M'$and N be right R-modules, where N is an M-slightly compressible module.

(1) If M is an epimorphism image of M', then N is an M'-slightly compressible module.
(2) If M is an M'-slightly compressible module, then N is also an M'-slightly compressible module.
(3) For any submodule A of N, A is an essential in N if and only if for any $0 \neq t \in Hom_R(M, N)$, $t(M) \cap A \neq 0$.
(4) For any submodule A of N, A is an uniform submodule of N if and only if for any $0 \neq t \in Hom_R(M, A)$, $t(M)$ is an essential in A.

Proof.

(1) Assume that M is an epimorphism image of M'. There exists an epimorphism α from M' to M, so $\alpha(M') = M$. Let $0 \neq A \hookrightarrow N$. Since N is an M-slightly compressible, there exists $0 \neq s \in Hom_R(M, N)$ such that $s(M) \hookrightarrow A$. Thus $s\alpha(M') \hookrightarrow A$. Therefore N is an M'-slightly compressible module.

(2) Assume that M is an M'-slightly compressible module. Let $0 \neq A \hookrightarrow N$. Since N is an M-slightly compressible module, there exists $0 \neq s \in Hom_R(M, N)$ such that $s(M) \hookrightarrow A$. Because M is an M'-slightly compressible module, there exists $0 \neq t \in Hom_R(M', M)$ such that $t(M') \hookrightarrow M$. Then, $st(M') \hookrightarrow s(M) \hookrightarrow A$. Thus N is an M'-slightly compressible module.

(3) ($\Rightarrow$) It is obvious.

(⇐) Assume that any $0 \neq t \in Hom_R(M,N)$, $t(M) \cap A \neq 0$ holds. Let $0 \neq B \hookrightarrow N$. Since N is an M-slightly compressible module, there exists $0 \neq s \in Hom_R(M,N)$ such that $s(M) \hookrightarrow B$. Thus $s(M) \cap A \neq 0$ and we have $B \cap A \neq 0$. Therefore A is an essential in N.

(4) (⇒) It is clear.

(⇐) Assume that for any $0 \neq s \in Hom_R(M,A)$ such that $t(M)$ is an essential in A. Let B and C be nonzero submodules of A. Since N is an M-slightly compressible module, there exists $u, v \in Hom_R(M,N)$ such that $0 \neq u(M) \hookrightarrow B$ and $0 \neq v(M) \hookrightarrow C$. By assumption we have $u(M)$ and $v(M)$ are essential in A. Then, $u(M) \cap v(M) \neq 0$ and we have $B \cap C \neq 0$. Therefore A is uniform.

Proposition 2.5 Let M and N be right R-modules such that $Hom_R(M,N) \neq 0$. Then, N is a simple module if and only if N is an N-slightly compressible module with every nonzero R-homomorphism from M to N is an epimorphism.

Proof.

(⇒) It is obvious.

(⇐) Assume that N is an M-slightly compressible module with every nonzero R-homomorphism from M to N is an epimorphism. Let $0 \neq A \hookrightarrow N$. There exists $0 \neq s \in S = Hom_R(M,N)$ such that $0 \neq s(M) \hookrightarrow A$. By assumption we have $N = s(M)$ and hence $N = A$. Therefore N is a simple module.

Corollary 2.6 ([10], Proposition 3.5) Let M be a right R-module. Then, M is a simple module if and only if M is a slightly compressible module with every nonzero endomorphism of M is an epimorphism.

Proposition 2.7 Let N be an M-slightly compressible module. Then,

(1) A is an M-slightly compressible module for all $A \hookrightarrow N$.
(2) N is an P-slightly compressible module for every right R-module P with $ker(s) \neq P \hookrightarrow M$ for all $s \in Hom_R(M,N)$.

Proof.

(1) Let $A \hookrightarrow N$. If $A = 0$, it is clear. We can suppose that $A \neq 0$. Let $0 \neq B \hookrightarrow A$, Then, $B \hookrightarrow N$ and there exists $0 \neq s \in Hom_R(M,N)$ such that $s(M) \hookrightarrow B$. Thus, $0 \neq s \in Hom_R(M,A)$. Hence, A is an M-slightly compressible.

(2) Let $P \hookrightarrow M$ such that $ker(s) \neq P$ for all $s \in Hom_R(M,N)$. Let $A \neq 0 \hookrightarrow N$. Since N is an M-slightly compressible module, there exists $0 \neq s \in$

$Hom_R(M,N)$ such that $s(M) \hookrightarrow A$ and $ker(s) \neq P$. Then, $0 \neq s|_p \in Hom_R(P,N)$ such that, $s|_p(P) \hookrightarrow A$, where $s|_p$ is an R-homomorphism with respect to P. Therefore N is an P-slightly compressible.

Proposition 2.8 Let M and N be right R-modules. If every nonzero submodule A of N containing nonzero submodule B such that $B \cong C$ where C is a direct summand of M, then N is an M-slightly compressible module.

Proof. Assume that every nonzero submodule A of N containing nonzero submodule B such that $B \cong C$ where C is a direct summand of M. Let $0 \neq A \hookrightarrow N$. By assumption there exists a nonzero submodule B such that $B \cong C$ where C is a direct summand of M. Since $B \cong C$ there exists a α that is an isomorphism from C to B. Let π_C be the canonical projection map from M to C. Thus, $\alpha\pi_C: M \to B$ is an R-homomorphism and $\alpha\pi_C(M) \hookrightarrow A$. Therefore N is an M-slightly compressible module.

Recall that $P \in \sigma[M]$ is called *hereditary* in $\sigma[M]$ if every submodule of P is a projective in $\sigma[M]$. We say that a ring R is *right* (*left*) *hereditary* if $R_R({}_RR)$ is a hereditary in Mod-R.

Theorem 2.9 Let R be a right hereditary ring and M an injective right R-module. If N is an M-slightly compressible module, then every nonzero submodule A of N contains a direct summand of N.

Proof. Assume that N is an M-slightly compressible module. Let $0 \neq A \hookrightarrow N$. By assumption there exists $0 \neq s \in Hom_R(M,N)$ such that $s(M) \hookrightarrow A$. Since M is an injective module, R is a hereditary ring and Theorem 3.22 in [11], $M|_{\ker(s)}$ is an injective. But $M|_{\ker(s)} \cong s(M)$, $s(M)$ is an injective. Therefore $s(M)$ is a direct summand of N.

Proposition 2.10 Let M and N be right R-modules such that N is an M-slightly compressible module. If every M-cyclic submodule of N is an injective module, then N is an M-generated module.

Proof. Assume that every M-cyclic submodule of N is an injective. Let $0 \neq A \hookrightarrow N$. There exists $0 \neq s \in Hom_R(M,N)$ such that $0 \neq s(M) \hookrightarrow A$. By assumption $s(M)$ is an injective and we have $s(M)$ is a direct summand of A. There exists $B \hookrightarrow A$ such that $s(M) \oplus B = A$. If $B = 0$, we are done. If $0 \neq B \hookrightarrow N$, there exists $0 \neq t \in Hom_R(M,N)$ such that $0 \neq t(M) \hookrightarrow B$. By assumption, $t(M)$ is an injective and we have $t(M)$ is a direct summand of B. Thus, there exists $C \hookrightarrow B$ such that $t(M) \oplus C = B$. Continuous in this process we have $A = \sum_{s \in Hom_R(M,N)} s(M)A$. Therefore N is an M-generated module.

Corollary 2.11 Let M be a slightly compressible module. If every M-cyclic submodule of M is an injective then M is a self-generator.

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