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Integral Operator Defined by k−th Hadamard Product

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Abstract. We introduce an integral operator on the class $\mathcal{A}$ of analytic functions in the unit disk involving $k-$th Hadamard product (convolution) corresponding to the differential operator defined recently by Al-Shaqsi and Darus. New classes containing this operator are studied. Characterization and other properties of these classes are studied. Moreover, subordination and superordination results involving this operator are obtained.

Keywords: *Hadamard product; integral operator; subordination; superordination.*

AMS Mathematics Subject Classification (2000): 30C45.

1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk U and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+....$ Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ z\in U. \tag{1}$$

The following differential operator is defined in [1] and studied in [2] $\mathcal{D}^k_{\lambda,\delta}:\mathcal{A}\to\mathcal{A}$ by

$$\mathcal{D}^k_{\lambda,\delta}f(z)=z+\sum_{n=2}^{\infty}[1+(n-1)\lambda]^kC(\delta,n)a_nz^n, k\in\mathbb{N}\cup\{0\},\lambda\geq 0,\delta\geq 0, \tag{2}$$

where

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$$C(\delta,n)=\binom{n+\delta-1}{\delta}=\frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

Remark 1.1. When $\lambda=1,\delta=0$ we get Sălăgean differential operator [3], $k=0$ gives Ruscheweyh operator [4], $\delta=0$ implies Al-Oboudi differential operator of order (k) [5] and when $\lambda=1$ operator (2) reduces to Al-shaqsi and Darus differential operator of order (k) [6].

Given two functions $f,g\in \mathcal{A},\ f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_n z^n$ their convolution or Hadamard product $f(z)*g(z)$ is defined by

$$f(z)*g(z)=z+\sum_{n=2}^{\infty}a_n b_n z^n,\ z\in U.$$

And for several functions $f_1(z),...,f_m(z)\in \mathcal{A}$

$$f_1(z)*...*f_m(z)=z+\sum_{n=2}^{\infty}(a_{1n}...a_{mn})z^n,,z\in U.$$

Analogous to $\mathcal{D}^k_{\lambda,\delta}f(z),z\in U$ we define an integral operator $\mathcal{J}^k_{\lambda,\delta}:\mathcal{A}\to\mathcal{A}$ as follows.

Let

$$\phi(z):=\frac{z}{1-z}+\frac{\lambda z}{(1-z)^2}-\frac{\lambda z}{1-z},\lambda\geq 0.$$

$$\begin{aligned}F_k(z)\quad &=\underbrace{\phi(z)*...*\phi(z)}_{k-times}*[\frac{z}{(1-z)^{\delta+1}}]\\ &=z+\sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)z^n,k\in\mathbb{N}_0.\end{aligned}$$

And let $F_k^{(-1)}$ be defined such that

$$F_k(z) * F_k^{(-1)} = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n.$$

Then

$$\begin{aligned} \mathrm{J}_{\lambda,\delta}^{k} f(z) &= F_k^{(-1)} * f(z) \\ &= [\underbrace{\phi(z) * \ldots * \phi(z)}_{k-times} * \frac{z}{(1-z)^{\delta+1}}]^{(-1)} * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{[1+(n-1)\lambda]^k C(\delta,n)} z^n, k \in \mathrm{N}_0, \lambda \geq 0, \delta \geq 0 \; z \in U. \end{aligned} \quad (3)$$

Remark 1.2. When $\lambda = 1, \delta = 0$ we get the integral operator [3], also $k = 0$ gives Noor integral operator [7,8].

Some of relations for this integral operator are discussed in the next lemma.

Lemma 1.1. Let $f \in \mathrm{A}$. Then

$$(i)\ \mathrm{J}_{\lambda,0}^{0} f(z) = f(z),$$

$$(ii)\ \mathrm{J}_{1,0}^{1} f(z) = \int_0^z \frac{f(t)}{t} dt.$$

Proof.

$$(i)\ \mathrm{J}_{\lambda,0}^{0} f(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z),$$

$$\begin{aligned} (ii) \int_0^z \frac{f(t)}{t} dt &= \int_0^z [1 + \sum_{n=2}^{\infty} a_n t^{n-1}] dt \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n \\ &= \mathrm{J}_{1,0}^{1} f(z). \end{aligned}$$

In the following definitions, we introduce new classes of analytic functions containing the integral operator (3):

Definition 1.1. Let $f(z)\in \mathcal{A}$. Then $f(z)\in \mathcal{S}^k_{\lambda,\delta}(\mu)$ if and only if

$$\Re\{\frac{z[\mathcal{J}^k_{\lambda,\delta} f(z)]'}{\mathcal{J}^k_{\lambda,\delta} f(z)}\} > \mu, 0\le \mu <1, z\in U.$$

Definition 1.2. Let $f(z)\in \mathcal{A}$. Then $f(z)\in \mathcal{C}^k_{\lambda,\delta}(\mu)$ if and only if

$$\Re\{\frac{[z(\mathcal{J}^k_{\lambda,\delta} f(z))']'}{(\mathcal{J}^k_{\lambda,\delta} f(z))'}\} > \mu, 0\le \mu <1, z\in U.$$

Let F and G be analytic functions in the unit disk U. The function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0)=G(0)$ and $F(U)\subset G(U)$. In general, given two functions $F(z)$ and $G(z)$, which are analytic in U, the function $F(z)$ is said to be subordination to $G(z)$ in U if there exists a function $h(z)$, analytic in U with

$$h(0)=0 \textit{ and } |h(z)|<1 \textit{ for all } z\in U$$

such that

$$F(z)=G(h(z)) \textit{ for all } z\in U.$$

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$. Let Φ be an analytic function in a domain containing $f(U)$, $\Phi(0)=0$ and $\Phi'(0)>0$.

The function $f\in \mathcal{A}$ is called $\Phi-$like if

$$\Re\{\frac{zf'(z)}{\Phi(f(z))}\} > 0, z\in U.$$

This concept was introduced by Brickman [9] and established that a function $f\in \mathsf{A}$ is univalent if and only if f is $\Phi-$like for some $\Phi.$

Definition 1.3. Let Φ be analytic function in a domain containing $f(U), \Phi(0)=0, \Phi'(0)=1$ and $\Phi(\omega)\neq 0$ for $\omega\in f(U)-0.$ Let $q(z)$ be a fixed analytic function in $U,$ $q(0)=1.$ The function $f\in \mathsf{A}$ is called $\Phi-$like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z), z\in U.$$

The paper is organized as follows: Section 2 discuses the characterization properties for functions belonging to the classes $\mathsf{S}_k(\mu), \mathsf{C}_k(\mu)$ and Section 3, gives the subordination and superordination results involving the integral operator $\mathsf{J}^k_{\lambda,\delta} f(z).$ For this purpose we need to the following lemmas in the sequel.

Definition 1.4. [10] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U}-E(f)$ where $E(f):=\{\zeta\in\partial U: lim_{z\to\zeta} f(z)=\infty\}$ and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U-E(f).$

Lemma 1.2. [11] Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U).$ Set $Q(z):=zq'(z)\phi(q(z)), h(z):=\theta(q(z))+Q(z).$ Suppose that

1. $Q(z)$ is starlike univalent in U , and

2. $\Re\frac{zh'(z)}{Q(z)} > 0$ for $z\in U.$

If

$$\theta(p(z))+zp'(z)\phi(p(z)) \prec \theta(q(z))+zq'(z)\phi(q(z))$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

Lemma 1.3. [12] Let $q(z)$ be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that

1. $zq'(z)\varphi(q(z))$ is starlike univalent in U, and

2. $\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ for $z \in U$.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi(z)$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

2 General Properties of $\mathcal{J}^k_{\lambda,\delta}$

In this section we study the characterization properties for the function $f(z) \in \mathcal{A}$ to belong to the classes $\mathcal{S}^k_{\lambda,\delta}(\mu)$ and $\mathcal{C}^k_{\lambda,\delta}(\mu)$ by obtaining the coefficient bounds.

Theorem 2.1. Let $f(z) \in \mathcal{A}$. If

$$\sum_{n=2}^{\infty} \frac{(n-\mu)\,|a_n|}{[1+(n-1)\lambda]^k C(\delta,n)} \le 1-\mu, \quad 0 \le \mu < 1, \quad (4)$$

then $f(z) \in \mathcal{S}^k_{\lambda,}(\mu)$. The result (4) is sharp.

Proof. Suppose that (4) holds. Since

$$
\begin{aligned}
1-\mu &\geq \sum_{n=2}^{\infty} \frac{|a_n||\mu-n|}{[1+(n-1)\lambda]^k} \\
&\geq \mu \sum_{n=2}^{\infty} \frac{|a_n|}{[1+(n-1)\lambda]^k} - \sum_{n=2}^{\infty} \frac{n|a_n|}{[1+(n-1)\lambda]^k}
\end{aligned}
$$

then this implies that

$$
\frac{1+\sum_{n=2}^{\infty} \frac{n|a_n|}{[1+(n-1)\lambda]^k}}{1+\sum_{n=2}^{\infty} \frac{|a_n|}{[1+(n-1)\lambda]^k}} > \mu,
$$

hence

$$
\Re\left\{\frac{z[\mathrm{J}_{\lambda,\delta}^k f(z)]'}{\mathrm{J}_{\lambda,\delta}^k f(z)}\right\} > \mu.
$$

We also note that the assertion (4) is sharp and the extremal function is given by

$$
f(z) = z + \sum_{n=2}^{\infty} \frac{[1+(n-1)\lambda]^k C(\delta,n)(1-\mu)}{(n-\mu)} z^n.
$$

Corollary 2.1. Let the assumption of Theorem 2.1. Then

$$
|a_n| \leq \frac{[1+(n-1)\lambda]^k C(\delta,n)(1-\mu)}{(n-\mu)}, \forall n \geq 2.
$$

Corollary 2.2. Let the assumption of Theorem 2.1. Then for $\mu = \delta = 0$ and $\lambda = 1$

$$
|a_n| \leq n^{k-1}, \forall n \geq 2, k \in \mathrm{N}_0.
$$

In the same way we can verify the following results:

Theorem 2.2. Let $f(z) \in \mathrm{A}$. If

$$\sum_{n=2}^{\infty}\frac{n|a_n||\mu+1-n|}{C(\delta,n)[1+(n-1)\lambda]^k}\le 1-\mu,\ 0\le\mu<1, \quad (5)$$

then $f(z)\in \mathsf{C}^k_{\lambda,\delta}(\mu)$. The result (5) is sharp.

Corollary 2.3. Let the assumption of Theorem 2.2. Then

$$|a_n|\le\frac{[1+(n-1)\lambda]^k C(\delta,n)(1-\mu)}{n|\mu-n+1|},\ \forall n\ge 2.$$

Also we have the following inclusion results

Theorem 2.3. Let $0\le\mu_1\le\mu_2<1$. Then $\mathsf{S}^k_{\lambda,\delta}(\mu_1)\supseteq\mathsf{S}^k_{\lambda,\delta}(\mu_2)$.

Proof. By Theorem 2.1.

Theorem 2.4. Let $0\le\mu_1\le\mu_2<1$. Then $\mathsf{C}^k_{\lambda,\delta}(\mu_1)\supseteq\mathsf{C}^k_{\lambda,\delta}(\mu_2)$.

Proof. By Theorem 2.2.

Theorem 2.5. Let $0\le\lambda_1\le\lambda_2$. Then $\mathsf{S}^k_{\lambda_1,\delta}(\mu)\supseteq\mathsf{S}^k_{\lambda_2,\delta}(\mu)$.

Proof. By Theorem 2.1.

Theorem 2.6. Let $0\le\lambda_1\le\lambda_2$. Then $\mathsf{C}^k_{\lambda_1\delta}(\mu)\supseteq\mathsf{C}^k_{\lambda_2,\delta}(\mu)$.

Proof. By Theorem 2.2.

Moreover, we introduce the following distortion theorems.

Theorem 2.7. Let $f\in\mathsf{A}$ and satisfies (4). Then for $z\in U$ and $0\le\mu<1$

$$|\mathsf{J}^k_{\lambda,\delta}f(z)|\ge|z|-\frac{(1-\mu)}{(2-\mu)}|z|^2$$

and

$$|\mathrm{J}_{\lambda,\delta}^{k} f(z)| \leq |z| + \frac{(1-\mu)}{(2-\mu)}|z|^2 .$$

Proof. By using Theorem 2.1, one can verify that

$$(2-\mu)\sum_{n=2}^{\infty}\frac{|a_n|}{[1+(n-1)\lambda]^k C(\delta,n)} \leq \sum_{n=2}^{\infty}\frac{(n-\mu)|a_n|}{[1+(n-1)\lambda]^k C(\delta,n)} \leq 1-\mu$$

then

$$\sum_{n=2}^{\infty}\frac{|a_n|}{[1+(n-1)\lambda]^k C(\delta,n)} \leq \frac{1-\mu}{2-\mu}.$$

Thus we obtain

$$\begin{aligned} |\mathrm{J}_{\lambda,\delta}^{k} f(z)| &= |z + \sum_{n=2}^{\infty}\frac{a_n}{[1+(n-1)\lambda]^k} z^n| \\ &\leq |z| + \sum_{n=2}^{\infty}\frac{|a_n|}{[1+(n-1)\lambda]^k}|z|^2 \\ &\leq |z| + [\frac{1-\mu}{2-\mu}]|z|^2 \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |\mathrm{J}_{\lambda,\delta}^{k} f(z)| &= |z + \sum_{n=2}^{\infty}\frac{a_n}{[1+(n-1)\lambda]^k C(\delta,n)} z^n| \\ &\geq |z - \sum_{n=2}^{\infty}\frac{a_n}{[1+(n-1)\lambda]^k C(\delta,n)} z^n| \\ &\geq |z| - \sum_{n=2}^{\infty}\frac{|a_n|}{[1+(n-1)\lambda]^k C(\delta,n)}|z|^2 \\ &\geq |z| - [\frac{1-\mu}{2-\mu}]|z|^2 . \end{aligned}$$

This complete the proof.

In the same way we can get the following results.

Theorem 2.8. Let $f(z)\in \mathsf{A}$ and satisfies (5). Then for $z\in U$ and $0\le \mu <1$

$$|\mathsf{J}^{k}_{\lambda,\delta} f(z)|\ge |z|-\frac{(1-\mu)}{2(2-\mu)}|z|^{2}$$

and

$$|\mathsf{J}^{k}_{\lambda,\delta} f(z)|\le |z|+\frac{(1-\mu)}{2(2-\mu)}|z|^{2} .$$

Also, we have the following distortion results

Theorem 2.9. Let $f(z)\in \mathsf{A}$ and and satisfies (4). Then for $m\ge [1+(n-1)\lambda]^{k} C(\delta,n), z\in U$ and $0\le \mu <1$

$$|f(z)|\ge |z|-\frac{m(1-\mu)}{(2-\mu)}|z|^{2}$$

and

$$|f(z)|\le |z|+\frac{m(1-\mu)}{(2-\mu)}|z|^{2} .$$

Proof. By using Theorem 2.1, one can show that

$$(2-\mu)\sum_{n=2}^{\infty}|a_n|\le \sum_{n=2}^{\infty}(n-\mu)|a_n|\le m\sum_{n=2}^{\infty}\frac{(n-\mu)|a_n|}{[1+(n-1)\lambda]^{k} C(\delta,n)}\le m(1-\mu)$$

then

$$\sum_{n=2}^{\infty}|a_n|\le \frac{m(1-\mu)}{2-\mu}.$$

Thus we obtain

$$
\begin{aligned}
|f(z)| &= |z + \sum_{n=2}^{\infty} a_n z^n| \\
&\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^2 \\
&\leq |z| + \frac{m(1-\mu)}{2-\mu} |z|^2
\end{aligned}
$$

The other assertion can be proved as follows

$$
\begin{aligned}
|f(z)| &\geq |z - \sum_{n=2}^{\infty} a_n z^n| \\
&\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^2 \\
&\geq |z| - \frac{m(1-\mu)}{2-\mu} |z|^2 .
\end{aligned}
$$

This completes the proof.

In the same way we can get the following results.

Theorem 2.10. Let $f(z) \in \mathcal{A}$ and and satisfies (5). Then for $z \in U$ and $0 \leq \mu < 1$

$$|f(z)| \geq |z| - \frac{m(1-\mu)}{2(2-\mu)} |z|^2$$

and

$$|f(z)| \leq |z| + \frac{m(1-\mu)}{2(2-\mu)} |z|^2 .$$

3 Sandwich Result.

By making use of lemmas 1.2 and 1.3, we prove the following subordination and superordination results involving the integral operator (3).

Theorem 3.1. Let $q \neq 0$ be univalent in U such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and

$$\Re\{1+(\frac{\alpha}{\gamma}+z)\frac{q''(z)}{q'(z)}-(\frac{\alpha}{\gamma}+z)\frac{q'(z)}{q(z)}\}>0, \alpha,\gamma \in \mathbb{C}, \gamma \neq 0. \tag{6}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(\alpha+\gamma z)\{\frac{1}{z}+\frac{[\mathrm{J}^k_{\lambda,\delta} f(z)]''}{[\mathrm{J}^k_{\lambda,\delta} f(z)]'}-\frac{\Phi'[\mathrm{J}^k_{\lambda,\delta} f(z)]}{\Phi[\mathrm{J}^k_{\lambda,\delta} f(z)]}\} \prec (\alpha+\gamma z)\frac{q'(z)}{q(z)},$$

then

$$\frac{z[\mathrm{J}^k_{\lambda,\delta} f(z)]'}{\Phi[\mathrm{J}^k_{\lambda,\delta} f(z)]} \prec q(z) \tag{7}$$

and q is the best dominant.

Proof. Our aim is to apply Lemma 1.2. Setting

$$p(z) := \frac{z[\mathrm{J}^k_{\lambda,\delta} f(z)]'}{\Phi[\mathrm{J}^k_{\lambda,\delta} f(z)]}.$$

By computation shows that

$$\frac{zp'(z)}{p(z)} = 1+\frac{z[J^k_{\lambda} f(z)]''}{[\mathrm{J}^k_{\lambda,\delta} f(z)]'}-\frac{z\Phi'[\mathrm{J}^k_{\lambda,\delta} f(z)]}{\Phi[\mathrm{J}^k_{\lambda,\delta} f(z)]}$$

which yields the following subordination

$$(\alpha+\gamma z)\frac{p'(z)}{p(z)} \prec (\alpha+\gamma z)\frac{q'(z)}{q(z)}, \alpha,\gamma \in \mathbb{C}.$$

By setting

$$\theta(\omega) := \frac{\alpha\omega'}{\omega} \; and \; \phi(\omega) := \frac{\gamma}{\omega}, \gamma \neq 0,$$

it can be easily observed that $\theta(\omega), \phi(\omega)$ are analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(\omega) \neq 0$ when $\omega \in \mathbb{C}\setminus\{0\}$. Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \gamma z \frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \frac{\alpha q'(z)}{q(z)} + \gamma z \frac{q'(z)}{q(z)} = (\alpha + \gamma z)\frac{q'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U and that

$$\Re\{\frac{zh'(z)}{Q(z)}\} = \Re\{1 + (\frac{\alpha}{\gamma} + z)\frac{q''(z)}{q'(z)} - (\frac{\alpha}{\gamma} + z)\frac{q'(z)}{q(z)}\} > 0, \alpha, \gamma \in \mathbb{C}, \gamma \neq 0.$$

Then the relation (7) follows by an application of Lemma 1.2.

Corollary 3.1. Let the assumptions of Theorem 3.1 hold. Then the subordination

$$1 + \frac{z[\mathrm{J}^k_{\lambda,\delta} f(z)]''}{[\mathrm{J}^k_{\lambda,\delta} f(z)]'} - \frac{z[\mathrm{J}^k_{\lambda,\delta} f(z)]'}{[\mathrm{J}^k_{\lambda,\delta} f(z)]} \prec \frac{zq'(z)}{q(z)},$$

implies

$$\frac{z[\mathrm{J}^k_{\lambda,\delta} f(z)]'}{[\mathrm{J}^k_{\lambda,\delta} f(z)]} \prec q(z) \tag{8}$$

and q is the best dominant.

Proof. By letting $\alpha = 0, \gamma = 1, \Phi(\omega) := \omega.$

Corollary 3.2. If $f \in \mathrm{A}$ and assume that (7) holds then

$$1+\frac{z[J^k_{\lambda,\delta}f(z)]''}{[J^k_{\lambda,\delta}f(z)]'}-\frac{z[J^k_{\lambda,\delta}f(z)]'}{[J^k_{\lambda,\delta}f(z)]}\prec\frac{(A-B)z}{(1+Az)(1+Bz)}$$

implies

$$\frac{z[J^k_{\lambda,\delta}f(z)]'}{[J^k_{\lambda,\delta}f(z)]}\prec\frac{1+Az}{1+Bz},\ -1\le B<A\le 1$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. By setting $\Phi(\omega):=\omega,\gamma=1,\alpha=0$ and $q(z):=\frac{1+Az}{1+Bz}$ where $-1\le B<A\le 1.$

Corollary 3.3. If $f\in\mathcal{A}$ and assume that (7) holds then

$$1+\frac{z[J^k_{\lambda,\delta}f(z)]''}{[J^k_{\lambda,\delta}f(z)]'}-\frac{z[J^k_{\lambda,\delta}f(z)]'}{[J^k_{\lambda,\delta}f(z)]}\prec\frac{2z}{1-z^2}$$

implies

$$\frac{z[J^k_{\lambda,\delta}f(z)]'}{J^k_{\lambda,\delta}f(z)}\prec\frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Proof. By setting $\Phi(\omega):=\omega,\ \alpha=0,\gamma=1,$ and $q(z):=\frac{1+z}{1-z}.$

Corollary 3.4. If $f\in\mathcal{A}$ and assume that (7) holds then

$$1+\frac{z[J^k_{\lambda,\delta}f(z)]''}{[J^k_{\lambda,\delta}f(z)]'}-\frac{z[J^k_{\lambda,\delta}f(z)]'}{[J^k_{\lambda,\delta}f(z)]}\prec Az$$

implies

$$\frac{z[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'}{\mathrm{J}_{\lambda,\delta}^{k} f(z)} \prec e^{Az},$$

and e^{Az} is the best dominant.

Proof. By setting $\Phi(\omega) := \omega,\ \alpha = 0, \gamma = 1,$ and $q(z) := e^{Az}, |A| < \pi.$

Theorem 3.2. Let $q(z) \neq 0$ be convex univalent in the unit disk U. Suppose that

$$\Re\{\frac{\alpha}{\gamma} q''(z) - \frac{\alpha}{\gamma}\frac{q'(z)}{q(z)}\} > 0, \alpha,, \gamma \in \mathbb{C} \ for\ z \in U \tag{9}$$

and $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $\frac{z[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]} \in \mathcal{H}[q(0),1] \cap Q$ where $f \in \mathcal{A}$,

$$(\alpha + \gamma z)\{\frac{1}{z} + \frac{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]''}{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'} - \frac{\Phi'[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}\}$$

is univalent is U and the subordination

$$(\alpha + \gamma z)\frac{q'(z)}{q(z)} \prec (\alpha + \gamma z)\{\frac{1}{z} + \frac{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]''}{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'} - \frac{\Phi'[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}\},$$

holds, then

$$q(z) \prec \frac{z[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]} \tag{10}$$

and q is the best subordinant.

Proof. Our aim is to apply Lemma 1.3. Setting

$$p(z) := \frac{z[J^k_{\lambda,\delta} f(z)]'}{\Phi[J^k_{\lambda,\delta} f(z)]}.$$

By computation shows that

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z[J^k_{\lambda,\delta} f(z)]''}{[J^k_{\lambda,\delta} f(z)]'} - \frac{z\Phi'[J^k_{\lambda,\delta} f(z)]}{\Phi[J^k_{\lambda,\delta} f(z)]}$$

which yields the following subordination

$$(\alpha + \gamma z)\frac{q'(z)}{q(z)} \prec (\alpha + \gamma z)\frac{p'(z)}{p(z)}, \alpha, \gamma \in \mathbb{C}.$$

By setting

$$\vartheta(\omega) := \frac{\alpha\omega'}{\omega} \; and \; \varphi(\omega) := \frac{\gamma}{\omega}, \gamma \neq 0,$$

it can be easily observed that $\vartheta(\omega), \varphi(\omega)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that $\varphi(\omega) \neq 0$ when $\omega \in \mathbb{C} \setminus \{0\}$. Also, we obtain

$$\Re\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\} = \Re\{\frac{\alpha}{\gamma} q''(z) - \frac{\alpha}{\gamma}\frac{q'(z)}{q(z)}\} > 0.$$

Then (10) follows by an application of Lemma 1.3.

Combining Theorems 3.1 and 3.2 in order to get the following Sandwich theorems

Theorem 3.3 Let $q_1(z) \neq 0, q_2(z) \neq 0$ be convex univalent in the unit disk U satisfy (9) and (6) respectively. Suppose that and $\frac{zq_i'(z)}{q_i(z)}, i = 1,2$ is starlike univalent in U. If $f \in \mathcal{A}$ and

$$\frac{z[J^k_{\lambda,\delta} f(z)]'}{\Phi[J^k_{\lambda,\delta} f(z)]} \in \mathcal{H}[q_1(0),1] \cap Q$$

$$(\alpha+\gamma z)\{\frac{1}{z}+\frac{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]''}{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'}-\frac{\Phi'[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}\}$$

is univalent in U and the subordination

$$(\alpha+\gamma z)\frac{q_1'(z)}{q_1(z)} \prec (\alpha+\gamma z)\{\frac{1}{z}+\frac{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]''}{[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'}-\frac{\Phi'[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]}\} \prec (\alpha+\gamma z)\frac{q_2'(z)}{q_2(z)}$$

holds, then

$$q_1(z) \prec \frac{z[\mathrm{J}_{\lambda,\delta}^{k} f(z)]'}{\Phi[\mathrm{J}_{\lambda,\delta}^{k} f(z)]} \prec q_2(z)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

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