Introduction of Interpolation and Extrapolation Model in Lanczos-type Algorithms A_{13}/B_6 and A_{13}/B_{13} to Enhance their Stability

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Abstract. A new method to treat the inherent instability of Lanczos-type algorithms is introduced. It enables us to capture the properties of the sequence of iterates generated by a Lanczos-type algorithm by interpolating on this sequence of points. The interpolation model found is then used to generate a point that is outside the range. It is expected that this new point will link up the rest of the sequence of points generated by the Lanczos-type algorithm if breakdown does not occur. However, because we assume that the interpolation model captures the properties of the Lanczos sequence, the new point belongs to that sequence since it is generated by the model. This paper introduces the so-called Embedded Interpolation and Extrapolation Model in Lanczos-type Algorithms (EIEMLA). The model was implemented in algorithms A_{13}/B_6 and A_{13}/B_{13}, which are new variants of the Lanczos algorithm. Individually, these algorithms perform badly on high dimensional systems of linear equations (SLEs). However, with the embedded interpolation and extrapolation models, EIEM A_{13}/B_6 and EIEM A_{13}/B_{13}, a substantial improvement in the performance on SLEs with up to 10^5 variables can be achieved.

Keywords : *breakdown; extrapolation; EIEM A_{13}/B_6; EIEM A_{13}/B_{13}; interpolation; Lanczos-type algorithms; patterns.*

AMS Subject Classification: 65F10, 65B05

1 Introduction

Systems of linear equations (SLEs) are ubiquitous in science and engineering applications. There are several approaches for solving them, broadly classified as direct and iterative methods. There are two general classes of iterative methods to solve SLEs: stationary and non-stationary methods. Stationary methods include Richardson [1], Jacobi, Gauss-Seidel [2], and Successive Over-relaxation (SOR) [3], among others. Non-stationary methods include

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Conjugate Gradient [4], BCG, BICG [5], GMRES, Arnoldi type [6], and Lanczos type [7,8]. Non-stationary methods are generally more efficient and suitable for large systems of linear equations. They are based on orthogonal polynomials and are often referred to as Krylov subspace methods [9,10].

Among the iterative methods, for solving large linear systems with a sparse non-symmetric matrix, those based on the Lanczos process are the most effective. This is because they feature short recurrence relations for the generation of the Krylov subspace, which means low cost and low memory requirement [5]. They are, however, prone to breakdown [11]. Ways to avoid breakdown in Krylov subspace algorithms have been researched extensively in the last decades, resulting in new, more robust and efficient algorithms using, for instance, restarting and switching strategies that allow us to restart and switch between Lanczos-type algorithms before they break down [12,13]. A recent work, discussed in Maharani & Salhi [14], considered three quality points for restarting Lanczos-types algorithms.

In this study, we propose a novel strategy for SLEs that have a unique solution, which takes advantage of the existence of patterns in the solutions generated by Lanczos-type algorithms. The patterns persist in the entries of the iterates during the iteration process. Interpolating every entry of the iterates allows to obtain model functions that can be used to predict the entries of a new iterate. This iterate is expected to be better than the previous iterates generated by the Lanczos-type algorithm.

This paper is organized as follows. In Section 1, we look at the background theory of Lanczos-type algorithms for solving SLEs. The motivation and derivation of the new approach are given in Sections 2 and 3, respectively. Section 4 discusses the numerical results of EIEMLA's implementation of algorithms A_{13}/B_6 and A_{13}/B_{13}. The conclusion is given in Section 5.

2 Review of Lanczos-type Algorithms for the Solution of SLEs

The Lanczos method to solve system

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where $A \in R^{nxn}$ and vectors $\mathbf{x}, \mathbf{b} \in R^n$, is based on the Krylov subspace method. If K_k and L_k are two Krylov subspaces in R^n, then we can choose an initial approximate solution $\mathbf{x}_0$ so that

$$\mathbf{x}_k - \mathbf{x}_0 \in K_k(A, \mathbf{r}_0) \tag{2}$$

and

$$\mathbf{r}_k = \mathbf{b} - \mathbf{x}_k \perp L_k = K_k(A^T, \mathbf{y}) \tag{3}$$

where $\mathbf{r}_0 = \boldsymbol{b} - \mathbf{x}_0$ is the corresponding residual vector, and $\mathbf{y}$ is an arbitrary non-zero vector. Moreover, the residual $\boldsymbol{r}_k$ satisfies the condition:

$$\left\langle \mathbf{y}, A^i P_k(A)\mathbf{r}_0 \right\rangle = 0 \text{, for } i = 0,1,...,n-1, \tag{4}$$

where $P_k(t) = 1 + \alpha_1 t + \cdots + \alpha_k t^k$ is a residual polynomial, i.e. ($\mathbf{r}_k = P_k(A)\mathbf{r}_0$), of degree k at most. This polynomial satisfies the normal condition, $P_k(0) = 1$. Let c be a linear functional in the vector space of polynomials, defined by

$$c_i = c(\mathbf{x}^i) \tag{5}$$

If we set $c_i = \left\langle \mathbf{y}, A^i \mathbf{r}_0 \right\rangle$, then Relation in Eq. (4) can be written as

$$c(t^i P_k(t)) = 0 \text{ , } i = 0,1,...,k-1 \tag{6}$$

Relation in Eq. (6) indicates that the highest degree of $P_k$is k, and it belongs to the family of orthogonal polynomials with respect to function c. Thus, by using formula $\mathbf{r}_k = P_k(A)\mathbf{r}_0$, we can set up the update solution, which is $\mathbf{x}_k = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{r}_k$. All of this is derived recursively.

Some formulas that Lanczos-type algorithms are based on have been discovered by Brezinski, et al. (see [15,16]). They include formulas A_i and B_j , for $i, j = 1,2,\ldots,10$. The extension of these formulas, which involve polynomials with a difference in degrees of at most two or three, were investigated in Farooq & Salhi [17]. They are formulas A_i and B_j, for $i, j = 11, 12, \ldots, 16$. New variants of the Lanczos-type algorithm, which combine Baheux-type and Farooq-type algorithms, have been investigated in Ulah, et al. [18] and Suharto, et al. [19], who discovered formulas , A13/B6 and A13/B10.

3 Motivation

Breakdown can be avoided in a variety of ways. Here, we suggest regression as a means for that purpose. When breakdown occurs it is possible to remove the last iterate that caused breakdown from the sequence of Lanczos-type algorithm points, regress on this sequence (their entries) to find a suitable model, and then use this model to generate new points that are within the sequence of points that would have been generated by the Lanczos-type algorithm if breakdown had not occurred.

The idea is to exploit patterns that may exist in the sequences generated by a given Lanczos-type algorithm. After running such an algorithm for a certain number of iterations it was pre-emptively stopped before it breaks down. We

then considered the generated iterates to see if any patterns existed. To illustrate what we mentioned above, we considered a Lanczos-type algorithm such as Orthodir [15], and ran it for 30 iterations to solve an SLE in 50 dimensions. We collected all iterates and saved them in Eq. (7) as follows:

$$data_{sol} = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(50)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(50)} \\ \vdots & & & \\ x_{30}^{(1)} & x_{30}^{(2)} & \dots & x_{30}^{(50)} \end{bmatrix} \tag{7}$$

We then plotted all the above iterates by using the Parallel Coordinate System (PCS) [20-23]. By visualizing high-dimensional data, PCS gives insight into the behavior of high-dimensional iterates generated over a number of iterations. This is the main motivation behind our idea to exploit patterns that may exist in iterates generated by Lanczos-type algorithms to improve the robustness of these algorithms.

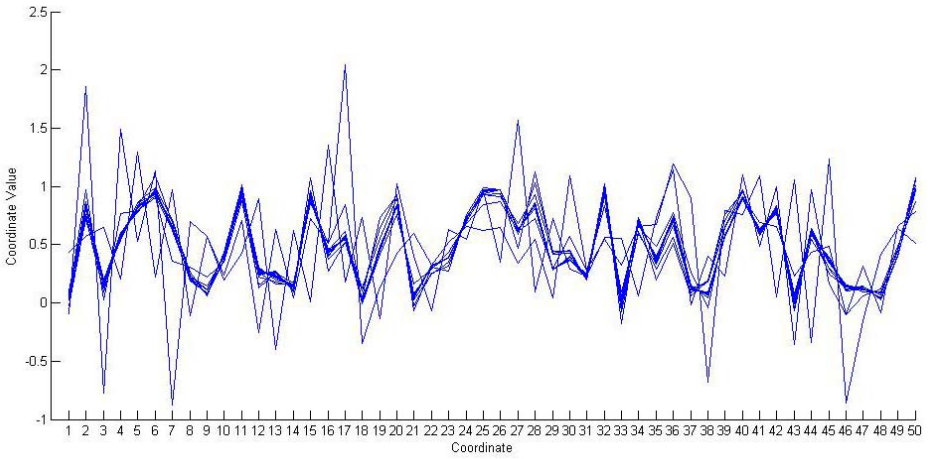


Figure 1 Behavior of EIEM Orthodir and Orthodir algorithms on SLEs in 50 dimensions. The blue line corresponding to EIEM Orthodir, has some very good points and grows much slower than the one corresponding to Orthodir.

Figure 1 illustrates the PCS representation of all iterates generated by Orthodir. The entries of the iterates are represented on the x-axis, while the values of the entries are represented on the y-axis. As we can see here, after some time several iterates start to form similar shapes. Our assumption is that these iterates are close enough to the true solution. In other words, these iterates have a small

residual norm. We investigated this further by exploiting the patterns at the level of a single coordinate. More precisely, we were interested in the patterns of the entries of the iterates. This allowed us to work in a single dimension. If we have k iterates of an n-dimension problem, the following sequence

$$S = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\} \tag{8}$$

holds them all. Now, consider the first entries of all iterates in S, the second entries of all iterates in S, etc. A regression model over all first entries will be used to generate the 1st entry of a new point; a regression model over all 2nd entries will be used to generate the second entry of the new point, etc. In this fashion we generate a new point that, as we will establish later, belongs to the sequence of iterates of the Lanczos-type algorithm that was stopped prematurely according to some norm. This idea should be applicable for instance when Lanczos process-based algorithms are used to generate eigenvalues and eigenvectors and in any iterative process where breakdown is an issue.

4 Generating the Approximate Solutions Using Interpolation and Extrapolation Model

A new approach to enhance the stability of Lanczos-type algorithms was developed, henceforth called Embedded Interpolation and Extrapolation Model in Lanczos-type Algorithms (EIEMLA).

4.1 Derivation of EIEMLA

Consider k iterates that form set *S* as in Eq. (8). As explained above, among the $\mathbf{x}_\mathrm{k}$, there are some iterates with small residual norms in interval $[m-j, k]$. Let $\mathbf{x}_\mathrm{m}$ have a corresponding minimum residual norm, $\|\boldsymbol{r}_m\|$ and set

$$V_1 = \{\mathbf{x}_{m-j}, \mathbf{x}_{m-j+1}, ..., \mathbf{x}_k\} \tag{9}$$

where

$$\begin{aligned}
&\mathbf{x}_{m-j} = [x_{m-j}^{(1)}, x_{m-j}^{(2)}, \cdots, x_{m-j}^{(n)}]^T \\
&\mathbf{x}_{m-j+1} = [x_{m-j+1}^{(1)}, x_{m-j+1}^{(2)}, \cdots, x_{m-j+1}^{(n)}]^T \\
&\vdots \\
&\mathbf{x}_k = [x_k^{(1)}, x_k^{(2)}, \cdots, x_k^{(n)}]^T
\end{aligned} . \tag{10}$$

We assume that the sequences $v_1 = \{x_{m-j}^{(1)}, x_{m-j+1}^{(1)}, \cdots, x_k^{(1)}\}$, $v_2 = \{x_{m-j}^{(2)}, x_{m-j+1}^{(2)}, \cdots, x_k^{(2)}\}$, …, and $v_n = \{x_{m-j}^{(n)}, x_{m-j+1}^{(n)}, \cdots, x_k^{(n)}\}$ have special properties, such as increasing or decreasing monotonically. In this case,

$v_1, v_2, \ldots, v_n$are obtained by rearranging entries in Eq. (10). According to [24], each v_i for i = 1, 2, …,n, is monotonic and convergent to its limit. This means that

$$\lim_{k\to\infty} x_k^{(i)} = x_i^* \tag{11}$$

If we set

$$\begin{aligned} w_1 &= \left\{\left(t_{m-j}, \mathbf{x}_{m-j}^{(1)}\right), \left(t_{m-j+1}, \mathbf{x}_{m-j+1}^{(1)}\right), \ldots, \left(t_k, \mathbf{x}_k^{(1)}\right)\right\} \\ w_2 &= \left\{\left(t_{m-j}, \mathbf{x}_{m-j}^{(2)}\right), \left(t_{m-j+1}, \mathbf{x}_{m-j+1}^{(2)}\right), \ldots, \left(t_k, \mathbf{x}_k^{(2)}\right)\right\} \\ &\vdots \\ w_n &= \left\{\left(t_{m-j}, \mathbf{x}_{m-j}^{(n)}\right), \left(t_{m-j+1}, \mathbf{x}_{m-j+1}^{(n)}\right), \ldots, \left(t_k, \mathbf{x}_k^{(n)}\right)\right\} \end{aligned}, \tag{12}$$

where $t \in R$, then we can use PCHIP to interpolate and extrapolate Eq. (12). Let f_i, for $i = 1,2, \ldots, n$, be functions of the interpolation process. This means they satisfy

$$f_i(t) \cong x_t^{(i)}, \text{ for } i = 1,2, \ldots, n, \tag{13}$$

for some $t = m - j, m - j + 1, \ldots, k$. For $t^* \in [k+1, s] \subset R$, we have

$$f_i(t^*) \cong x_r^{(i)}, \text{ for } i = 1,2, \ldots, n \tag{14}$$

where $s \geq k + 1$. It is guaranteed that $x_r^{(i)}$ has a similar property as $x_t^{(i)}$ since the later vectors are the extrapolation results of the first one. In fact, we use PCHIP to produce, $x_t^{(i)}$, where it preserves the properties of the data and captures the persistent patterns of the data. This process terminates by rearranging vectors $\boldsymbol{x}_r$, with $x_r^{(i)}$ being the i^{th} entries of the vector. This whole process is called Embedded Interpolation and Extrapolation Model in Lanczos-type Algorithms (EIEMLA).

Theoretically, since PCHIP captures the persistent pattern of the i^{th} entry, $i = 1,2, \ldots, n$, of the iterates generated by a Lanczos-type algorithm, the entries of the new iterate, as a result of the model function, are likely to behave the same as those entries. Furthermore, we can produce as many vector solutions as we want by applying the functions f_i over $t^* \in R$ without running the Lanczos-type algorithm again. However, we should be aware that the quality of these generated solutions may not be good enough, in which case we must either restart the iterative process from the best point or take the best solution found so far as the candidate solution. It is therefore, reasonable to choose the integer s

such that the residual norms of the iterates generated by these functions, $\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \ldots, \mathbf{x}_s$, are small enough. In this case, we obtain another sequence of the iterates generated by EIEMLA. It is expected that these iterates replace the 'missing' iterates not generated by the Lanczos-type algorithm due to breakdown.

4.2 Implementation of the EIEMLA Method

The above results suggest a procedure for EIEMLA (see Algorithm 1). As an illustration of the implementation of EIEMLA, consider System in Eq. (1) with $n = 5$. Orthodir is used to solve it over 50 iterates. Following the above procedures, we first collect all of the 50 iterates as. $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{150}\}$. A visualization of several entries of S is shown in Figure 1, Section 1. Secondly, we calculate index m of iterate $\mathbf{x}_m$ associated with the lowest residual norm, $\|\boldsymbol{r}_m\|$. In this particular example, $\|\mathbf{r}_m\| - 0{,}0067$ with $m = 17$. Thus re-arranging S as in Eq. (1) yields the following sequence:

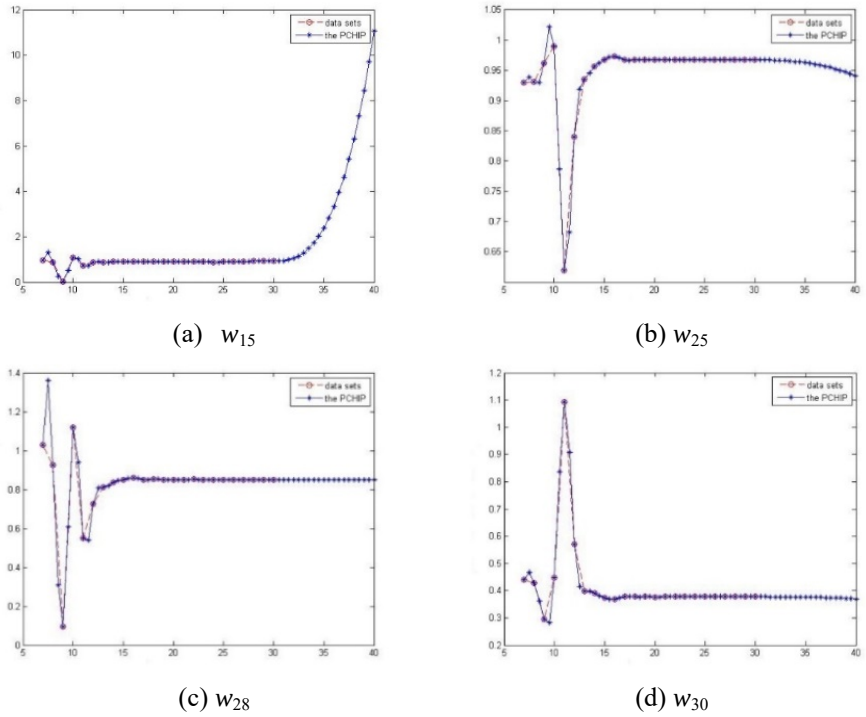
$$\begin{aligned} w_1 &= \left\{\left(t_7, \mathbf{x}_7^{(1)}\right), \left(t_8, \mathbf{x}_8^{(1)}\right), \ldots, \left(t_{50}, \mathbf{x}_{50}^{(1)}\right)\right\} \\ w_2 &= \left\{\left(t_7, \mathbf{x}_7^{(2)}\right), \left(t_8, \mathbf{x}_8^{(2)}\right), \ldots, \left(t_{50}, \mathbf{x}_{50}^{(2)}\right)\right\} \\ w_{50} &= \left\{\left(t_7, \mathbf{x}_7^{(n)}\right), \left(t_8, \mathbf{x}_8^{(n)}\right), \ldots, \left(t_{50}, \mathbf{x}_{50}^{(n)}\right)\right\} \end{aligned} \tag{18}$$

where t is an integer within the range $[m - 10, 50]$. The choice of this range is based on our observation that some good iterates can be found in it. Also, based on Figure 1, the entries of the iterates in that range are increasing or decreasing monotonically.

The next step is to interpolate each w_i in Eq. (18) using PCHIP to get functions f_i. We use these functions to extrapolate and generate points that are outside the range. In this case, we choose $t^* \in [17{,}50]$. The interpolation and extrapolation results are captured in Figure 2. As can be seen, PCHIP interpolates the data accurately and smoothly (see the blue curves). However, the extrapolation results in some points are of poor quality. In Figure 2(a), for instance, the blue curve, which represents the PCHIP data, goes up after the $\{50\}^{th}$ iteration. The PCHIP curve, on the other hand, seems monotone after the $\{50\}^{th}$ iteration in Figures 2(b), (2c), and (2d).

Among the iterates generated by Orthdir, some entries behave as in Figure 2(a). This means their trend goes up or down after k iterations. However, there are many other entries that behave monotonically. Since we put all entries in one vector, overall, the effect of the fluctuating entries is smaller, which means that a good approximate solution may be generated.

After calculating functions f_i over t^*, we now have the approximate solutions as in Eq. (16). We compute the residual norms accordingly. The behavior of EIEM Orthodir and Orthodir for this case, is represented in Figure 3. As can be seen, generally, the residual norms of the iterates generated by EIEM Orthodir (the blue curve) are found below those generated by Orthodir (the red curve) from iteration 17 to 50. In fact, at iteration 31, the residual norm hits the value of 0.003. However, it goes up after iteration 71.



(a) w_{15} (b) w_{25}

(c) w_{28} (d) w_{30}

Figure 2 The interpolation and extrapolation results of sequences w_i, for $i = 7,8, \ldots 50$ as in Eq. (18). The red curve represents all of the entries of the i^{th} sequence, while the blue curve represents their interpolation and extrapolation results using PCHIP.

We can safely say that some iterates generated by EIEM Orthodir have smaller residual norms than all previous iterates generated by the Orthodir. On its own, it does not find solutions with such low residual norms, even with a high number of iterations.

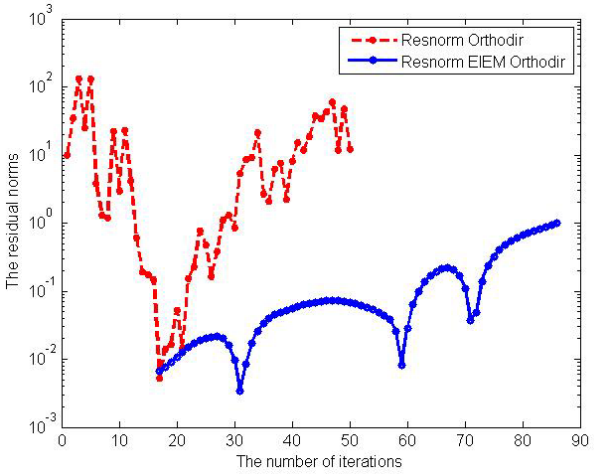


Figure 3 Behavior of EIEM Orthodir and Orthodir algorithms on SLEs in 50 dimensions. The blue line, corresponding to EIEM Orthodir, has some very good points and grows much slower than that corresponding to Orthodir.

Algorithm 1. The EIEMLA method

1. Initialization. Choose $\mathbf{x}_0$ and $\mathbf{y}$, Set $\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$, $\boldsymbol{y}_0 = \boldsymbol{y}$, and $\mathbf{z}_0 = \mathbf{r}_0$.
2. Fix the number of iterations to, say, k, and the tolerance, ϵ, to E-13 and run the Lanczos-type algorithm.
3. IF $\{\|\mathbf{r}_m\| < \in\}$
4. The solution is obtained
5. Stop
6. ELSE
7. Collect all k vector solutions as in Eq. (7).
8. Choose some j such that $m - j < k$.
9. Set w_i as in Eq. (18), for $i = 1,2, \dots, n$.
10. Interpolate w_i using PCHIP to get f_i.
11. Choose $t^* \in [m, s] \subset R$, where $s \leq m \leq k$ is an integer, and calculate $f_i(t^*)$
12. FOR $\{q = 1, 2, \dots, l\}$ DO
13. Arrange vectors

14. $$\mathbf{x}_*^{(q)} = \begin{bmatrix} f_1^{(q)}(t^*) \\ f_2^{(q)}(t^*) \\ \vdots \\ f_n^{(q)}(t^*) \end{bmatrix} \quad (16)$$

15. Where $l = length([m, s])$.
16. Calculate the residual norms of Eq. (16) as follows
17. $\left\|\mathbf{r}_*^{(q)}\right\| = \mathbf{b} - A\mathbf{x}_*^{(q)}$ (17)
18. ENDFOR
19. ENDIF
20. The solutions of the systems are $\left\{x_*^{(1)}, x_*^{(2)}, \ldots, x_*^{(l)}\right\}$
21. Stop.

4.3 Formal Basis of EIEMLA

As mentioned in the previous section, the sequences generated by Lanczos-type algorithms have the property of monotonicity. Since we consider PCHIP, which preserves monotonicity [25], to interpolate the sequences, we can assume that points returned by the function are also monotonic. This leads to the theorem below, which guarantees the monotonicity of sequences generated by Lanczos-type algorithms. The proofs of the theorems can be seen in the Appendix.

Theorem 1. *Given a sequence $\boldsymbol{x}_k$ of k iterates generated by a Lanczos-type algorithm, sequences of $\boldsymbol{x}_k^{(i)}, i = 1, 2, n,$* and $k = 1,$ *... namely the entries of k iterates, are monotonic.*

Lemma 1 [26]. Let $A = VAV^{-1}$ be a nonsingular diagonalizable matrix, where V is a matrix that consists of the eigenvectors of A, S is a diagonal matrix with $A = \begin{bmatrix} B & -I & \cdots & \cdots & 0 \\ -I & B & -I & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & -I & B & -I \\ 0 & \cdots & \cdots & -I & B \end{bmatrix}$ $a_{ii} \in \sigma(A)$ being the diagonal entries, and $\sigma(A)$ is the set of the eigenvalues of A. Then, $\|P_k(A)\| \leq \kappa_2(V) \max_{\lambda \epsilon \sigma(A)} |\lambda|$where $\kappa_2(V)$ is the condition of matrix V.

Theorem 2. *Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k, \boldsymbol{x}_{k+1}$ be the iterates generated by the Orthodir algorithm. Let x_{model} be a vector returned by EIEMLA as explained in the previous section. Then, for some $\in > 0$, $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_{model}\| \leq \in$, where $\|.\|$ is the Euclidean norm.*

Based on the above theorem, we finally show that the residual norm of an iterate generated by EIEMLA is always smaller or equal to that of an iterate

generated by the Lanczos-type algorithm considered. In other words, better solutions are generated through this process.

Theorem 3. *Let* $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ *be the iterates generated by the Orthodir algorithm. Let* r_{model} *be a residual vector that corresponds to the iterate generated by EIEMLA. Then,* $\|r_{model}\| \leq (1 + |\lambda|)\|\boldsymbol{r}_k\|$.

5 Numerical Results and Discussion

Our test problems used in this study arise in the 5-point discretization of the operator $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \gamma \frac{\partial}{\partial x}$ on a rectangular region [27]. This yields System in Eq. (1) with

$$A = \begin{bmatrix} B & -I & \cdots & \cdots & 0 \\ -I & B & -I & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & -I & B & -I \\ 0 & \cdots & \cdots & -I & B \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 4 & \alpha & \cdots & \cdots & 0 \\ \beta & 4 & \alpha & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \beta & 4 & \alpha \\ 0 & \cdots & \cdots & \beta & 4 \end{bmatrix}$$

and $\alpha = -1 + \delta$, $\beta = -1 - \delta$, δ takes values 0; 0.2; 0.5; 0.8; 5; 8. The parameter δ plays an important role in the singularity of matrix *A*. For instance, when $\delta = 0$, A is a symmetric matrix. The condition number of matrix *A* for this particular value of δ is large enough, i.e. 119.9999. Any system built by this matrix tends to be ill-conditioned [28-30]. When δ is large enough, matrix *A* can become well-conditioned. We can also say here that the SLEs that involve matrix *A* have a unique solution.

Furthermore, we use iterations $k = 100 \leq n$. The choice of *k* is based on the observation that Lanczos-type algorithms generally fail to find a good approximate solution after 100 iterations. In addition, the right-hand side $\boldsymbol{b}$ is taken to be $\boldsymbol{b} = A\boldsymbol{x}$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is the solution of the system, and x_i, $i = 1, 2, \ldots, n$ is a random value between 0 and 1. We use formulas A_{13}/B_6 and A_{13}/B_{13} in this experiment as representative of Lanczos-type algorithms. Note, that the first formula is a new type of Lanczos algorithm, investigated in Ullah, *et al*. [18], while the second one is found in Suharto, *et al*. [19].

5.1 Numerical Results

The test problems were carried out under MATLAB 2015b on a computer with an Intel 2.40 GHz processor, Core i7, and 16 GB RAM. The aim of the

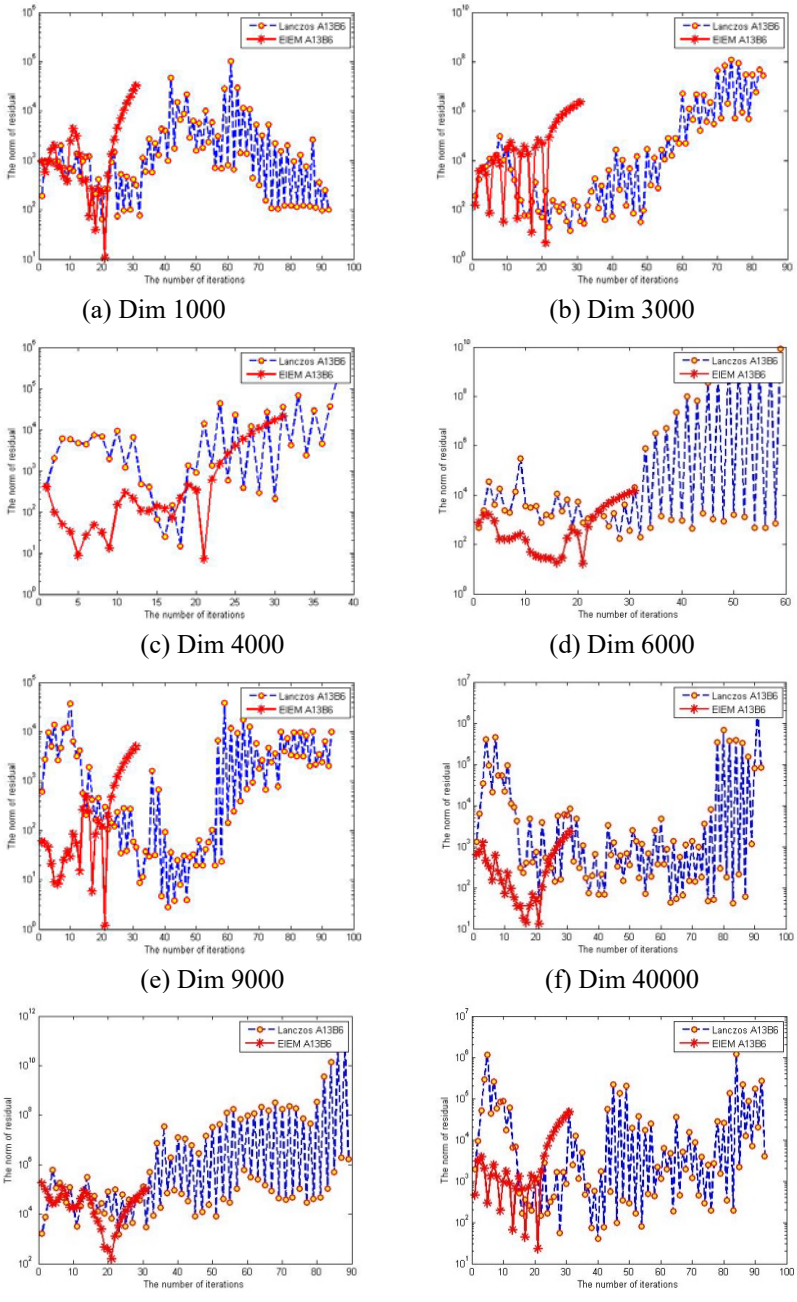
experiment was to examine the reduction in the residual norm of the iterates generated by EIEMLA compared with the original Lanczos-type algorithms. All of the results are recorded in Tables 1 and 2, and visualized graphically.

Table 1 Comparison of residual norms of iterates generated by A_{13}/B_6 and generated by EIEM A_{13}/B_6 over 100 iterations.

Dim n	**$A_{13}B_6$** $\|\mathbf{r}_k\|$	**EIEM $A_{13}B_6$** $\|\mathbf{r}_{model}\|$	**Time (s)**
1000	3.62E+02	10.6498	0.3232
2000	6.42E+04	8.4998	0.3903
3000	NaN	4.9849	0.6783
4000	1.87E+09	12.3174	17.8791
5000	1.87E+09	12.3174	0.6783
6000	NaN	16.6164	0.7714
7000	4.69E+03	12.7736	0.8029
8000	5.03E+08	2.2334	1.0650
9000	3.54E+03	1.1718	0.9489
10000	9.56E+05	14.9270	1.0276
20000	NaN	9.6835	1.6685
30000	7.24E+10	152.9614	2.6506
40000	1.15E+03	13.1240	3.3747
50000	1.13E+10	53.2363	4.3543
60000	1.67E+06	154.0032	5.1728
70000	NaN	138.9045	5.8259
80000	NaN	82.3836	6.6018
90000	7.13E+03	23.2396	7.1726
100000	1.53E+07	4.4710	8.0383

Table 2 Comparison of residual norms of iterates generated by A_{13}/B_{13} and generated by EIEM A_{13}/B_{13} over 100 iterations.

Dim n	**$A_{13}B_{13}$** $\|\mathbf{r}_k\|$	**EIEM $A_{13}B_{13}$** $\|\mathbf{r}_{model}\|$	**Time (s)**
1000	NaN	7.0391	0.3607
2000	NaN	3.6217	0.4311
3000	NaN	57.3744	0.7274
4000	NaN	50.1365	1.0101
5000	NaN	40.5467	1.5103
6000	NaN	3.5261	2.0745
7000	NaN	29.2577	2.6821
8000	NaN	78.5510	2.9997
9000	NaN	713.7055	4.2056
10000	NaN	9.9809	4.9926
20000	NaN	19.6627	20.0117
30000	NaN	24.7834	54.4602



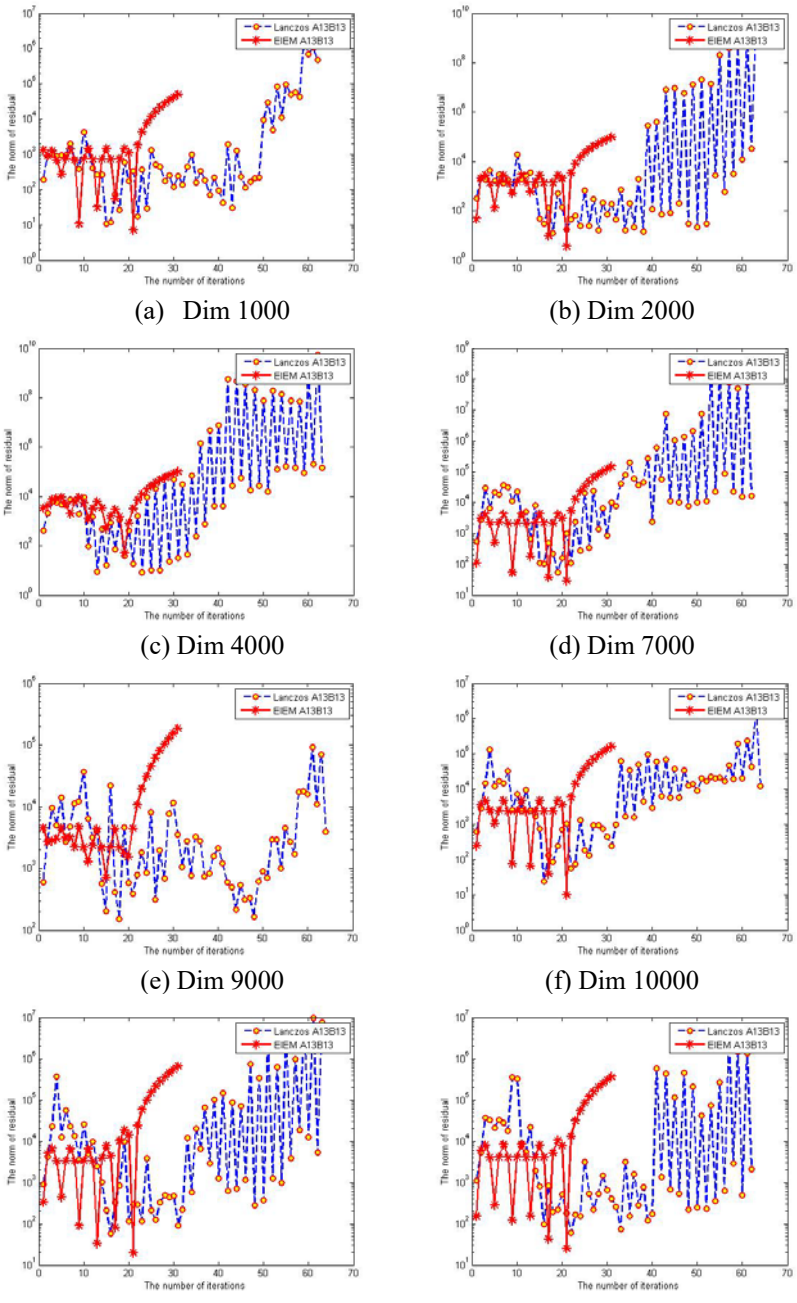
(a) Dim 1000 (b) Dim 3000

(c) Dim 4000 (d) Dim 6000

(e) Dim 9000 (f) Dim 40000

(g) Dim 60000 (h) Dim 90000

Figure 4 Performance of Lanczos A_{13}/B_6 and EIEM A_{13}/B_6 on a variety of SLEs. Blue and red curves illustrate the residual norms of respectively Lanczos A_{13}/B_6 and EIEM A_{13}/B_6.



(a) Dim 1000 (b) Dim 2000

(c) Dim 4000 (d) Dim 7000

(e) Dim 9000 (f) Dim 10000

(g) Dim 20000 (h) Dim 30000

Figure 5 Performance of Lanczos A_{13}/B_{13} and EIEM A_{13}/B_{13} on a variety of SLEs. Blue and red curves illustrate the residual norms of respectively Lanczos A_{13}/B_{13} and EIEM A_{13}/B_{13} .

5.2 Discussion

Overall, the approximate solutions generated by the new algorithms, EIEM A_{13}/B_6 and EIEM A_{13}/B_{13}, were better than those from all of the previous iterates generated by the Lanczos-type algorithms alone. The residual norms of the iterates generated by both algorithms were smaller than all residual norms of the iterates generated by algorithms A_{13}/B_6 and A_{13}/B_{13} individually. This is clearly visible in Tables 1 and 2. This was even worse for Lanczos A_{13}/B_{13}, where breakdown occurred in every SLE. There were, however, some cases where the new method seemed unsuccessful. It occurred, for instance, on the SLEs of dimensions 30000, 60000 and 70000 for algorithm EIEM A_{13}/B_6, and on the SLE of dimension 9000 for algorithm EIEM A_{13}/B_{13}.

Our above observations are supported by Figures 5 and 6. As can be seen in both figures, the residual norms of the iterates generated by either A_{13}/B_6 or A_{13}/B_{13} individually, became larger and larger so that convergence was never reached. In contrast, both EIEM A_{13}/B_6 and EIEM A_{13}/B_{13} generated approximate solutions with small residual norms, even smaller than the smallest residual norm of the iterates generated by A_{13}/B_6 and A_{13}/B_{13} alone. Note that the residual norms of the solutions generated by EIEM A_{13}/B_6 and A_{13}/B_{13} are still larger than the ambitious tolerance of 1E-13 we have imposed. One way of improving accuracy is perhaps to restart the algorithms with the final iterates, which are rather good. A discussion of the restarting strategy can be found in [31].

6 Summary

We have introduced a new approach to solving high-dimension SLEs by embedding an interpolation model in Lanczos-type algorithms, called EIEMLA. We implemented this approach for algorithms A_{13}/B_6 and A_{13}/B_{13}, which are new Lanczos-type algorithms. Experimentally, the idea enables us to find a good solution in terms of the residual norm, i.e. the solutions must have smaller residual norms than all of the previous iterates generated by algorithms A_{13}/B_6 and A_{13}/B_{13}. This helps avoid breakdown in both algorithms since we interpolate over a low number of iterations. Further investigation is required to find a prediction strategy, such as a regression, that fits the data in such a way that the embedded model produces better iterates still.

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