Countable Fuzzy Topological Space and Countable Fuzzy Topological Vector Space

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Abstract: This paper deals with countable fuzzy topological spaces, a generalization of the notion of fuzzy topological spaces. A collection of fuzzy sets F on a universe X forms a countable fuzzy topology if in the definition of a fuzzy topology, the condition of arbitrary supremum is relaxed to countable supremum. In this generalized fuzzy structure, the continuity of fuzzy functions and some other related properties are studied. Also the class of countable fuzzy topological vector spaces as a generalization of the class of fuzzy topological vector spaces has been introduced and investigated.

Keywords: *countable fuzzy topological space; countable fuzzy topological vector space; fuzzy topology.*

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1 Introduction and Preliminaries

In this research the notions of countable fuzzy topology and countable fuzzy topological vector space as the generalizations of fuzzy topology and fuzzy topological vector space respectively have been introduced and studied. Some of the generalizations of fuzzy topology found in the literature are that of fuzzy supra-topology, introduced by Monsef and Ramadan [1], and the fuzzy minimal structure introduced by Alimohammady and Roohi [2]. We recall that, a family $\mathcal{F}' \subset I^X$ is called a fuzzy supra-topology on X if $\mathbf{0}_X$, $\mathbf{1}_X \in \mathcal{F}'$ and $\mathcal{F}'$ is closed under arbitrary union. Again, a family $\mathcal{M}$ of fuzzy sets in X is a fuzzy minimal structure on X if $\alpha\mathbf{1}_X \in \mathcal{M}$ for any $\alpha \in I$. On the other hand, Das and Baishya [3] defined another type of fuzzy topological space, viz. mixed fuzzy topological space with respect to a fuzzy point x_λ. Tripathy and Ray [4] generalized this concept and defined a new type of mixed fuzzy topological space. To find out more about mixed fuzzy topological space and its properties see references [5-7]. It can be seen that these generalized forms of fuzzy structures have important applications in many areas of science and technology. Many properties of topological spaces such as boundedness and continuity can be discussed through its generalized forms. Under this perspective, the motivation

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of the present paper is to study a generalized concept of fuzzy topological space introduced in [8], which is restrictive enough to include the basic properties of fuzzy topological space and at the same time has the potential of generalizing the theory further. Also, due to the relaxation of the stringent condition regarding preservation of arbitrary supremum in this generalized structure, the results obtained by generalizing the existing theories may be applied in various fields of science and technology. It may be further mentioned here that this new structure is capable of cardinal extension [9]. The proposed structure generalizes the notion of fuzzy topology but stronger than that of the fuzzy minimal structure. So it would be interesting to study the properties of this new generalized space.

For easy understanding of the material presented in this paper we recall some basic definitions. For details we refer to [10], [11] and [12]. The fuzzy set on a universe X is a function with domain X and values in I = [0, 1]. The class of all fuzzy sets on X will be denoted by I^X and symbols λ, δ, . . . are used to denote fuzzy sets on X. $\mathbf{01_X}$ is called empty fuzzy set, whereas $\mathbf{1_X}$ is the characteristic function on X. A family $\mathscr{F}$ of fuzzy sets in X is called a fuzzy topology for X if (i) $\mathbf{01_X}$, $\mathbf{1_X} \in \mathscr{F}$, (ii) $\lambda \wedge \delta \in \mathscr{F}$, where $\lambda, \delta \in \mathscr{F}$ and (iii) $\vee_{j\in J}\lambda_j \in \mathscr{F}$ whenever, $\lambda_j \in \mathscr{F}$ for all j in J. The pair (X, $\mathscr{F}$) is called a fuzzy topological space [13]. Further, Lowen [14] suggested an alternative and more natural definition of fuzzy topology for achieving more results that are compatible with the general case in topology by incorporating all constant functions instead of only $\mathbf{01_X}$ and $\mathbf{1_X}$ in the condition (i) above.

Every member of $\mathscr{F}$ is called a fuzzy open set. Its complement is called fuzzy closed set. In a fuzzy topological space X, the interior and the closure of a fuzzy set λ (denoted by Int(λ) and Cl(λ) respectively) are defined by

$$\mathrm{Int}(\lambda) = \vee\{\delta : \delta \le \lambda, \delta \text{ is fuzzy open in X}\},$$

$$\mathrm{Cl}(\lambda) = \wedge\{\gamma : \lambda \le \gamma, \gamma \text{ is a fuzzy closed set in X}\}.$$

Let f be a function from X to Y. Then f(λ) is a fuzzy set in Y defined by

$$f(\lambda)(y) = \begin{cases} \vee_{x\in f^{-1}(\{y\})}\lambda(x), & f^{-1}(\{y\}) \neq \varnothing \\ 0, & f^{-1}(\{y\}) = \varnothing \end{cases}$$

for all y in Y, where λ is an arbitrary fuzzy set in X and $f^{-1}(\delta)$ is a fuzzy set in X, defined by $f^{-1}(\delta)(x) = \delta(f(x))$, $x \in X$, where δ is a fuzzy set in Y.

In [8] and [15] we discuss the r-countably induced fuzzy topological space and define the notion of countable fuzzy topology in the following way:

A family $\mathscr{F}$ of fuzzy sets in X is said to form a countable fuzzy topology if

i) $0\mathbf{1}_X$, $\mathbf{1}_X \in \mathscr{F}$.
ii) For countable family $\{\lambda_i : i \in N\}$ of fuzzy subsets of $\mathscr{F}$, $\bigvee_i \lambda_i \in \mathscr{F}$.
iii) For any two fuzzy subset λ and μ of $\mathscr{F}$, $\lambda \wedge \mu \in \mathscr{F}$.

The space (X, $\mathscr{F}$) is called countable fuzzy topological space.

In Section 2, we treat the modified definition of countable fuzzy topology by incorporating all constant functions instead of only $0\mathbf{1}_X$ and $\mathbf{1}_X$ in the condition (i) above and studied some of its properties. In Section 3, countable fuzzy initial topology, countable fuzzy product space and some related concepts are discussed. Finally in Section 4, the concept of linear countable fuzzy topological space, as a generalization of linear fuzzy topological space, is introduced. We have examined some of its properties.

2 Countable Fuzzy Topological Space

Throughout this paper we take the following definition of countable fuzzy topological space:

Definition 2.1 A family $\mathcal{C}$ of fuzzy sets in X is said to form a countable fuzzy topology if

i) $r\mathbf{1}_X \in \mathcal{C}$ for any $r \in I$.
ii) For countable family $\{\lambda_i : i \in N\}$ of fuzzy subsets of $\mathcal{C}$, $\bigvee_{i\in N} \lambda_i \in \mathcal{C}$.
iii) For any two fuzzy subsets λ and μ of $\mathcal{C}$, $\lambda \wedge \mu \in \mathcal{C}$.

The space (X, $\mathcal{C}$) is called countable fuzzy topological space. Every member of $\mathcal{C}$ is called a c-open set of X and the complement of a c-open set is called a c-closed set.

To provide an example of countable fuzzy topology, we recall the definitions of regular G_δ-subsets and regular lower semi continuous (RLSC) functions.

Definition 2.2 [16] A subset H of a topological space X is called a regular G_δ-subset if H is an intersection of a sequence of closed sets whose interiors contain H. Equivalently, if $H = \bigcap_i G_i = \bigcap_i cl_X G_i$ for $i \in N$, where each G_i is open in X, then H is a regular G_δ-subset of X.

Definition 2.3 [17] A function f: X $\rightarrow$ R is RLSC if for each real number r, the Lebesgue set $L_r(f) = \{x : f(x) \leq r\}$ is a regular G_δ-subset of X.

Now, we give an example of countable fuzzy topological space.

Example 2.4 [8] Let (X, T) be a topological space and let $\mathcal{C}$ = {f: f is a function from X to I and f is RLSC}. Then $\mathcal{F}$ is a countable fuzzy topology on X and (X, $\mathcal{C}$) is the countable fuzzy topological space.

Remark 2.5 It may be worth mentioning here that every fuzzy topology is a countable fuzzy topology and every countable fuzzy topology is a fuzzy minimal structure.

Definition 2.6 A fuzzy point x_t is said to be quasi-coincident with a fuzzy set β, denoted by $x_t\, q\, \beta$ iff $t > 1-\beta(x)$ i.e. $t + \beta(x) > 1$. Again, a fuzzy set α is said to be quasi-coincident with another fuzzy set β, denoted by $\alpha q \beta$ iff there exists $x \in X$ such that $\alpha(x) > 1-\beta(x)$ i.e. $\alpha(x) + \beta(x) > 1$.

Definition 2.7 We set the definition of interior and closure of a fuzzy set λ in countable fuzzy topology denoted by c-Int(λ) and c-Cl(λ) respectively as follows:

$$\text{c-Int}(\lambda) = \bigvee\{\delta: \delta \le \lambda, \delta \in \mathcal{C}\} \text{ and } \text{c-Cl}(\lambda) = \bigwedge\{\gamma: \lambda \le \gamma, (1-\gamma) \in \mathcal{C}\}.$$

Remark 2.8 It should be noted here that in a countable fuzzy topological space, arbitrary union of c-open sets may not be c-open [8]. Thus, c-interior of a fuzzy set in (X, $\mathcal{C}$) may not be c-open, and dually c-closure of a fuzzy set in (X, $\mathcal{C}$) may not be c-closed.

Proposition 2.9 For any two fuzzy sets λ and μ

i) c-Int(λ) $\le \lambda$ and c-Int(λ) = λ, if λ is a c-open set. Specially c-Int($r1_X$) = $r1_X$ for all $r \in I$.
ii) $\lambda \le$ c-Cl(λ) and λ = c-Cl(λ), if λ is a fuzzy c-closed set. Specially c-Cl($r1_X$) = $r1_X$ for all $r \in I$.
iii) c-Int(λ) $\le$ c-Int(μ) and c-Cl(λ) $\le$ c-Cl(μ), if $\lambda \le \mu$.
iv) c-Int($\lambda \wedge \mu$) = c-Int(λ)$\wedge$ c-Int(μ) and c-Int(λ)$\vee$ c-Int(μ) $\le$ c-Int($\lambda \vee \mu$).
v) c-Cl($\lambda \vee \mu$) = c-Cl(λ)$\vee$ c-Cl(μ) and c-Cl($\lambda \wedge \mu$) $\le$ c-Cl(λ)$\wedge$ c-Cl(μ).
vi) c-Int(c-Int(λ)) = c-Int(λ) and c-Cl(c-Cl(μ)) = c-Cl(μ).
vii) $1-$ c-Cl(λ) = c-Int($1-\lambda$) and $1-$ c-Int(λ) = c-Cl($1-\lambda$).

Proof. The proof of (i), (ii) and (iii) can be obtained directly from the definition. Here we prove (v) and (vii). The proof of (iv) is similar to (v) and (vi) is similar to (vii).

(v) We note that $\lambda \le \lambda \vee \mu$ and $\mu \le \lambda \vee \mu$. Therefore from (iii), we have

$\text{c-Cl}(\lambda) \le \text{c-Cl}(\lambda \vee \mu)$ and $\text{c-Cl}(\mu) \le \text{c-Cl}(\lambda \vee \mu)$.

Thus $\text{c-Cl}(\lambda) \vee \text{c-Cl}(\mu) \le \text{c-Cl}(\lambda \vee \mu)$.

Again, $\text{c-Cl}(\lambda) \vee \text{c-Cl}(\mu) = (\wedge\{\gamma: \lambda \le \gamma, (1-\gamma) \in \mathcal{C}\}) \vee (\wedge\{\delta: \mu \le \delta, (1-\delta) \in \mathcal{C}\})$

$= \wedge\{\gamma \vee \delta : \lambda \vee \mu \le \gamma \vee \delta, 1-(\gamma \vee \delta) \in \mathcal{C}\}$

$\ge \wedge\{\beta: \lambda \vee \mu \le \beta, 1-\beta \in \mathcal{C}\}$, where β is any closed set containing $\lambda \vee \mu$.

$= \text{c-Cl}(\lambda \vee \mu)$.

Thus, $\text{c-Cl}(\lambda) \vee \text{c-Cl}(\mu) = \text{c-Cl}(\lambda \vee \mu)$.

Since $\lambda \wedge \mu \le \lambda$ and $\lambda \wedge \mu \le \mu$, so $\text{c-Cl}(\lambda \wedge \mu) \le \text{c-Cl}(\lambda)$ and $\text{c-Cl}(\lambda \wedge \mu) \le \text{c-Cl}(\mu)$

Hence, $\text{c-Cl}(\lambda \wedge \mu) \le \text{c-Cl}(\lambda) \wedge \text{c-Cl}(\mu)$.

(vii) We have

$$1-\text{c-Cl}(\lambda) = 1-(\wedge\{\gamma: \lambda \le \gamma, (1-\gamma) \in \mathcal{C}\}) = \vee\{(1-\gamma): \lambda \le \gamma, (1-\gamma) \in \mathcal{C}\}$$

$$= \vee\{(1-\gamma): 1-\gamma \le 1-\lambda, (1-\gamma) \in \mathcal{C}\} = \vee\{\delta: \delta \le 1-\lambda, \delta \in \mathcal{C}\}$$

$$= \text{c-Int}(1-\lambda)$$

Also, $1-\text{c-Int}(\lambda) = 1-\vee\{\delta: \delta \le \lambda, \delta \in \mathcal{C}\} = \wedge\{(1-\delta): \delta \le \lambda, \delta \in \mathcal{C}\}$

$= \wedge\{(1-\delta): 1-\lambda \le 1-\delta, \delta \in \mathcal{C}\} = \wedge\{\gamma: 1-\lambda \le \gamma, 1-\gamma \in \mathcal{C}\}$

$= \text{c-Cl}(1-\lambda)$

Definition 2.10 Let $(X, \mathcal{C})$ and $(X, \mathcal{D})$ be two countable fuzzy topological spaces. Then a fuzzy function $f: (X, \mathcal{C}) \to (X, \mathcal{D})$ is said to be countable fuzzy continuous (briefly fuzzy c-continuous) if $f^{-1}(\lambda) \in \mathcal{C}$ for any $\lambda \in \mathcal{D}$.

Remark 2.11 Consider $\mathcal{C} = \mathcal{F}_1$ and $\mathcal{D} = \mathcal{F}_2$, where $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ are fuzzy topological spaces in Lowen's sense. It is easily seen that $f: (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is fuzzy continuous if and only if $f: (X, \mathcal{C}) \to (Y, \mathcal{D})$ is fuzzy c-continuous.

We state the following results without proof.

Proposition 2.12 Suppose (X, C) and (Y, D) are two countable fuzzy topological spaces. Then

a) The identity map $id_X: (X, \mathcal{C}) \to (X, \mathcal{C})$ is fuzzy c-continuous.
b) $id_X: (X, \mathcal{C}) \to (X, \mathcal{D})$ is fuzzy c-continuous where $\mathcal{D} \le \mathcal{C}$.
c) Any constant function $f: (X, \mathcal{C}) \to (Y, \mathcal{D})$ is fuzzy c-continuous.

Theorem 2.13 The composition of any two fuzzy c-continuous functions is fuzzy c-continuous.

Proof. The proof follows directly from the fact that $(fog)^{-1}(\lambda) = g^{-1}(f^{-1})(\lambda)$ and by the c-continuity of f and g.

The following theorem characterizes fuzzy c-continuous function:

Theorem 2.14 Consider the following properties for a fuzzy function $f: (X, \mathcal{C}) \to (Y, \mathcal{D})$ between two countable fuzzy topological spaces.

(a) f is a fuzzy c-continuous function.
(b) $f^{-1}(\delta)$ is a fuzzy c-closed set in $(X, \mathcal{C})$ for each fuzzy c-closed set $\delta \in (Y, \mathcal{D})$.
(c) $c\text{-}Cl(f^{-1}(\delta)) \leq f^{-1}(c\text{-}Cl(\delta))$ for each $\delta \in I^Y$.
(d) $f(c\text{-}Cl(\lambda)) \leq c\text{-}Cl(f(\lambda))$ for any $\lambda \in I^X$.
(e) $f^{-1}(c\text{-}Int(\delta)) \leq c\text{-}Int(f^{-1}(\delta))$for each $\delta \in I^Y$.

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e).

Proof. (a) $\Rightarrow$ (b)

We consider a fuzzy c-closed set δ in Y, then $(1-\delta) \in \mathcal{D}$.
So $1 - f^{-1}(\delta) = f^{-1}(1-\delta) \in \mathcal{C}$. i. e. $f^{-1}(\delta)$ is a fuzzy c-closed set in X.

(b) $\Rightarrow$ (a) is straightforward.

(b) $\Rightarrow$(c)

$$c\text{-}Cl(f^{-1}(\delta)) = \wedge\{\gamma: f^{-1}(\delta) \leq \gamma, (1-\gamma) \in \mathcal{C}\})$$

Let $1-\beta \in \mathcal{D}$ be such that $\delta \leq \beta$. Then, $f^{-1}(\delta) \leq f^{-1}(\beta)$. Now, by (b), $1 - f^{-1}(\beta) \in \mathcal{C}$.

Hence, by definition of c-closure,

$c\text{-}Cl(f^{-1}(\delta)) = \wedge\{\gamma: f^{-1}(\delta) \leq \gamma, (1-\gamma) \in \mathcal{C}\} \leq \wedge\{f^{-1}(\beta): \delta \leq \beta, 1-\beta \in \mathcal{D}\}$(since, $\{f^{-1}(\beta): 1-\beta \in \mathcal{D}\}$ is a subfamily of the closed sets of $\mathcal{C}$).

Thus $c\text{-}Cl(f^{-1}(\delta)) \leq \wedge\{f^{-1}(\beta): \delta \leq \beta, 1-\beta \in \mathcal{D}\} = f^{-1}(\wedge\{\beta: \delta \leq \beta, 1-\beta \in \mathcal{D}\} = f^{-1}(c\text{-}Cl(\delta))$.

(c) $\Rightarrow$ (d)

Since $\lambda \leq f^{-1}(f(\lambda))$, then

$c\text{-}Cl(\lambda) \leq c\text{-}Cl(f^{-1}(f(\lambda))) \leq f^{-1}(c\text{-}Cl(f(\lambda)))$.

Consequently, $f(c\text{-}Cl(\lambda)) \leq c\text{-}Cl(f(\lambda))$.

(d) $\Rightarrow$ (e)

$f(1-(c\text{-}Int(f^{-1}(\delta))) = f(c\text{-}Cl(1- f^{-1}(\delta))) = f(c\text{-}Cl(f^{-1}(1- \delta))) \leq c\text{-}Cl(f(f^{-1}(1- \delta)))$

$\leq c\text{-}Cl(1- \delta) = 1- c\text{-}Int(\delta)$.

Which implies, $1-(c\text{-}Int(f^{-1}(\delta))) \leq f^{-1}(1- c\text{-}Int(\delta)) = 1- f^{-1}(c\text{-}Int(\delta))$

Consequently, $f^{-1}(c\text{-}Int(\delta)) \leq c\text{-}Int(f^{-1}(\delta))$.

(e) ⇒ (c)

$1- f^{-1}(c\text{-}Cl(\delta)) = f^{-1}(1- (c\text{-}Cl(\delta))) = f^{-1}(c\text{-}Int(1- \delta)) \leq c\text{-}Int\ (f^{-1}(1- \delta))$, by (e)

$= c\text{-}Int(1- f^{-1}(\delta)) = 1- (c\text{-}Cl(f^{-1}(\delta)))$.

Thus, $c\text{-}Cl(f^{-1}(\delta)) \leq f^{-1}(c\text{-}Cl(\delta))$.

3 Product Countable Fuzzy Topology

To define the notion of product countable fuzzy topology, at the outset we present some concepts that will be used in the following discussion without any specific reference.

Lemma 3.1 The intersection of any collection of countable fuzzy topological spaces is a countable fuzzy topological space.

Proof. Let $\mathcal{B} = \{\mathcal{C}: \mathcal{C}$ is a countable fuzzy topology on X$\}$ and set $\mathcal{C}_0 = \bigcap\{\mathcal{C}: \mathcal{C}\in\mathcal{B}\}$. To prove $\mathcal{C}_0$ is a countable fuzzy topology, we have the following:

(i) Clearly $r1_X \in C_0$.

(ii) Let us consider a countable family of functions (fuzzy sets) $\{f_i: i\in N\} \subseteq C_0$, i.e. $\{f_i: i\in N\} \subseteq C$ for each $C\in B$ which implies that $\sup\{f_i: i\in N\}\in C$ for each $C\in B$ and hence $\sup\{f_i: i\in N\}\in \bigcap C = C_0$.

(iii) Let $\{f_1, f_2\} \subseteq C_0$,which implies $\{f_1, f_2\} \subseteq C$ for each $C\in B$, which in turn implies that $\inf\{f_1, f_2\}\in C$ for each $C\in B$. Thus, $\inf\{f_1, f_2\}\in \bigcap C = C_0$.

Therefore, the intersection of any collection of countable fuzzy topological spaces is again a countable fuzzy topological space.

Using the above lemma, we prove the following theorem which will lead to the definition of initial countable fuzzy topological space.

Theorem 3.2 Suppose $(Y, \mathcal{D})$ is a countable fuzzy topological space and $f: X \to (Y, \mathcal{D})$ a fuzzy function. Then, there exists a weakest countable fuzzy topology on X for which f is fuzzy c-continuous.

Proof. Let $\mathscr{B}$ = {$\mathcal{C}$: $\mathcal{C}$ is a countable fuzzy topology on X and f: (X, $\mathcal{C}$) → (Y, $\mathcal{D}$) be a fuzzy c-continuous function}. Clearly, $I^X \in \mathscr{B}$ and so $\mathscr{B} \neq \varnothing$. Then, by Lemma 3.1, $\mathcal{C}_0 = \bigcap\{\mathcal{C}: \mathcal{C} \in \mathscr{B}\}$ is a countable fuzzy topology on X, which is the weakest countable fuzzy topology on X with the property f is a fuzzy c-continuous function.

The following result is a consequence of the above theorem.

Corollary 3.3 Suppose (X, $\mathcal{C}$) is a countable fuzzy topological space and Y ⊆ X. Then there is a weakest countable fuzzy topology on Y, say $\mathcal{D}$, such that the inclusion map *i*: (Y, $\mathcal{D}$) → (X, $\mathcal{C}$) is fuzzy c-continuous.

We call $\mathcal{D}$ the induced countable fuzzy topology by $\mathcal{C}$ on Y.

Corollary 3.4 Suppose Y ⊆ X and f: (X, $\mathcal{C}$) → (Z, $\mathcal{E}$) are fuzzy c-continuous. Then f |Y is fuzzy c-continuous if Y is endowed with induced countable fuzzy topology.

Proof. Clearly, f |Y = f ∘ *i* where *i*: Y → X is the inclusion map and Y is endowed with induced countable fuzzy topology. Then by Corollary 3.3, *i*: Y → X is fuzzy c-continuous. Again, by Theorem 2.13, the composition of two fuzzy c-continuous function is fuzzy c-continuous and hence f |Y is fuzzy c-continuous.

Theorem 3.5 Consider a family $\{(X_j, \mathcal{D}_j), j \in J\}$ of countable fuzzy topological spaces and $f_j: X \to X_j$, a family of fuzzy functions. Then there is a weakest countable fuzzy topology on X such that each f_j is fuzzy c-continuous.

Proof. From the Theorem 3.2, for any $j \in J$ there is a weakest countable fuzzy topology $\mathcal{C}_j$ on X for which $f_j: (X, \mathcal{C}_j) \to (X_j, \mathcal{D}_j)$ is a fuzzy c-continuous function. Then, clearly, $\mathcal{C}* = \bigcup\{\mathcal{C}_j: j \in J\}$ is the subbase for a countable fuzzy topology on X making each f_j fuzzy c-continuous. Moreover, it can be easily verified that this is the smallest countable fuzzy topology with this property.

Definition 3.6 The weakest countable fuzzy topology generated on X by the family $\{(X_j, \mathcal{D}_j), j \in J\}$ of countable fuzzy topological spaces and a family of fuzzy functions $f_j: X \to X_j$ is called the initial countable fuzzy topological space that makes each f_j fuzzy c-continuous.

Remark 3.7 For any set Y, a countable fuzzy topological space (X, $\mathcal{C}$) and a function f: (X, $\mathcal{C}$) → Y it can be easily verified that $\mathcal{D} = \{\delta \in I^Y: f^{-1}(\delta) \in \mathcal{C}\}$ is the finest countable fuzzy topology on Y making f a fuzzy c-continuous function. More generally, consider a family of countable fuzzy topological spaces $(X_j, \mathcal{C}_j)$, $j\in J$ and for each $j\in J$, a function f_j: $X_j \to Y$, then the intersection $\cap_j\mathcal{D}_j$ is the finest countable fuzzy topology on Y making all f_j fuzzy c-continuous functions, where for each $j\in J$, $\mathcal{D}_j$ is the finest countable fuzzy topology on Y for which f_j: $(X_j, \mathcal{C}_j) \to (Y, \mathcal{D}_j)$ is fuzzy c-continuous. This topology $\cap_j\mathcal{D}_j$ on Y is called the final countable fuzzy topology.

Theorem 3.8 Consider a family f_j: $X \to (X_j, \mathcal{C}_j)$, $j\in J$, of fuzzy functions, where $(X_j, \mathcal{C}_j)$ are countable fuzzy topological spaces and X is a given set equipped with the initial countable fuzzy topology $\mathcal{C}$. Then, a fuzzy function f: $(Y, \mathcal{D}) \to (X, \mathcal{C})$ is fuzzy c-continuous if and only if $f_j \circ f$ is a fuzzy c-continuous function for all $j\in J$.

Proof. If f is a fuzzy c-continuous function, then from Theorem 2.13, for each $j\in J$, $f_j \circ f$ is a fuzzy c-continuous function. Conversely, let each $f_j \circ f$ be a fuzzy c-continuous function for each $j\in J$, with f is not a fuzzy c-continuous function, which means that there is $\lambda\in\mathcal{C}$ such that $f^{-1}(\lambda)\notin\mathcal{D}$. Then one of the following two statements holds:

i) For any particular $i\in J$, there exists $\lambda_i\in\mathcal{C}_i$ such that $\lambda = f_i^{-1}(\lambda_i)$
ii) For all $j\in J$ and for all $\lambda_j\in\mathcal{C}_j$, $\lambda\neq f_j^{-1}(\lambda_j)$.

In case (i), we have

$$f^{-1}(\lambda) = f^{-1}(f_i^{-1}(\lambda_i)) = (f_i \circ f)^{-1}(\lambda_i).$$

Therefore $(f_i \circ f)^{-1}\notin\mathcal{D}$, which is a contradiction because $(f_i \circ f)$ is fuzzy c-continuous.

In case (ii), since $f^{-1}(r1_X) = r1_Y$, so $\lambda \neq r1_X$, for each $r\in I$. Then $\mathcal{C} -\{\lambda\}$ is a countable fuzzy topology on X. But for each $j\in J$, f_j: $(X, \mathcal{C} - \{\lambda\}) \to (X_j, \mathcal{C}_j)$ is fuzzy c-continuous, which is a contradiction with the choice of $\mathcal{C}$, as $\mathcal{C}$ is the initial (weakest) countable fuzzy topology on X.

Thus f is fuzzy c-continuous.

As a consequence of Theorem 3.5 we present a product countable fuzzy topology for an arbitrary family $\{(X_j, \mathcal{C}_j): j\in J\}$ of countable fuzzy topological spaces. The product countable fuzzy topology on $X = \prod_{j\in J}X_j$ is the weakest countable fuzzy topology on X (denoted by $\mathcal{C} = \prod_{j\in J}\mathcal{C}_j$), such that for each $k\in J$, the canonical projection π_k: $\prod_{j\in J}X_j \to X_k$ is a fuzzy c-continuous function.

Corollary 3.9 For any family $\{(X_j, \mathcal{C}_j): j\in J\}$ of countable fuzzy topological spaces, a fuzzy product countable topology on $X = \prod_{j\in J} X_j$ exists.

Proof. In Theorem 3.5, if we replace f_j by π_j, then we get the required theorem.

Corollary 3.10 Suppose the family $(X_j, \mathcal{E}_j)$ consists of countable fuzzy topological spaces and $X = \prod_{j\in J} X_j$ is equipped with the product countable fuzzy topology generated by projection maps $\{\pi_j: X \to X_j, j\in J\}$. Then $f: (Y, \mathcal{D}) \to (X, \mathcal{C})$ is a fuzzy c-continuous function if and only if $\pi_j \circ f$ is a fuzzy c-continuous function for all $j\in J$.

Proof. The proof follows immediately from Theorem 3.8.

Corollary 3.11 Suppose $f: (X, \mathcal{C}) \to (Y, \mathcal{D})$ and $g: (X, \mathcal{C}) \to (Z, \mathcal{E})$ are two fuzzy c-continuous functions. Then the function $f \times g: (X, \mathcal{C}) \to (Y\times Z, \mathcal{D}\times\mathcal{E})$ defined by $(f\times g)(x) = (f(x), g(x))$ is fuzzy c-continuous.

Proof. Consider π_1 and π_2 such that $(\pi_1 \circ (f \times g))(x) = \pi_1(f(x), g(x)) = f(x)$ and $(\pi_2 \circ (f \times g))(x) = \pi_2(f(x), g(x)) = g(x)$.

Clearly, $(\pi_1 \circ (f \times g))$ and $(\pi_2 \circ (f \times g))$ are fuzzy c-continuous functions. Hence by Corollary 3.10, $(f \times g)$ is fuzzy c-continuous.

4 Countable Fuzzy Topological Vector Space

Katsaras and Liu [18] introduced the concept of fuzzy vector spaces and fuzzy topological vector spaces. Later in 1981, Katsaras [19] introduced a new definition for fuzzy topological vector spaces that are compatible with the general case in topology. Recently, in 2006, Alimohammady and Roohi [2] introduced the fuzzy minimal vector space as a generalization of fuzzy topological vector space. In this section we introduced countable fuzzy topological vector spaces as another generalization of the concept of fuzzy topological vector spaces introduced by Katsaras [19].

We recall that the fuzzy usual topology of R is the fuzzy topology consisting of all lower semi-continuous functions from R into the unit interval I.

Definition 4.1 A countable fuzzy topological vector space (or a linear space) on a vector space X over the real line **R** is a countable fuzzy topological space $(X, \mathcal{C})$ such that the two mappings

$$+ : X\times X \to X, \quad (x, y) \to x + y$$

$$\cdot : \mathbf{R}\times X \to X, \qquad (t, x) \to tx$$

are fuzzy c-continuous where **R** has the fuzzy usual topology and $X\times X$, $\mathbf{R}\times X$ the corresponding product countable fuzzy topology.

Proposition 4.2 Suppose f, g: $X \to Y$ are fuzzy c-continuous functions where X and Y are a countable fuzzy topological space and a countable fuzzy topological vector space respectively. Then $f + g$: $X \to Y$ is also a fuzzy c-continuous function.

Proof. Let h: $Y\times Y\to Y$ be a function defined by $h(y_1, y_2) = y_1 + y_2$. Then from definition h is fuzzy c-continuous. Again, by Corollary 3.11, it follows that $f + g = h\circ (f \times g)$. Again $(f \times g)$ and h are fuzzy c-continuous and hence by Theorem 3.8, $f + g$ is fuzzy c-continuous.

Theorem 4.3 Let X be a countable fuzzy topological vector space, $a\in\mathbf{R}$ and $x_0\in X$. Then f: $X \to X$ defined by $f(x) = ax + x_0$ is fuzzy c-continuous. If $a \neq 0$, then f is a fuzzy countable homeomorphism (c-homeomorphism).

Proof. We consider f_1: $X \to \mathbf{R}$ and f_2: $X \to X$ defined by $f_1(x) = a$ and $f_2(x) = x$ respectively. Then from Proposition 2.12, we have f_1 and f_2 are fuzzy c-continuous. Again from Corollary 3.11, it follows that $f_1\times f_2$: $X \to \mathbf{R}\times X$ is fuzzy c-continuous. Then it can be verified that

$f = g\circ (f_1\times f_2) + x_0$. Then from Proposition 4.2, f is fuzzy c-continuous.

For the second assertion, we must note that $f^{-1}(x) = (x-x_0)/a$ and in a similar fashion as above it can be shown that $f^{-1}$is also continuous and hence f is a homeomorphism.

The notion of countable supremum is more realistic in nature than arbitrary supremum. It is a general trend to fuzzify the concepts that are already prevailing in the literature, but here, in this paper, we have introduced a concept directly in fuzzy setting. The concepts of countable fuzzy topology and that of countable fuzzy topological vector space introduced in this paper may further be studied to explore the applicability of these new concepts in other fields especially in science and technology. The present work may be considered as an initial step in this regard. Finally, it may be remarked here that both the concepts are capable of cardinal extension. We conclude this paper by introducing a generalized topological space to be called countable topological space.

Definition Let X be any set and C_T be a subset of power set of X. Then C_T is called a countable topology if

i) $\varnothing$, X $\in C_T$
ii) Countable union of members of C_T is also in C_T.
iii) Finite intersection of members of C_T is also in C_T.

Then the space (X, C_T) is called countable topological space. The members of C_T are called c-open sets and the complement of a c-open set is called a c-closed set.

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