J. Math. Fund. Sci., Vol. 47, No. 2, 2015, 143-153

Hyers-Ulam-Rassias Stability for a First Order Functional Differential Equation

Cemil Tunç[1] & Emel Biçer[2]

[1]Department of Mathematics, Faculty of Sciences,
Yüzüncü Yıl University, 65080, Van, Turkey
[2]Department of Mathematics, Faculty of Arts and Sciences,
Bingöl University 12000, Bingöl, Turkey
Email: cemtunc@yahoo.com

Abstract. In this paper, by using the fixed point method, we prove two new results on the Hyers-Ulam-Rassias and the Hyers-Ulam stability for the first order delay differential equation of the form

$$y'(t) = F(t, y(t), y(t-\tau)).$$

Our results improve some related results in the literature.

Keywords: *first order; fixed point; functional differential equation; generalized metric; Hyers-Ulam-Rassias stability.*

1 Introduction

In 1940, S.M. Ulam gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (see S.M. Ulam [1]). Among those was the question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. Given any $\varepsilon > 0,$ does there exist a $\delta > 0$ such that if a function $h: G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1,$ then there exists a homomorphisim $H: G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, the functional equation for homomorphisms is said to have the Hyers-Ulam stability because the first result concerning the stability of functional equations was presented by D.H. Hyers. Indeed, he has answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces (see D.H. Hyers [2]).

Received October 10th, 2014, 1St Revision January 31st, 2015, 2nd Revision April 6th, 2014, Accepted for publication April 13th, 2015.
 ISSN: 2337-5760, DOI: 10.5614/j.math.fund.sci.2015.47.2.3

In 1978, Th.M. Rassias [3] addressed the Hyers's Stability Theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference $h(xy)-h(x)h(y)$ and proved a considerably generalized result of Hyers. Since then, the stability problems of various functional equations have been investigated by many authors (see [1-17]). By regarding the large influence of D.H. Hyers, S.M. Ulam, and Th.M. Rassias on the study of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called the Hyers-Ulam-Rassias stability.

Alsina and Ger [4] were the first authors who investigated the Hyers-Ulam stability of the first order linear differential equation $y'(t) = y(t).$ They proved that if a differentiable function $y: I \to \Re$ satisfies $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I,$ $(I \subset \Re),$ then there exist a differentiable function $g: I \to \Re$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \leq 3\varepsilon$ for every $t \in I.$

In the literature, first, Obłoza [5] initiated the study of the Hyers stability for the ordinary differential equation $x' = f(t, x).$ The author also studied the particular case of this equation, the linear differential equation $x' + p(t)x = p(t)$ for stability in the sense of Hyers ([2,6,7]).

Later, Obłoza [8] considered two kinds of stability, the Hyers stability and Lyapunov stability, for the ordinary differential equation $x' = f(t, x),$ where $f: \Re^2 \to \Re$ is a continuous function, Lipschitzian with respect to the second variable. The author discussed the Hyers stability and the Lyapunov stability.

In 2004, Cădariu and Radu [9] applied a well-known fixed point theorem to show the Hyers-Ulam-Rassias stability of an additive Cauchy functional equation. The method used in [9] is a new and attractive one in researching stability problems and seems to be applied to investigating stability phenomena of other functional equations.

After that, in 2010, Jung [10] considered the first order differential equation without delay

$$y'(x) = F(x, y(x)), \tag{1}$$

where $F: I \times \Re \to \Re$ is a bounded and continuous function that satisfies Lipschitz condition,

$$|F(x, y) - F(x, z)| \leq L|y - z|$$

for any $x \in I$, $I = [a,b]$, a, $b \in \Re$ and y, $z \in \Re$. By using the fixed point method and adopting the idea used in Cădariu and Radu [9,11], the author proved the Hyers-Ulam-Rassias stability of Eq. (1). In the same work, Jung [10] also discussed the Hyers-Ulam stability for Eq. (1) defined on a finite and closed interval.

In 2010, Jung and Brzdęk [12] investigated approximate solutions $y:[-\tau,\infty) \to \Re$ of the delay differential equation $y'(t) = \lambda y(t-\tau)$ with an initial condition, where $\lambda > 0$ and $\tau > 0$ are real constants. The authors showed that approximate solutions can be "approximated" by solutions of the equation that are constants on the interval $[-\tau, 0]$ and, therefore, have quite simple forms. Their results correspond to the notion of stability introduced by Ulam.

Finally, in 2013, Otrocol and Ilea [13] concerned themselves with the stability analysis of delay differential equation

$$x'(t) = f(t, x(t), x(g(t))).$$

Otrocol and Ilea [13] obtained two results on the Ulam-Hyers stability of the solution on a compact interval $[a,b]$ and they generalized the Ulam-Hyers-Rassias stability on an interval $[a,\infty)$.

Motivated by the mentioned papers and those of Rassias [14], Takahasi, *et al*. [15], Tunc and Biçer [16] and Ulam [17], the aim of this paper is to investigate the Hyers-Ulam-Rassias and the Hyers-Ulam stability of the delay differential equation of the form

$$y'(t) = F(t, y(t), y(t-\tau)), \tag{2}$$

where $F:\Re^3 \to \Re$ is a bounded and continuous function, $\tau > 0$ is a real constant.

We will apply the fixed point method to establish the Hyers-Ulam-Rassias and Hyers-Ulam stability of Eq. (2). Our results extend and improve the results of Jung and Brzdęk [12] and Obłoza [5].

It should be noted that in spite of our Eq. (2) being included by the equation discussed in Otrocol and Ilea [13], the assumptions to be established here and the procedure to be used in this paper are different from those in [13]. Meanwhile, in recent years the Hyers-Ulam-Rassias and Hyers-Ulam stability problems for various differential equations have been obtained and are still attracting much attention from researchers. However, we would not like to discuss the details of these works here. Therefore, it is of merit and important to investigate the Hyers-Ulam-Rassias and Hyers-Ulam stability of Eq. (2). The results to be obtained here provide a contribution to the subject in the literature and may be useful for researchers working on the qualitative behaviors of functional differential equations.

2 Preliminaries

Definition 1 (Jung [10]). Let X be anon-empty set. A function $d : X \times X \to [0,\infty]$ is called generalized metric on X if and only if d satisfies

m1) $d(x,y) = 0$ if and only if $x = y$,

m2) $d(x,y) = d(y,x)$ for all $x, y \in X$,

m3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$

It should be remarked that the only difference of the generalized metric from the usual metric is that the range of the former is permitted to include the unbounded interval (Jung [10]).

Definition 2 (Jung [12]). For some $\varepsilon \ge 0$, $\Psi \in C[t_0 - \tau, t_0]$ and $t_0, T \in \Re$ with $T > t_0$, assume that for any continuous function $f : [t_0 - \tau, T] \to \Re$ satisfying

$$\begin{cases} \left| f'(t) - F(t, f(t), f(t-\tau)) \right| < \varepsilon, \ t \in [t_0, T], \\ \left| f(t) - \Psi(t) \right| < \varepsilon, \ t \in [t_0 - \tau, t_0]. \end{cases}$$

There exists a continuous function $f_0 : [t_0 - \tau, T] \to \Re$ satisfying:

$$\begin{cases} f_0'(t) = F(t, f_0(t), f_0(t-\tau)), \ t \in [t_0, T], \\ f_0(t) = \Psi(t), \ t \in [t_0 - \tau, t_0], \end{cases}$$

$$\left| f(t) - f_0(t) \right| < K(\varepsilon), \ t \in [t_0 - \tau, T],$$

where $K(\varepsilon)$ is an expression of ε only.

Then, we say that Eq. (2) has the Hyers-Ulam stability. If the above statement is also true when we replace ε and $K(\varepsilon)$ by φ and Φ, where ϕ, $\Phi \in C[t_0 - \tau, T]$ are functions not depending on f and f_0 explicitly, then we say that Eq. (2) has the Hyers-Ulam-Rassias stability (or generalized the Hyers-Ulam stability).

Definition 3 (Otrocol and Ilea [13]). Equation (2) is generalized Hyers-Ulam-Rassias stable with respect to ϕ if there exists $c_\phi > 0$ such that for each solution $y \in C^1([t_0-\tau,T],\Re)$ to $\left|y'(t)-F(t,y(t),y(t-\tau))\right| \le \phi(t),\quad t\in[t_0-\tau,T]$, there exists a solution $x \in C^1([t_0-\tau,T],\Re)$ to Eq. (4) with $\left|y(t)-x(t)\right| \le c_\phi \phi(t)$, $t\in[t_0-\tau,T]$.

To prove our main result, we need the following lemma; this lemma is a result of the fixed point theorem.

Lemma 1 (Jung [10]). Let (X,d) be a generalized complete metric space. Assume that $\Lambda: X \to X$ is a strictly contractive operator with the Lipschitz constant $L<1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:

a) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ,

b) x^* is the unique fixed point of Λ in

$$X^* = \{y \in X : d(\Lambda^k x, y) < \infty\}.$$

If $y \in X^*$, then

$$d(y,X^*) \le \frac{1}{1-L} d(\Lambda y, y).$$

3 Hyers-Ulam-Rassias Stability

We will prove the Hyers-Ulam-Rassias stability of Eq. (2) by using the fixed point method.

Theorem 1 For $I=[t_0-\tau,\ T]$, we suppose that $F: I\times\Re\times\Re \to \Re$ is a continuous function with the following Lipschitz condition:

$$\left|F(t,x,y)-F(t,z,w)\right| \le L_1\left|x-z\right| + L_2\left|y-w\right|$$

for all $(t,x,y),\ (t,z,w) \in I\times\Re\times\Re$.

Let $\phi: I \to (0,\infty)$ be a continuous function. Assume that $\Psi \in C[t_0-\tau,\ t_0]$, K, L_1 and L_2 are positive constants with

$$0 < K(L_1+L_2) < 1, \text{ and } \left|\int_{t_0}^{t} \phi(u)du\right| \le K\phi(t), \text{ for all } t \in I = [t_0-\tau,\ T].$$

Then, if a continuous function $y: I \to \Re$ satisfies

$$\begin{cases} |y'(t) - F(t, y(t), y(t-\tau))| < \varphi(t), \ t \in [t_0, \ T], \\ |y(t) - \Psi(t)| < \varphi(t), \ t \in [t_0 - \tau, \ t_0], \end{cases}$$

then there exists a unique continuous function $y_0 : I \to \Re$ such that

$$\begin{cases} y_0'(t) = F(t, y_0(t), y_0(t-\tau)), \ t \in [t_0, T], \\ \\ y_0(t) = \Psi(t), \ t \in [t_0 - \tau, t_0], \end{cases} \tag{3}$$

and

$$|y(t) - y_0(t)| \le \frac{1}{1 - K(L_1 + L_2)} K\phi(t), \text{ for all } t \in I. \tag{4}$$

Proof. Let C be the space of all continuous functions from $I \to \Re$ and define the set S by

$$S = \{\varphi : I \to \Re : \quad \varphi \in C, \ \varphi(t) = \Psi(t), \ \text{ if } t \in [t_0 - \tau, t_0]\}.$$

We define a generalized metric on S as the following:

$$d(\varphi, \mu) = \inf\{M \in [0, \infty) : |\varphi(t) - \mu(t)| \le M\phi(t), \ \ \forall t \in I\}. \tag{5}$$

Then, (S, d) is generalized complete metric space.

Define the mapping $\Lambda : S \to S$ by

$$\begin{cases} (\Lambda\varphi)(t) = \Psi(t), \ t \in [t_0 - \tau, t_0], \\ (\Lambda\varphi)(t) = \Psi(t_0) + \int\limits_{t_0}^{t} F(u, \varphi(u), \varphi(u-\tau)) du, \ t \in [t_0, \ T]. \end{cases} \tag{6}$$

It is clear that for $\varphi \in S$, $\Lambda\varphi$ is continuous. Thus, we can write $\Lambda\varphi \in S$.

We show that Λ is strictly contractive on S. Let $\varphi, \mu \in S$. Then,

$$|(\Lambda\phi)(t) - (\Lambda\mu)(t)|$$
$$= \left| \int\limits_{t_0}^{t} \{F(u, \varphi(u), \varphi(u-\tau)) - F(u, \mu(u), \mu(u-\tau))\} du \right|$$

$$\leq \int_{t_0}^{t} \left| F(u,\varphi(u),\varphi(u-\tau)) - F(u,\mu(u),\mu(u-\tau)) \right| du$$

$$\leq L_1 \int_{t_0}^{t} \left| \varphi(u) - \mu(u) \right| du + L_2 \int_{t_0}^{t} \left| \varphi(u-\tau) - \mu(u-\tau) \right| du$$

$$\leq (L_1 + L_2) M \left| \int_{t_0}^{t} \phi(u) du \right|, \ t \in [t_0, \ T],$$

and

$$\left| (\Lambda\varphi)(t) - (\Lambda\mu)(t) \right| = \Psi(t) - \Psi(t) = 0, \ t \in [t_0 - \tau, \ t_0],$$

which implies that $d(\Lambda\varphi, \Lambda\mu) \leq K(L_1 + L_2) d(\varphi, \mu)$. Since $0 < K(L_1 + L_2) < 1$, is s then Λ is strictly contractive on S.

For an arbitrary $\xi \in S$ and all $t \in I$, since $\min_{t \in I} \phi(t) > 0$ and $F(t, \xi(t), \xi(t-\tau))$ and $\xi(t)$ are bounded on I, $I = [t_0 - \tau, T]$, then there exists a constant $0 < M < \infty$ such that

$$\left| (\Lambda\xi)(t) - \xi(t) \right| = \left| \psi(t_0) + \int_{t_0}^{t} F(u, \xi(u), \xi(u-\tau)) du - \xi(t) \right| \leq M \varphi(t),$$

where φ is given by $\varphi : I \to (0, \infty)$.

Thus, from Eq. (5), it follows that $d(\Lambda\xi, \xi) < \infty$. Hence, according to Lemma 1a), there exists a continuous function $y_0 : I \to \Re$ such that $\Lambda^n \xi \to y_0$ in (S, d) and $\Lambda y_0 = y_0$, so y_0 satisfies

$$\begin{cases} y_0'(t) = F(t, y_0(t), y_0(t-\tau)), \ t \in [t_0, T], \\ y_0(t) = \Psi(t), \ t \in [t_0 - \tau, t_0]. \end{cases}$$

If we show that $\{g \in S : d(\xi, g) < \infty\} = S$, then from Lemma 1b), we can say that y_0 is the unique continuous function with the property (3). For any $g \in S$, since g and ξ are bounded on I, there exists a constant $0 < M_g < \infty$ such that

$$\left| \xi(t) - g(t) \right| \leq M_g \varphi(t), \text{ for any } t \in I.$$

Then, for all $g \in S$,

$$d(\xi, g) < \infty, \text{ and } \{g \in S : d(\xi, g) < \infty\} = S.$$

Furthermore, it is clear that

$$-\phi(t) \le y'(t) - F(t, y(t), y(t-\tau)) \le \phi(t), \text{ for all } t \in [t_0,\ T].$$

If we integrate each term in this inequality from t_0 to t, then we obtain

$$\left| y(t) - \psi(t_0) - \int_{t_0}^{t} F(u, y(u), y(u-\tau)du \right| \le \left| \int_{t_0}^{t} \phi(u)du \right|,\ t \in [t_0,\ T].$$

Hence, it follows from the definition of Λ that

$$|y(t) - (\Lambda y)(t)| \le \left| \int_{t_0}^{t} \varphi(u)du \right| \le K\varphi(t), \text{ for each } t \in I.$$

Then, $d(y, \Lambda y) \le K.$ From Lemma 1c) and the last estimate, we get

$$d(y, y_0) \le \frac{1}{1 - K(L_1 + L_2)} d(\Lambda y, y) \le \frac{1}{1 - K(L_1 + L_2)} K\phi(t), \quad \text{for all } t \in I.$$

The proof of Theorem 1 is now complete.

4 Hyers-Ulam Stability

Theorem 2 Suppose that $F : I \times \Re \times \Re \to \Re$ is a continuous function with the following Lipschitz condition:

$$|F(t, x, y) - F(t, z, w)| \le L_1 |x - z| + L_2 |y - w|,$$

where $(t, x, y), (t, z, w) \in I \times \Re \times \Re$ and $0 < T(L_1 + L_2) < 1$.

Let

$$\Psi \in C[t_0 - \tau,\ t_0] \text{ and } \varepsilon \ge 0.$$

If a continuous function $y : I \to \Re$ satisfies

$$\begin{cases} |y'(t) - F(t, y(t), y(t-\tau))| < \varepsilon,\ t \in [t_0,\ T], \\ |y(t) - \Psi(t)| < \varepsilon,\ t \in [t_0 - \tau,\ t_0], \end{cases}$$

then there exists a unique continuous function $y_0 : I \to \Re$ such that

$$\begin{cases} y_0'(t) = F(t, y_0(t), y_0(t-\tau)),\ t \in [t_0,\ T], \\ y_0(t) = \Psi(t),\ t \in [t_0 - \tau,\ t_0], \end{cases} \tag{7}$$

and

$$|y(t) - y_0(t)| \le \frac{\varepsilon T}{1-(L_1+L_2)}, \text{ for all } t \in I. \tag{8}$$

Proof. We introduce a generalized metric on S as follows:

$$d_1(\varphi, \mu) = \inf\{M \in [0,\infty] : |\varphi(t) - \mu(t)| \le M,\ \ \forall t \in I\}.$$

Then, $(S,\ d_1)$ is a generalized complete metric space. For any $\varphi, \mu \in S$ and $M_{\varphi,\mu} \in \{M \in [0,\infty] : |\varphi(t) - \mu(t)| \le M,\ \ \forall t \in I\}$, from Eq. (6), we obtain

$$\begin{aligned} &|(\Lambda\phi)(t) - (\Lambda\mu)(t)| \\ &\qquad \le \left| \int_{t_0}^{t} \{F(u, \varphi(u), \varphi(u-\tau)) - F(u, \mu(u), \mu(u-\tau))\} du \right| \\ &\qquad \le \left| \int_{t_0}^{t} |F(u, \varphi(u), \varphi(u-\tau)) - F(u, \mu(u), \mu(u-\tau))| du \right| \\ &\qquad \le L_1 \int_{t_0}^{t} |\varphi(u) - \mu(u)| du + L_2 \int_{t_0}^{t} |\varphi(u-\tau) - \mu(u-\tau)| du \\ &\qquad \le (L_1 + L_2) T M_{\varphi,\mu},\ t \in [t_0,\ T], \end{aligned}$$

and

$$|(\Lambda\varphi)(t) - (\Lambda\mu)(t)| = \Psi(t) - \Psi(t) = 0,\ t \in [t_0 - \tau,\ t_0],$$

which imply that $d_1(\Lambda\varphi, \Lambda\mu) \le (L_1 + L_2) T d_1(\varphi, \mu).$

Since $0 < T(L_1 + L_2) < 1,$ then Λ is strictly contractive on $S.$

For an arbitrary $\zeta \in S$, from the boundedness of $F(t, \zeta(t), \zeta(t-\tau))$ and $\zeta(t),$ we can show $d_1(\Lambda\zeta, \zeta) < \infty.$ Hence, from Lemma 1a), there exists a continuous function $y_0 : I \to \Re$ such that $\Lambda^n \xi \to y_0$ in (S, d_1) and $\Lambda y_0 = y_0,$ so y_0 satisfies

$$\begin{cases} y_0'(t) = F(t, y_0(t), y_0(t-\tau)),\ t \in [t_0,\ T], \\ y_0(t) = \Psi(t),\ t \in [t_0 - \tau,\ t_0], \end{cases}$$

Since g and ζ are bounded on I, then $d_1(\zeta, g) < \infty$ $(\forall g \in S)$. That is, $\{d_1(\zeta, g) < \infty\} = S.$

Thus, due to Lemma 1b) y_0 is the unique continuous function with the property (7).

Furthermore, it is clear that

$$-\varepsilon \le y'(t) - F(t, y(t), y(t-\tau)) \le \varepsilon, \text{ for all } t \in [t_0,\ T].$$

Hence,

$$-\varepsilon\int_{t_0}^{t} du \le \int_{t_0}^{t} y'(u)du - \int_{t_0}^{t} F\left(u, y(u), y(u-\tau)\right)du \le \varepsilon\int_{t_0}^{t} du, \text{ for all } t \in [t_0,\ T],$$

and

$$\left|(\Lambda y)(t) - y(t)\right| \le \varepsilon(t - t_0), \text{ for all } t \in [t_0,\ T].$$

It follows from the definition of Λ that

$$\left|y(t) - (\Lambda y)(t)\right| \le T\varepsilon, \text{ for each } t \in I.$$

From Lemma 1c) and the last estimate, we get

$$d_1(y, y_0) \le \frac{1}{1 - T(L_1 + L_2)} d_1(\Lambda y, y) \le \frac{T\varepsilon}{1 - T(L_1 + L_2)}, \quad \text{for all } t \in I.$$

This estimate completes the proof of Theorem 2.

Remark 2 Since the equations considered in Jung [10] and Jung and Brzdęk [12] are all special cases of Eq. (2), the results of this paper are new and complementary to the previous known results in Jung [10] and Jung and Brzdęk [12].

Acknowledgements

The authors of this paper would like to expresses their sincere appreciation to the anonymous referees for their valuable comments and suggestions which have led to an improvement in the presentation of this paper.

References

[1] Ulam, S.M., *Problems in Modern Mathematics*, Science Editions, John Wiley & Sons, Inc., New York, 1964.

[2] Hyers, D.H., *On the Stability of the Linear Functional Equation*, Proc. Nat. Acad. Sci. U.S.A., **27**, pp. 222-224, 1941.

[3] Rassias, T.M., *On the Stability of the Linear Mapping in Banach Spaces*, Proc. Amer. Math. Soc., **72**(2), 297-300, 1978.

[4] Alsina, C. & Ger, R., *On some Inequalities and Stability Results Related to the Exponential Function*, J. Inequal. Appl., **4**, pp. 373-380, 1998.

[5] Obłoza, M., *Hyers Stability of the Linear Differential Equation*, Rocznik Nauk.-Dydakt. Prace Mat., **13**, pp. 259-270, 1993.

[6] Hyers, D.H., George, I. & Rassias, T.M., *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, **34**, 1998.

[7] Hyers, D.H. & Rassias, T.M., *Approximate Homomorphisms*, Aequationes Math, **44**, pp. 125-153, 1992.

[8] Obłoza, M., *Connections between Hyers and Lyapunov Stability of the Ordinary Differential Equations*, Rocznik Nauk-Dydakt, Prace Mat., **14**, pp. 141-146, 1997.

[9] Cădariu, L. & Radu, V., *On the Stability of the Cauchy Functional Equation: A Fixed Point Approach*, Iteration theory (ECIT ’02), Grazer Math. Ber., **346**, pp. 43-52, 2004.

[10] Jung, S.M., *A Fixed Point Approach to the Stability of Differential Equations* $y'=F(x,y)$, Bull. Malays. Math. Sci. Soc., **33**, pp. 47-56, 2010

[11] Cădariu, L. & Radu, V., *Fixed Points and the Stability of Jensen's Functional Equation*., J. Inequal. Pure Appl. Math., **4**, pp. 1-7, 2003.

[12] Jung, S.M., & Brzdęk, J., *Hyers-Ulam Stability of the Delay Equation* $y'(t)=\lambda y(t-\tau)$, Abstr. Appl. Anal., pp. 1-10, 2010.

[13] Otrocol, D. & Ilea, V., *Ulam Stability for a Delay Differential Equation*. Cent. Eur. J. Math., **7**, pp. 1296-1303, 2013.

[14] Rassias, T.M., *On the Stability of Functional Equations and a Problem of Ulam*, Acta Appl. Math., **1**, pp. 23-130, 2000.

[15] Takahasi, S.E., Miura, T. & Miyajima, S., *On the Hyers-Ulam Stability of the Banach Space-Valued Differential Equation* $y'=\lambda y$, Bull. Korean Math. Soc., **39**, pp. 309-315, 2002.

[16] Tunc, C. & Biçer, E., *On the Hyers-Ulam Stability of Non-Homogeneous Euler Equations of Third and Fourth Order*, Scientific Research and Essays, **8**, pp. 220-226, 2013.

[17] Ulam, S.M., *A Collection of Mathematical Problems*. Interscience Tracts in Pure and Applied Mathematics, **8**, Interscience Publishers, New York-London, 1960.