On Eigenvalues and Eigenvectors of Perturbed Pairwise Comparison Matrices

Pudji Astuti[1] & Agah D. Garnadi[2]

[1]Algebra Research Group, FMIPA ITB, Indonesia
[2]Scientific Computing Research Group, Dept. Mathematics, FMIPA IPB, Indonesia

Abstract. This work studied eigenvalues and eigenvectors of a class of perturbed pairwise comparison matrices (PCMs). This type of matrices arises from Analytical Hierarchical Process with inconsistency comparison. By employing some nice structures of the PCMs, we show that the object dimension of size $n \geq 3$ can be reduced into a case of size 3, hence simplify the studies.

Keywords: *image; kernel; pairwise comparison matrix; principal eigenvalue and eigenvector.*

1 Introduction

A pairwise comparison matrix (PCM) is a positive symmetrically reciprocal matrix arising in Analytical Hierarchy Process (AHP). The AHP is a multicriteria decision model introduced by Saaty which solves decision problems by prioritizing alternatives [1]. The application of the AHP as a decision problem tools gives rise to pairwise comparison matrices (PCMs). The core of the AHP is the priority vector corresponding to any PCM. This vector is the normalized principal right eigenvector of the PCM corresponding to the largest eigenvalue which is simple and its existence is guaranteed by Perron's Theorem [2].

Several approaches have been proposed for investigating the priority vector of a PCM [3]. Saaty's eigenvector method (EM) derived the priority vector as a positive vector minimizing the distance between the PCM and the ratio matrix formed by the positive vector with respect to a certain norm. Chu [4] proposed the Least-squares method (LSM) and the Weighted Least-squares method (WLSM). Another method to estimate the priority vector proposed by Gass and Rapcsák [3] is by using the Singular Value Decomposition (SVD) of the PCM and the theory of low rank approximation. It was observed that the SVD approach has much to over from theoretical point of view.

Meanwhile several results were proposed regarding the study and analysis of the principal eigenvector of a PCM. For example, Farkas [5] developed the spectral properties of PCM and shows how the perturbed PCM results in a

Received December 10th, 2008, Revised May 11th, 2009, Accepted for publication May 15th, 2009.

reversal of the rank order of the decision alternatives. Studying those results, it is observed that the nice property and structure of the PCM, particularly for simply perturbed PCMs, have not been fully exploited.

In this note we study the eigenvalues and eigenvectors of PCMs which are perturbed on one row and its corresponding column. The notion of this class perturbed PCMs is adopted from Farkas [5]. By employing the well structured of these perturbed PCMs, we demonstrates that investigating eigenvalues and eigenvectors of the PCMs can be transformed into a similar problem of the corresponding 3×3 matrices. Hence, the investigation becomes simpler.

2 Perturbed PCM

Let $\Re$ denote the real field. An $n\times n$ matrix $A=(a_{ij})$ whose components belong to $\Re$ is called transitive if $a_{ij}=a_{ik}a_{kj}$ for all $i,j,k=1,2,...,n$. An $n\times n$ matrix $A=(a_{ij})$ whose components belong to $\Re-\{0\}$ is called symmetrically reciprocal (SR) if $a_{ij}a_{ji}=1$ for $i\neq j$ and $a_{ii}=1$ for all $i,j=1,2,...,n$. Any nonzero transitive matrix is SR but the converse is not true. It has been shown that an SR matrix is transitive if and only if it is of rank one [5] [3].

A pairwise comparison matrix (PCM) is an SR matrix which is positive. This kind of matrices arises in Analytical Hierarchy Process (AHP); a multicriteria decision model introduced by Saaty which solves decision problems by prioritizing alternatives. In AHP, the-*ij*th component of a PCM $A=(a_{ij})$ represents relative importance ratios of the-*i*th alternative over the-*j*th alternative with respect to a certain criterion. The core of the AHP is the priority vector; a positive vector whose *i*th component represents the weight or the score priority of the-*i*th alternative. This vector is the normalized principal right eigenvector of the PCM corresponding to the largest eigenvalue which is simple and its existence is guaranteed by Perron's Theorem [2]. Several methods have been proposed to study, analyse, and approximate the priority vector of a PCM. In this note we study the eigenvalues and eigenvectors of a PCM which is perturbed at one row.

An ideal decision problem produces transitive PCMs called specific PCMs. As explained above, we have that a specific PCM has one-rank. Any normalized column vector of a specific PCM will be the priority vector. Particularly, suppose $\mathbf{A}_c=(a_{ij})$ is a specific PCM. We have

$$\mathbf{A}_c = \mathbf{u}\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_{n-1} \\ \frac{1}{x_1} & 1 & \frac{x_2}{x_1} & \cdots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2} & \frac{x_1}{x_2} & 1 & \cdots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \cdots & 1 \end{bmatrix} \tag{1}$$

for some $\mathbf{u} = (1 \ \frac{1}{x_1} \ \frac{1}{x_2} \cdots \frac{1}{x_{n-1}})$, $\mathbf{v} = (1 \ x_1 \ x_2 \cdots x_{n-1})$, with x_1, $x_2, \cdots, x_{n-1}$, positive real numbers. In this case, the priority vector is $\mathbf{u}/\mathrm{c}$ where $\mathrm{c} = \sum_{\tau=0}^{\mathrm{n-1}} x_\tau$. The-τth component of $\mathbf{u}/\mathrm{c}$, say u_τ/c is the weight of the-τth alternative where the sum of the weights of all alternatives is 1.

However, a real decision problem most likely contains a subjective judgment on the relative importance ratios between alternatives, resulting in PCMs which are not transitive. A PCM is called perturbed PCM if it is not transitive. A PCM which is perturbed at the first row (and its associated first column) can be denoted as

$$\mathbf{A} = \begin{bmatrix} 1 & x_1\delta_1 & x_2\delta_2 & \cdots & x_{n-1}\delta_{n-1} \\ \frac{1}{x_1\delta_1} & 1 & \frac{x_2}{x_1} & \cdots & \frac{x_{n-1}}{x_1} \\ \frac{1}{x_2\delta_2} & \frac{x_1}{x_2} & 1 & \cdots & \frac{x_{n-1}}{x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}\delta_{n-1}} & \frac{x_1}{x_{n-1}} & \frac{x_2}{x_{n-1}} & \cdots & 1 \end{bmatrix} \tag{2}$$

where x_i, δ_i are positive real numbers such that there are $1 \le i < j \le n-1$ such that $\delta_i \ne \delta_j$. A PCM which is perturbed at another row can be transformed using permutation matrices to the above form (2). Hence, in this note it is sufficient if our discussion strictly on PCM of the form (2).

The analysis of the algebraic eigenvalue-eigenvector problem of the simply perturbed PCM was addressed in Farkas [5]. It was shown that rank reversals can occur even only arbitrarily small departure from specific PCM. However, the nice structure of the perturbed PCM have not been fully employed. In this note we propose another approach to study eigenvalue-eigenvector of PCM of the form (2). In the following section, by employing the nice structures of the perturbed PCM, we transform the object domain from matrices of size $n \times n$ into 3×3 matrices. Consequently, the study of the eigenvalues and eigenvectors of the perturbed PCM can be studied by studying a more simple case.

3 Eigenvalue and Eigenvector of Perturbed PCM

Let us consider a perturbed PCM of the form $\mathbf{A}$ (2). When $\delta_i = \delta_j$ for all i, j = 1, 2,…, n-1, we obtain $\mathbf{A}$ is transitive. Hence it has one-rank and can be written as (1). For the case there exist $1 \le i < j \le n-1$ such that $\delta_i \ne \delta_j$, we have the following theorem.

Theorem 3.1 If $n \ge 3$ *and there exist* $1 \le i < j \le n-1$ *such that* $\delta_i \ne \delta_j$ *then* Rank(**A**)=**3**. *Particularly we have* $B = \{\mathbf{e}_1, \mathbf{u}, \mathbf{w}\}$ *is a basis of the image* **A,** $\mathrm{Im}(\mathbf{A})$, *where*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ \frac{1}{x_1} \\ \frac{1}{x_2} \\ \\ \frac{1}{x_{n-1}} \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ \frac{1}{x_1}\left(\frac{1}{\delta_1} - 1\right) \\ \frac{1}{x_2}\left(\frac{1}{\delta_2} - 1\right) \\ \\ \frac{1}{x_{n-1}}\left(\frac{1}{\delta_{n-1}} - 1\right) \end{pmatrix}.$$

Proof: We have that the first, the-(i+1)th, and the-(j+1)th columns of **A** are linearly independent. Further, each of the other columns can be written as a linear combination of these three columns of **A**. Hence we have Rank(**A)** = 3 and those three columns of **A** form a basis of $\mathrm{Im}(\mathbf{A})$.

Since each vector in B can be written as a linear combination of the obtained basis above and B is linearly independent, we can conclude that B is a basis of $\mathrm{Im}(\mathbf{A})$. □

It is well known that the subspace $\mathrm{Im}(\mathbf{A})$ is **A**-invariant. For the case of the perturbed PCM we have the following nice property of the **A**-invariant subspace $\mathrm{Im}(\mathbf{A})$.

Theorem 3.2 *If* $n \geq 3$ *then the restriction of the linear transformation* **A** *on* $\mathrm{Im}(\mathbf{A})$ *is an isomorphism*.

Proof: To prove the theorem it is enough to show that the restriction of the linear transformation **A** on $\mathrm{Im}(\mathbf{A})$ is injective. It is obvious if for all *i*, *j*= 1, 2, … *n*-1 $\delta_i = \delta_j$ then the restriction of the linear transformation **A** on $\mathrm{Im}(\mathbf{A})$ is injective. Hence, suppose and there exist $1 \leq i < j \leq n-1$ such that $\delta_i \neq \delta_j$. Let $\mathbf{x} \in \mathrm{Im}(\mathbf{A})$ such that **A***x* = **0**. Suppose $\mathbf{x} = \alpha\mathbf{e}_1 + \beta\mathbf{u} + \gamma\mathbf{w}$. Then we have the restriction of **A** on $\mathrm{Im}(\mathbf{A})$ is injective if we can show that $\alpha = \beta = \gamma = 0$. Consider that

$$\alpha\mathbf{A}\mathbf{e}_1 + \beta\mathbf{A}\mathbf{u} + \gamma\mathbf{A}\mathbf{w} = 0$$

$$\alpha(\mathbf{u} + \mathbf{w}) + \beta(a\mathbf{e}_1 + n\mathbf{u} + \mathbf{w}) + \gamma(b\mathbf{e}_1 + c\mathbf{u}) = 0$$

where $a = \sum_{\tau=1}^{n-1}(\delta_\tau - 1), b = \sum_{\tau=1}^{n-1}(\delta_\tau - 1)\left(\frac{1}{\delta_\tau} - 1\right), c = \sum_{\tau=1}^{n-1}\left(\frac{1}{\delta_\tau} - 1\right)$ Since B is linearly independent we have the following equation

$$\begin{pmatrix} 0 & a & b \\ 1 & n & c \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Thus, we obtain the theorem if we can show that the matrix

$$Z = \begin{pmatrix} 0 & a & b \\ 1 & n & c \\ 1 & 1 & 0 \end{pmatrix} \tag{4}$$

is nonsingular which will be done by showing $\det(Z) > 0$.

First, note that we have *a*+*b*+*c* = 0 and $\det(Z) = b(1-n) + ac$. By substituting the above formula of *a*, *b*, *c*, it is a routine manipulation to finally obtain

$$\det(Z) = \left(\sum_{\tau=1}^{n-1}\delta_\tau\right)\left(\sum_{\mu=1}^{n-1}\frac{1}{\delta_\mu}\right) - (n-1)^2$$

Now, we shall show $\det(Z) > 0$ using mathematical induction on n and employing the fact that the function $f(t) = t + \frac{1}{t}$ on the interval $(0, \infty)$ has only one minimum value $f(1) = 2$ at $t = 1$. For n=3 we have

$$\det(Z) = (\delta_1 + \delta_2)\left(\tfrac{1}{\delta_1} + \tfrac{1}{\delta_2}\right) - 4 = t + \tfrac{1}{t} - 2$$

with $t = \frac{\delta_1}{\delta_2}$. Hence we have that for n =3, $\det(Z) > 0$ provided $\delta_1 \neq \delta_2$.

Further, suppose we have $\det(Z) > 0$ for $3 \leq n = k$ provided there are $1 \leq i < j \leq n-1$ such that $\delta_i \neq \delta_j$. Let $n = k + 1$ and there be $1 \leq i < j \leq k$ such that $\delta_i \neq \delta_j$. Consider

$$\det(Z) = \left(\sum_{\tau=1}^{n-1=k} \delta_\tau\right)\left(\sum_{\mu=1}^{n-1=k} \tfrac{1}{\delta_\mu}\right) - k^2.$$

We have

$$\det(Z) = \left(\sum_{\tau=1}^{k-1} \delta_\tau + \delta_k\right)\left(\sum_{\mu=1}^{k-1} \tfrac{1}{\delta_\mu} + \tfrac{1}{\delta_k}\right) - \{(k-1)^2 + 2(k-1) + 1\}$$

$$= \left(\sum_{\tau=1}^{k-1} \delta_\tau\right)\left(\sum_{\mu=1}^{k-1} \tfrac{1}{\delta_\mu}\right) + \left(\sum_{\tau=1}^{k-1} \delta_\tau\right)\left(\tfrac{1}{\delta_k}\right) + \delta_k\left(\sum_{\mu=1}^{k-1} \tfrac{1}{\delta_\mu}\right) + 1 - \{(k-1)^2 + 2(k-1) + 1\}.$$

Hence

$$\det(Z) = \left\{\left(\sum_{\tau=1}^{k-1} \delta_\tau\right)\left(\sum_{\mu=1}^{k-1} \tfrac{1}{\delta_\mu}\right) - (k-1)^2\right\} + \sum_{\tau=1}^{k-1}\left(t_\tau + \tfrac{1}{t_\tau} - 2\right) \tag{5}$$

where $t_\tau = \frac{\delta_\tau}{\delta_k}$ for all $\tau = 1, 2, \cdots, k-1$. It is clear that both parts of the right side of the equation (5) are nonnegative. If $j \leq k-1$ then, according to the hypothesis above, we obtain

$$\left\{\left(\sum_{\tau=1}^{k-1} \delta_\tau\right)\left(\sum_{\mu=1}^{k-1} \tfrac{1}{\delta_\mu}\right) - (k-1)^2\right\} > 0$$

resulting in $\det(Z) > 0$. On the other hand, if $j = k$, then $t_i = \frac{\delta_i}{\delta_k} \neq 1$. Hence, according to the property of the function $f(t)$ mentioned above, $(t_i + \frac{1}{t_i} - 2) > 0$. As a result,

$$\sum_{\tau=1}^{k-1} (t_\tau + \tfrac{1}{t_\tau} - 2) > 0$$

implying $\det(Z) > 0$. Thus, we proved the theorem. □

The nice structures of the perturbed PCM **A** on the two theorems above and their proofs result in the following facts.

1. The space $\Re^n$ can be decomposed as a direct sum of two **A**-invariant subspaces

   $$\Re^n = \mathrm{Im}(\mathbf{A}) \oplus \mathrm{Ker}(\mathbf{A})$$

2. The matrix (4) is the transformation matrix with respect to the basis B of the restriction of **A** on $\mathrm{Im}(\mathbf{A})$, denoted as $[\mathbf{A}|_{\mathrm{Im}(\mathbf{A})}]_B$.

3. The characteristics polynomial of **A** is the product of the characteristics polynomial of the restriction of **A** respectively on $\mathrm{Im}(\mathbf{A})$ and $\mathrm{Ker}(\mathbf{A})$. Hence, according to 2. we have the characteristic polynomial of **A** is $p(\lambda) = n^{n-3}(\lambda^2(\lambda - n) + b(n-1) - ac)$

4. All of the nonzero eigenvalues and eigenvectors of **A** are all of eigenvalues and eigenvectors of the restriction of **A** on $\mathrm{Im}(\mathbf{A})$. Hence, suppose r is an eigenvalue of the transformation matrix (4) corresponding with the eigenvector $(\alpha_1, \alpha_2, \alpha_3)^T$. We obtain r is a nonzero eigenvalue of **A** with the corresponding eigenvector is

   $$\mathbf{x} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{u} + \alpha_3 \mathbf{w}$$

The above facts and Perron's Theorem about positive matrices lead us to a method for obtaining the principal eigenvalue and its corresponding eigenvector of the perturbed PCM **A** by the help of corresponding transformation matrix (4) as the following. Let r be the maximal eigenvalue of $[\mathbf{A}|_{\mathrm{Im}(\mathbf{A})}]_B$. That is, r is the maximal root of the characteristic polynomial

$$p_1(\lambda) = (\lambda^2(\lambda - n) + b(n-1) - ac)$$

According to Perron's Theorem we have r is simple. Hence, the solution subspace of the linear equation

$$\begin{pmatrix} r & -a & -b \\ -1 & r-n & -c \\ -1 & -1 & r \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

is a one dimensional subspace. Following the argument in [5] we obtain that any column vector of the $\mathrm{adj}(rI - [\mathbf{A}|_{\mathrm{Im}(\mathbf{A})}]_B)$ is a basis of the solution subspace of (6). Particularly, assuming $\delta_i, i = 1, 2, \cdots, n-1$ are closed to 1 and so $r + c >> 0$, we have

$$\mathbf{x} = \tfrac{(r(r-n)-c)}{r+n}\mathbf{e}_1 + \mathbf{u} + \tfrac{(1+r-n)}{r+n}\mathbf{w} \tag{7}$$

is a principal eigenvector of **A**.

Note that we can consider the PCM **A** obtained from a specific (transitive) PCM (1) which is perturbed by a matrix formed by the vector $\mathbf{e}_1$ and **w.** In this case, the transitive part of the PCM has the principal eigenvalue n with a principal eigenvector **u**. Meanwhile the perturbed PCM **A** has the principal eigenvalue r with a principal eigenvector **x**. From equation (7) above, we observe that the closed form of the principal eigenvector **x** obtained from **u** perturbed by the vectors $\mathbf{e}_1$ and **w.**

4 Concluding Remarks

By using basic results in linear algebra we can show some nice structures of a class perturbed PCMs. These properties made us able to obtain a closed form of the principal eigenvector of the perturbed PCMs as the principal eigenvector of the transitive part perturbed by the set of vectors that perturb the PCMs. The work on extension of current results to a general perturbed PCM, will be relegated in the future.

Acknowledgement

This work is supported by Fundamental Research Grant 2008, according to Surat Perjanjian Pelaksanaan Riset No. 0365/K01.03/Kontr-WRRIM/PL2.1.5/IV/2008 tanggal 11 April 2008. The authors would like to thank the referees for their useful comments.

References

[1] Saaty, T.L., *The Analytical Hierarchy Process*, McGraw-Hill, New York, 1980.

[2] Horn, R.A. & Johnson, C.R., *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[3] Gass, S.I. & Rapcsák, T., *Singular Value Decomposition in AHP*, European Journal of Operational Research, **154**, 573-584, 2004.

[4] Chu, M.T., *On the Optimal Consistent Approximation to Pairwise Comparison Matrices*, Linear Algebra and Its Applications, **272**, 155-168, 1998.

[5] Farkas, András, *The Analysis of The Principal Eigenvector of Pairwise Comparison Matrices*, Acta Polytechnica Hungarica, **4**(2), 2007.

[6] Ding, Jiu & Pye, Wallace C., *On The Spectrum And Pseudoinverse of A Special Bordered Matrix*, Linear Algebra and Its Applications, **331**, 11 - 20, 2001.

[7] Laub, A.J., *Matrix Analysis for Scientist & Engineers*, SIAM, USA, 2005.