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Approximate Solutions of Multi-Pantograph Type Delay Differential Equations Using Multistage Optimal Homotopy Asymptotic Method

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Abstract. In this paper, a numerical procedure called multistage optimal homotopy asymptotic method (MOHAM) is introduced to solve multi-pantograph equations with time delay. It was shown that the MOHAM algorithm rapidly provides accurate convergent approximate solutions of the exact solution using only one term. A comparative study between the proposed method, the homotopy perturbation method (HPM) and the Taylor matrix method are presented. The obtained results revealed that the method is of higher accuracy, effective and easy to use.

Keywords: *approximate solutions; multistage optimal homotopy asymptotic method* (*MOHAM*); *optimal homotopy asymptotic method* (*OHAM*); *pantograph equation; series solution.*

Mathematical Subject Classification 2010 : 65C20, 65L03

1 Introduction

The pantograph equation is one of the most well-known and important types of delay differential equations. It plays a significant role in modeling various phenomena in applied science and engineering, such as electrodynamics, biology, engineering, physics, and economy. For example, Aiello, *et al.* [1] have proposed a stage-structured model of population growth, Ockendon & Taylor [2] have analyzed the dynamics of the current collection system of an electric locomotive, etc. [3-5]. In this work, the following multi-pantograph equation is considered:

$$u'(t) + a(t)u(t) + \sum_{i=1}^{l} b_i(t)u(\tau_i t) + f(t) = 0\,, u(0) = \alpha_i,\ 0 < t < T \qquad (1)$$

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where f is a given function and $0 < \tau_1 < \tau_2 \ldots < \tau_l \leq 1$. In the past few years, several studies on delay differential equations (DDEs) have been reported. For instance, Ahmadzadehed, *et al.* [6] reported analytical solutions of delay differential equations using modified Adomian decomposition with Pad approximation. Moreover, an approximate solution of a delay differential equation based on the differential transform method (DTM) and the Adomian decomposition method (ADM) has been introduced by Raslan & Sheer [7]. In 2013, the optimal homotopy asymptotic method (OHAM) was employed successfully, and accurate approximate solutions for linear, nonlinear and system of delay equations were obtained by Anakira, *et al.* [8]. Multi-pantograph equations are among the most important types of DDEs. They have been considered by numerous researchers analytically and numerically. For example, an analytic study on multi-pantograph delay equations with variable coefficients has been considered by Feng [9]. A brief review of the recent literature on solution methods for multi-pantograph-type delay equations can be found in [10-13].

In most real-life situations, the equation that models the problem is too complicated to solve exactly. Therefore, methods for approximating the solution are used to obtain a solution to an initial value problem that satisfies a given initial condition.

The method considered in this paper is based on the optimal homotopy asymptotic method (OHAM). OHAM was introduced in 2008 by Marinca and Herisanu [14-17] for finding an approximate solution of nonlinear problems of thin-film flow of a fourth-grade fluid down a vertical cylinder. In their work, they employed this procedure to understand the behavior of nonlinear mechanical vibration in an electrical machine. Furthermore, they used the same procedure to obtain an approximate solution of nonlinear equations that arise in steady state flow of a fourth-grade fluid past a porous plate, and for the solution of a nonlinear equation arising in heat transfer. OHAM has been successfully employed to solve different kinds of differential equations in science and engineering [18-24].

Recently, OHAM has been modified by Anakira, *et al.* [25] resulting in an effective algorithm called the multistage optimal homotopy asymptotic method (MOHAM) to find approximate solutions of linear, nonlinear and system of initial value problems. On the other hand, MOHAM has been successfully applied to obtain approximate solutions for the Quadrature Riccati equation [26]. This procedure is easy and effective in obtaining approximate series solutions for linear and nonlinear differential equations without linearization and discretization.

The present study attempted to effectively employ MOHAM to solve multi-pantograph delay equations. The results can be obtained in only one iteration with higher degree of accuracy compared with other methods from the literature. Moreover, the procedure is simpler.

The rest of this paper is organized as follows. Section 2 explains the basic principles of OHAM and MOHAM. To present a clear overview of the procedure, in Section 3 several examples with exact solutions are given and a comparison is made with existing results. Finally, a brief conclusion is given in Section 4.

2 Description of OHAM and MOHAM

2.1 OHAM

The basic idea of OHAM, as explained by Marinca, *et al*. [14] and other researchers [15,19], is to define the map $h(v(t,p),p): R \times [0,1] \rightarrow R$

$$(1-p)[L(v(t,p) - u0(t))] = H(p)\left[\frac{dv(t,p)}{dt} + a(t)v(t,p) + \sum_{i=1}^{l} b_i v(\tau_i t, p) + f(t)\right] \quad (2)$$

Here, $t \in R$ and $p \in [0,1]$ are embedding parameters, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$, and $v(t,p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds that $v(t,0) = u0(t)$ and $v(t,1) = u(t)$ respectively. Thus, as p varies from 0 to 1, the solution $v(t,p)$ approaches from $u_0(t)$ to $u(t)$, where $u_0(t)$ is the initial guess that satisfies the linear operator:

$$L(u_0(t)) = 0, u_0(0) = \alpha \quad (3)$$

next, we choose the auxiliary function $H(p)$ in the form:

$$H(p) = (\sum_{i=1}^{n} C_i t^i)p = (C_1 t + C_2 t^2 + \cdots + C_n t^n)p \quad (4)$$

where C_1, C_2, C_3, … are convergence control parameters that can be determined later. To get an approximate solution, we expand $v(t,p,C_k)$ in Taylor's series about p in the following manner:

$$v(t,p) = u_0(t) + \sum_{k=1}^{\infty} u_k(t)pk \quad (5)$$

Substituting Eq. (5) into Eq. (2) and equating the coefficient of like powers of p, we obtain the following linear equations:

$$L[u_m(t) - \chi_m u_{m-1}(t)] = \left(\sum_{i=1}^{n} C_i t^i\right)\left(u'_{m-1}(t) + a(t)u_{m-1}(t) + \sum_{i-1}^{l} b_i(t)\, u_{m-1}(t) + (1-\chi_m)f(t)\right), \quad (6)$$

subject to $u_m\ (a)\ =\ 0$, where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

And the m^{th}-order approximation is given by:

$$\tilde{u}(t, C_1, C_2, C_3, \ldots, C_n) = u_0(t) + \sum_{i=1}^{n} u_i\ (t, C_1, C_2, C_{23}, \ldots, C_n\ ). \quad (7)$$

Substituting Eq. (7) into Eq. (1) yields the following residual:

$$R(t, C_1, C_2, C_3, \ldots, C_n) = \tilde{u}'(t, C_1, C_2, C_2, \ldots, C_n) + a(t)\tilde{u}(t, C_1, C_2, C_3, \ldots, C_n) + \sum_{i=1}^{l} b_i(t)\,\tilde{u}(\tau_i t, C_1, C_2, C_3, \ldots, C_n) \quad (8)$$

If $R = 0$, then $\tilde{u}$ will be the exact solution. Generally, such a case will not arise for nonlinear problems. The values of the convergence control parameters C_i can be obtained by several methods, such as, the method of least squares collocation method, Galerkin's method and the Ritz method. Using the least squares method, we obtain the following equation:

$$J(C_1, C_2, C_3, \ldots, C_n) = \int_a^b R^2(C_1, C_2, C_2, \ldots, C_n)dt \quad (9)$$

where a and b are the endpoints of the given problem. The unknown convergence controls parameters $C_i(i = 1,2,3, \ldots, m\ )$ can be identified from the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \cdots = \frac{\partial J}{\partial C_n} = 0 \quad (10)$$

Making use of these known convergence control parameters, the approximate solution (of order m) is well determined.

2.2 MOHAM

To ensure the validity of the approximations for multi-pantograph equations, we further developed Multistage OHAM. This development is based on dividing the interval $[0, T]$ into subintervals as $[t_0\ , t_1], \ldots, [t_{n-1}, t_n]$, where $t_n = T$ and standard OHAM is applied to each subinterval. The initial approximation of each interval is taken from the solution of the previous one. Firstly, consider the initial condition as:

$$u(t_j)\ =\ a \quad (11)$$

Thus, we can choose the initial approximation, $u_0(x) = a_j$. According to OHAM we construct a homotopy, $h(v(t, p), p): R\ \times\ [0, 1]\ \rightarrow\ R$, which satisfies

$$(1-p)\big[L\big(v(t,p)-u_0(t)\big)\big] = H(p)\Big[\frac{dv(t,p)}{dt} + a(t)v(p,t) + \sum_{i=1}^{l} b_i(t)v(\tau_i t,p) + f(t)\Big] \quad (12)$$

where

$$H(p) = \big(C_{1,j} + C_{2,j}\mathrm{t} + C_{3,j}t^2 + \cdots + C_{n,j}t^{n-1}\big) \quad (13)$$

Then, the first, second and m^{th} order can be generated subject to the initial condition, $u_1(t_j) = u_2(t_j) = \cdots = u_m(t_j) = 0$. The approximate solution is:

$$\tilde{u}(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}) = u_{i,0}(t) + \sum_{i=1}^{n} u_i\,(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}) \quad (14)$$

Substituting Eq. (14) into Eq. (1) yields the following residual:

$$R_i\big(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}\big) = \frac{d\tilde{u}\big(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}\big)}{dt} + a(t)\tilde{u}\big(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}\big) + \sum_{i=1}^{l} b_i(t)\,\tilde{u}\big(\tau_i t, C_{1,j}, C_{2,j}, \dots, C_{n,j}\big) + f(t) \quad (15)$$

If $R = 0$, then $\tilde{u}$ will be the exact solution. Generally, such a case will not arise for nonlinear problems, but we can minimize the function

$$J_i\big(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}\big) = \int_{t_j}^{t_{j+1}} R_i^2\big(t, C_{1,j}, C_{2,j}, \dots, C_{n,j}\big)dt \quad (16)$$

where t_j and t_{j+1} are the endpoints of the given problem in the subinterval. The unknown convergence control parameters $C_{i,j}(i = 1, 2, 3, \dots, n)$ can be identified from the conditions

$$\frac{\partial J}{\partial C_{1,j}} = \frac{\partial J}{\partial C_{2,j}} = \cdots = \frac{\partial J}{\partial C_{n,j}} \quad (17)$$

Let h be the length of subinterval $[t_j, t_{j+1})$ and $N = T/h$ the number of subintervals. Now, we can solve Eq. (17) at $j = 0, 1, \cdots, N$ by changing initial approximation a in each subinterval from the previous one. For example, in the subinterval $[t_j, t_{j+1})$ we define $a_j = \tilde{u}(t_j)$. Therefore, the approximate analytic solution will be:

$$\tilde{u}(t) = \begin{cases} \tilde{u}_1(t), & t_0 \le t < t_1, \\ \tilde{u}_2(t), & t_1 \le t < t_2, \\ \quad\vdots \\ \tilde{u}_N(t), & t_{N-1} \le t < T \end{cases} \quad (18)$$

In this way, we successfully obtain the solution of the initial value problem for a large value of T analytically. It is worth mentioning that when $j = 0$, MOHAM gives the standard OHAM, so the new algorithm is a generalization of standard OHAM.

3 Examples

In this section, several examples are presented to demonstrate the efficiency of the new algorithm.

3.1 Example 1

Given the following multi-pantograph delay equation [9]:

$$y'(t) = -\frac{5}{6}y(t) + 4y\left(\frac{1}{2}t\right) + 9y\left(\frac{1}{3}t\right) + t^2 - 1, \ \ 0 \le t \le 1 \tag{19}$$

$$y(0) = 1$$

Based on OHAM formulated in Section 2, a homotopy equation is constructed in the following form:

$$(1-p)\left[\frac{dv(t,p)}{dt}\right] =$$

$$H(p)\left[\frac{dv(t,p)}{dt} + \frac{5}{6}v(t) - 4v\left(\frac{1}{2}t\right) - 9v\left(\frac{1}{3}t\right) - t^2 + 1\right] \tag{20}$$

where

$$v(t,p) = \sum_{j=1}^{1} y_i \ (t, C_{1,j}, C_{2,j}, \dots, C_{n,j})\, p^i, \tag{21}$$

and

$$H(p) = \left(C_{1,j} + C_{2,j}t + C_{3,j}t^2 + C_{4,j}t^3\right)p \tag{22}$$

Substituting Eq. (21) and Eq. (22) into Eq. (20), and equating the coefficients of the same powers of p yields the following set of linear differential equations:

$$y_0'(t_j) = 0, \ \ y_0(t_j) = \alpha \tag{23}$$

$$y_1'(t_j) = -\frac{1}{6}(-6 + 73\alpha + 6t^2)\left(C_{1,j} + t\left(C_{2,j} + t\left(C_{3,j} + tC_{4,j}\right)\right)\right) \tag{24}$$

$$y_1'(t_j) = 0$$

Now, by using $m = 1$ into Eq. (14), the MOHAM approximate of the first order is:

$$\tilde{y}_i(t) = y_0(\mathrm{t}) + y_1(t) \tag{25}$$

Substituting the solutions of Eq. (23) and (24) into Eq. (25), we obtain a first-order approximate solution:

$$\tilde{y}(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) = \alpha + tC_{1,j} - \frac{73}{6}\alpha t C_{1,j} - \frac{1}{3}t^3 C_{1,j} - t_j C_{1,j} + \frac{73}{6}\alpha t_j C_{1,j} - \frac{1}{3}t_j^{\,3} + \frac{1}{2}t^2 C_{2,j} - \frac{73}{12}\alpha t^3 C_{2,j} - \frac{1}{4}t^4 C_{2,j} - \frac{1}{2}t_j^{\,2} C_{2,j} + \frac{73}{12}\alpha t_j^{\,2} C_{2,j} + \frac{1}{4}t_j^{\,4} C_{2,j} + \frac{1}{3}t^3 C_{3,j} - \frac{73}{18}\alpha t^3 C_{3,j} - \frac{1}{5}t_j^{\,5} C_{3,j} - \frac{1}{3}t_j^{\,3} C_{3,j} + \frac{73}{18}\alpha t_j^{\,3} C_{3,j} + \frac{1}{5}t_j^{\,5} C_{3,j} + \frac{1}{4}t^4 C_{4,j} - \frac{73}{24}\alpha t^4 C_{4,j} - \frac{1}{6}t^6 C_{4,j} + \frac{73}{24}\alpha t_j^{\,4} C_{4,j} + \frac{1}{6}\alpha t_j^{\,6} C_{4,j} \quad (26)$$

Substituting Eq. (26) into Eq. (15) yields the following residual:

$$\tilde{R}(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) = \tilde{y}'(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) + \frac{5}{6}\tilde{y}'(t, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) - 4\tilde{y}\left(\frac{t}{2}, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}\right) - 9\tilde{y}\left(\frac{t}{3}, C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}\right) - t^2 + 1 \quad (27)$$

Table 1 Values of C_i, j for Example 1.

j	$C_{1,j}$	$C_{2,j}$	$C_{3,j}$	$C_{4,j}$
1	−0.40318	−1.67947	−1.00586	0.0813752
2	−0.220828	−0.919465	−0.555676	0.0285004
3	−0.138789	−0.57757	−0.350671	0.012845
4	−0.0945971	−0.393436	−0.239529	0.00669195
5	−0.0680883	−0.283013	−0.172603	0.00383648
6	−0.0509878	−0.211804	−0.129327	0.00235634
7	−0.0393567	−0.16339	−0.0998435	0.00152335
8	−0.0311188	−0.129111	−0.078942	0.00102716
9	−0.0250926	−0.10405	−0.0636403	0.000714712
10	−0.0205657	−0.085245	−0.0521379	0.000508176

According to Eq. (16), we can minimize the functional:

$$J_i(C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j}) = \int_{t_j}^{t_{j+1}} R_i^2(C_{1,j}, C_{2,j}, C_{3,j}, C_{4,j})dt \quad (28)$$

Hence, from the solutions of the systems of equations:

$$\frac{\partial J}{\partial C_{1,j}} = \frac{\partial J}{\partial C_{2,j}} = \frac{\partial J}{\partial C_{3,j}} = \frac{\partial J}{\partial C_{4,j}} \quad (29)$$

The convergence control parameters, j, are obtained as presented in Table 1, with $h = 0.1$ and starting with $t_0 = 0$ to $t_j = T = 1$. Table 2 presents a comparison between MOHAM's approximate solution of the first-order and HPM's solution of the third-order, along with the exact solution. It is obvious that the approximate solutions obtained by our algorithm are more accurate along with an increased number of terms in the auxiliary convergence control functions. Table 3 displays the residual error obtained from MOHAM's approximate solution.

Table 2 Numerical results for Example 1.

t	Exact Solution	MOHAM Solution	HPM Relative Error [9]	MOHAM Relative Error	MOHAM Absolute Error
0.1	1.00000	1.00000	4.09×10^{-12}	0.00000	0.00000
0.2	4.23893	4.23893	1.63×10^{-3}	3.59×10^{-8}	1.52×10^{-7}
0.4	9.78923	9.78924	1.08×10^{-3}	6.73×10^{-7}	6.59×10^{-6}
0.6	18.1012	18.1012	2.84×10^{-3}	2.98×10^{-6}	5.40×10^{-5}
0.8	29.6252	29.6252	5.27×10^{-3}	6.28×10^{-6}	1.86×10^{-4}
1.0	44.8114	44.8114	8.17×10^{-2}	9.83×10^{-6}	4.39×10^{-4}

Table 3 Residual Obtained using the Approximate Solution given by Eq. (15).

t	0.05	0.25	0.45	0.65	0.85	0.95
Residual Error	1.26×10^{-6}	1.40×10^{-5}	$1. \times 10^{-4}$	5.1×10^{-4}	1.26×10^{-3}	1.67×10^{-3}

3.2 Example 2

Given the following multi-pantograph delay equation [10]

$$y'(t) = -y(t) + y_1(t)y\left(\frac{1}{2}t\right) + y_2(t)y\left(\frac{1}{4}t\right) + t^2 - 1,$$
$$0 \le t \le 1 \tag{30}$$
$$y(0) = 1$$

where

$$y_1(t) = -e^{-0.5t}sin(0.5t),$$
$$y_2(t) = -2e^{-0.75t}cos(0.5t)sin(0.25t)$$
$$(1-p)\left[\frac{dv(t,p)}{dt}\right] = H(p)\left[\frac{dv(t,p)}{dt} + v(t) + \frac{1}{2}e^{-0.5t}sin(0.5t)v\left(\frac{1}{2}t\right) + 2e^{-0.75t}cos(0.5t)sin(0.25t)v\left(\frac{1}{4}t\right)\right] \tag{31}$$

where

$$v(t,p) = \sum_{j=1}^{1} y_i\,(t, C_{1,j}, C_{2,j}, \dots, C_{5,j})\,p^i \tag{32}$$

and

$$H(p) = \left(C_{1,j} + C_{2,j}t + C_{3,j}t^2 + C_{4,j}t^3 + C_{5,j}t^{53}\right)p \tag{33}$$

Substituting Eq. (32) and Eq. (33) into Eq. (31) and equating the coefficients of the same powers of p yields the following set of linear differential equations:

$$y_0(t_j) = 0, y_0(t_j) = \alpha \tag{34}$$

$$y_1'(t_j) = \alpha(1+t)\left(C_{1,j} + t\left(C_{2,j} + tC_{3,j} + \left(C_{4,j} + tC_{5,j}\right)\right)\right) \tag{35}$$

$$y_1'(t_j) = 0$$

The first-order approximate solution is given by Eq. (14) for $m = 1$ as follows:

$$\tilde{y}_i(t) = y_0(t) + y_1(t) \tag{36}$$

Table 4 Values of $C_{i,j}$ for Example 2.

j	$C_{1,j}$	$C_{2,j}$	$C_{3,j}$	$C_{4,j}$	$C_{5,j}$
1	−1.11075	1.11185	−0.016291	−0.620752	0.458202
2	−1.24657	1.25318	−0.060435	−0.555298	0.334237
3	−1.41437	1.43441	−0.136241	−0.478299	0.251391
4	−1.62368	1.66795	−0.245053	−0.399309	0.194177
5	−1.88781	1.97011	−0.390731	−0.320047	0.152874
6	−2.22633	2.36692	−0.588629	−0.231245	0.11867
7	−2.66825	2.89558	−0.856532	−0.128689	0.0888091
8	−3.2466	3.56002	−1.13007	−0.0773363	0.0821022
9	−3.97549	4.24009	−1.15925	−0.247713	0.137218
10	−5.22898	6.05518	−2.46661	0.358075	0.00183246

By applying the same procedure as in Example 3.1, the desired approximate solution is obtained The values of the convergent control parameters are displayed in Table 4. If we compare the results presented in Table 5, we can reach the conclusion that the results obtained by MOHAM's approximate solutions of the first-order are more accurate than the results obtained by Taylor's method of order 7. This proves MOHAM's validity and potential for solutions of this type of differential equations. Table 6 displays the residual error obtained from MOHAM's approximate solution.

Table 5 Numerical Results for Example 2.

t	Exact Solution	MOHAM solution	MOHAM Absolute Error	MOHAM Relative Error	Absolute Error [10]
0.1	1.00000	1.00000	0.000000	0.00000	0.00000
0.2	0.802411	0.802411	8.10553×10^{-10}	8.10554×10^{-10}	1.91535×10^{-8}
0.4	0.617406	0.617405	2.55076×10^{-7}	2.55076×10^{-7}	2.35133×10^{-6}
0.6	0.452954	0.452952	4.80133×10^{-6}	4.80133×10^{-6}	3.82097×10^{-5}
0.8	0.313051	0.313038	4.14924×10^{-5}	4.14924×10^{-5}	2.72162×10^{-4}
1.0	0.198766	0.198778	6.06934×10^{-5}	6.06934×10^{-5}	1.23389×10^{-3}

Table 6 Absolute Residual Error Obtained from the Approximate Solution of Example 2.

t	0.05	0.25	0.45	0.65	0.85	0.95
Residual Error	2.96×10^{-10}	2.08×10^{-7}	6.27×10^{-6}	4.4×10^{-5}	1.17×10^{-4}	2.61×10^{-4}

4 Conclusion

In this paper, we have proved the potential of MOHAM for obtaining approximate analytic solutions of multi-pantograph equations. High approximate solutions were obtained in one iteration, which is sufficient to achieve extremely accurate results compared with other methods from the literature. Three numerical experiments were solved to illustrate that the present algorithm is effective and accurate and converges rapidly to exact solutions. Hence, we can say that these numerical results show that the MOHAM is an acceptable and reliable technique for the solution of multi-pantograph equations.

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