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Total Edge Irregularity Strength of the Disjoint Union of Helm Graphs

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Abstract. The total edge irregular k-labeling of a graph *G=(V,E)* is the labeling of vertices and edges of *G* in such a way that for any different edges their weights are distinct. The total edge irregularity strength, *tes (G),* is defined as the minimum *k* for which *G* has a total edge irregular *k*-labeling. In this paper, we consider the total edge irregularity strength of the disjoint union of m special types of helm graphs.

Keywords: *disjoint union; edge irregular total labeling; helm graph; irregularity strength; total edge irregularity strength.*

1 Introduction

In this paper, we consider a graph *G* as a finite graph (without loop and multiple edges) with the vertex-set *V* and the edge-set *E*. In [1], Baca, Jendrol, Miller and Ryan introduced the notion of the total edge irregular *k*-labeling of a graph *G=(V,E* namely the labeling $\psi : V \cup E \rightarrow \{1, 2, \ldots, k\}$ such that all edge weights are different. The weight *wt(uv)* of an edge *uv* is defined as $wt_{\psi}(uv) = \psi(u) + \psi(uv) + \psi(v)$. The total edge irregularity strength of *G*, denoted by *tes(G)*, is the smallest *k* for which *G* has a total edge irregular *k*-labeling.

The basic idea of the total edge irregularity strength came from irregular assignments and the irregularity strength of graphs introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [2]. An *irregular assignment* is a *k*-labeling of the edges such that the sum of the labels of edges incident to a vertex is different for all the vertices of *G*. The smallest integer *k* for which *G* has an *irregular assignment* is called the *irregularity strength* of *G*, and is denoted by *s(G).*

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It is not an easy task to compute the irregularity strength of graphs with simple structures, see [3-6]. Karonski, Luczak and Thomason [7] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from {1, 2, 3} such that for all pairs of adjacent vertices the sums of the labels of the incident edges are distinct. Baca, Jendrol, Miller and Ryan [1] gave a lower bound on the total edge irregularity strength of a graph:

$$tes(G) \geq \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\} \tag{1}$$

where $\Delta(G)$ is the maximum degree of *G*. The authors of [1] determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs. Recently, Ivanco and Jendrol [8] posed the following conjecture:

Conjecture 1. Let *G* be an arbitrary graph different from *K_5*. Then

$$tes(G) = \max\left\{\left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\right\} \tag{2}$$

Conjecture 1 has been verified for all trees in [8], for complete graphs and complete bipartite graphs in [9] and [10], for the Cartesian product of two paths $P_n \square P_m$ in [11], for the corona product of a path with certain graphs in [12], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [13], for hexagonal grids in [14], for the zigzag graph [15], for the categorical product of two paths $P_n \times P_m$ [16], for the categorical product of a cycle and a path $C_n \times P_m$ in [17,18], for a subdivision of stars in [19], for the categorical product of two cycles in [20], and for the strong product of two paths in [21].

Motivated by [22], we investigated the total edge irregularity strength of the disjoint union of helm graphs. A *helm graph* H_n is obtained from a wheel on *n+1* vertices by adding a pendant edge to every vertex of its cycle C_n. In this study, we determined the total edge irregularity strength of the disjoint union of *m* copies of a certain helm graph. We also determined the total edge irregularity strength of the disjoint union of non-isomorphic helm graphs.

This paper adds further support to Conjecture 1 by demonstrating that the disjoint union of helm graphs has a total edge irregularity strength equal to

$$\left\lceil \frac{\left| E\left(\bigcup_{j=1}^{m} H_{n+j} \right) \right| + 2}{3} \right\rceil.$$

2 Main Results

First, we determine the total edge irregularity strength of a disjoint union mH_n of *m* copies of a helm graph H_n. Let

$$V(H_n) = \{c^j, x_i^j, y_i^j;\ 1 \le i \le n, 1 \le j \le m\}$$

$$E(H_n) = \{c^j x_i^j, x_i^j y_i^j,\ x_i^j x_{i+1}^j\ ;\ 1 \le i \le n, 1 \le j \le m\}$$

Moreover, the subscript *n*+1 is replaced by 1.

Lemma 1. *For* $n \ge 3$ $tes(2H_n) = 2n+1.$

Proof. From (1) it follows that $tes(2H_n) \ge 2n+1$. Now the existence of an optimal labeling φ_1 proves the converse inequality for $1 \le i \le n$ as follows:

$$\varphi_1(x_i^1) = \varphi_1(y_i^1) = 1,\ \ \varphi_1(c^1) = \varphi_1(c^2) = 2n+1,$$

$$\varphi_1(c^1 x_i^1) = \varphi_1(x_i^2 x_{i+1}^2) = \varphi_1(x_i^1 y_i^1) = \varphi_1(x_i^2 y_i^2) = i,$$

$$\varphi_1(x_i^2) = 2n+1,\ \ \varphi_1(y_i^2) = n+1,\ \ \varphi_1(c^2 x_i^2) = \varphi_1(x_i^1 x_{i+1}^1) = n+i.$$

It is easy to see that the weights of the edges are pair-wise distinct. This concludes the proof.

Theorem 1. Let $m, n \ge 3$ be two integers. Then, the total edge irregularity strength of a disjoint union mH_n of *m* copies of a helm graph H_n is *mn*+1.

Proof. As |E(mH_n)| = 3*mn* then (1) implies that $tes(H_n) \ge mn+1$. Let *k* = *mn*+1. To prove the converse inequality, we define the total edge irregular *k*-labeling ψ_1 for $1 \le i \le n$ and $1 \le j \le m$ as follows:

$$\psi_1(c^j) = \psi_1(x_i^j) = \psi_1(y_i^j) = \min\left\{\left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor,\ k\right\}.$$

Case I: For $1 \le j \le m$ such that $\left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor < k$

(i) When n is even,

$\psi_1\left(x_i^j y_i^j\right) = i,\ \psi_1\left(x_i^j x_{i+1}^j\right) = n+i,\ \psi_1\left(c^j x_i^j\right) = 2n+i,$

(ii) When n is odd,

(a) If j is odd, then the edges $c^j x_i^j, x_i^j y_i^j$ and $x_i^j x_{i+1}^j$ receive the same labels as in Case I (i)

(b) If j is even,

$\psi_1\left(x_i^j y_i^j\right) = 1+i,\quad \psi_1\left(x_i^j x_{i+1}^j\right) = n+1+i,\quad \psi_1\left(c^j x_i^j\right) = 2n+1+i,$

Case II: For $1 \le j \le m$ such that $\left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor \ge k$

Let

$$w = \min\left\{j;\ 1 \le j \le m \ \ such\ \ that\ \left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor \ge k\right\}$$

$$l = \max\left\{t_j;\ 1 \le j \le m \ \ such\ \ that\ \left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor < k\right\}$$

$$t_j = \min\left\{\left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor,\ \mathrm{k}\right\} \quad for \quad 1 \le j \le m \ \ such\ \ that\ \left\lfloor \frac{3n(j-1)+2}{2} \right\rfloor < k$$

(i) When n is even,

$$\psi_1\left(x_i^j y_i^j\right) = \begin{cases} 3n+2(l-k)+i, & if \quad j = w \\ 3n+2(l-k)+i+(j-w)3n, & if \quad w+1 \le j \le m \end{cases}$$

$$\psi_1\left(x_i^j x_{i+1}^j\right) = \begin{cases} 4n+2(l-k)+i, & if \quad j = w \\ 4n+2(l-k)+i+(j-w)3n, & if \quad w+1 \le j \le m \end{cases}$$

$$\psi_1\left(c^j y_i^j\right) = \begin{cases} 5n+2(l-k)+i, & if \quad j = w \\ 5n+2(l-k)+i+(j-w)3n, & if \quad w+1 \le j \le m \end{cases}$$

(ii) When n is odd,

(a) If w is odd

$$\psi_1\left(x_i^j y_i^j\right) = \begin{cases} 3n+2(l-k)+1+i, & if \quad j = w \\ 3n+2(l-k)+i+1+(j-w)3n, & if \quad w+1 \le j \le m \end{cases}$$

$$\psi_1\left(x_i^j x_{i+1}^j\right)=\begin{cases}4n+2(l-k)+1+i, & if \quad j=w\\ 4n+2(l-k)+i+1+(j-w)3n, & if \quad w+1\le j\le m\end{cases}$$

$$\psi_1\left(c^j y_i^j\right)=\begin{cases}5n+2(l-k)+1+i, & if \quad j=w\\ 5n+2(l-k)+i+1+(j-w)3n, & if \quad w+1\le j\le m\end{cases}$$

(b) If w is even, then the edges $c^j x_i^j, x_i^j y_i^j$ *and* $x_i^j x_{i+1}^j$ receive the same labels as in Case II (i)

Under the labeling ψ_1 the total weights of the edges are described as follows:

(i) The edges $x_i^j y_i^j$ for $1\le i\le n, 1\le j\le m$ receive consecutive integers from the interval [3*n*(*j*-1)+3, 3*n*(*j*-1)+*n*+2],

(ii) The edges $x_i^j x_{i+1}^j$ for $1\le i\le n, 1\le j\le m$ receive consecutive integers from the interval [3*n*(*j*-1)+*n*+3, 3*n*(*j*-1)+2*n*+2],

(iii) The edges $c^j x_i^j$ for $1\le i\le n, 1\le j\le m$ receive consecutive integers from the interval [3*n*(*j*-1)+2*n*+3, 3*n*(*j*-1)+3*n*+2].

It is not difficult to see that all vertex and edge labels are at most *k* and the edge-weights of the edges $c^j x_i^j, x_i^j y_i^j$ and $x_i^j x_{i+1}^j$ are pairwise distinct. Thus, the resulting labeling is a total edge irregular *k*-labeling. This concludes the proof.

For $m, n\ge 2$, let us consider the disjoint union of m non-isomorphic helm graphs: $H_{n+1}, H_{n+2}, H_{n+3},..., H_{n+m}$, where

$$V\left(\bigcup_{j=1}^{m} H_{n+j}\right)=\left\{c^j, x_i^j, y_i^j;\ 1\le i\le n+j, 1\le j\le m\right\}$$

is the corresponding vertex set and

$$E\left(\bigcup_{j=1}^{m} H_{n+j}\right)=\left\{c^j x_i^j,\ x_i^j y_i^j,\ x_i^j x_{i+1}^j\ ;\ 1\le i\le n+j, 1\le j\le m\right\}$$

is the corresponding edge set. Note that the subscript *n*+*j*+1 is replaced by 1.

Now, we determine the exact value of the total edge irregularity strength of the graph $\bigcup_{j=1}^{m} H_{n+j}$.

Theorem 1. Let $m, n \geq 2$ be two integers and $G \cong \bigcup_{j=1}^{m} H_{n+j}$. Then $tes(G) = mn + 1 + \frac{m(m+1)}{2}$.

Proof. As $|E(\bigcup_{j=1}^{m} H_{n+j})| = 3\sum_{j=1}^{m}(n+j)$ then from (1) it follows that $tes(G) \geq mn + 1 + \frac{m(m+1)}{2}$. Let $k = mn + 1 + \frac{m(m+1)}{2}$. To prove the converse inequality, we define the total edge irregular k-labeling ψ_2 for $1 \leq i \leq n+j$ and $1 \leq j \leq m$ as follows.

For $1 \leq i \leq n+1$

$$\psi_2(c^1) = \psi_2(x_i^1) = \psi_2(y_i^1) = 1,$$

$$\psi_2\left(x_i^1 y_i^1\right) = i, \quad \psi_2\left(x_i^1 x_{i+1}^1\right) = n + 1 + i, \quad \psi_2\left(c^1 x_i^1\right) = 2n + 2 + i,$$

For $1 \leq i \leq n+j$

$$\psi_2(c^j) = \psi_2(x_i^j) = \psi_2(y_i^j) = \min\left\{\left\lfloor \frac{3\sum_{s=1}^{j-1}(n+s+2)}{2} \right\rfloor,\ k\right\} \quad for \quad 2 \leq j \leq m.$$

- For $2 \leq j \leq m$ such that $\left\lfloor \frac{3\sum_{s=1}^{j-1}(n+s)+2}{2} \right\rfloor < k$

(i) When $\sum_{s=1}^{j-1}(n+s) \equiv 0 \pmod 2$

$$\psi_2\left(x_i^j y_i^j\right) = i, \quad \psi_2\left(x_i^j x_{i+1}^j\right) = n + j + i, \quad \psi_2\left(c^j x_i^j\right) = 2n + 2j + i,$$

(ii) When $\sum_{s=1}^{j-1}(n+s) \equiv 1 \pmod 2$

$$\psi_2\left(x_i^j y_i^j\right) = 1 + i, \quad \psi_2\left(x_i^j x_{i+1}^j\right) = 1 + n + j + i, \quad \psi_2\left(c^j x_i^j\right) = 1 + 2n + 2j + i,$$

- For $2 \le j \le m$ such that $\left\lfloor \frac{3\sum_{s=1}^{j-1}(n+s)+2}{2} \right\rfloor \ge$ k

Let

$$w = \min\left\{ j;\quad 2 \le j \le m \ \ such\ \ that\ \left\lfloor \frac{3\sum_{s=1}^{j-1}(n+s)+2}{2} \right\rfloor \ge \mathrm{k} \right\}$$

$$\psi_2\left(x_i^j y_i^j\right) = \begin{cases} 3\sum_{s=1}^{w-1}(n+s) - 2k + i, & if \quad j = w \\ 3\sum_{s=1}^{w-1}(n+s) - 2k + i + 3n + 3j - 1, & if \quad w+1 \le j \le m \end{cases}$$

$$\psi_2\left(x_i^j x_{i+1}^j\right) = \begin{cases} 3\sum_{s=1}^{w-1}(n+s) - 2k + 2 + n + w + i, & if \quad j = w \\ 3\sum_{s=1}^{w-1}(n+s) - 2k + i + 4n + 2j + w, & if \quad w+1 \le j \le m \end{cases}$$

$$\psi_2\left(c^j x_i^j\right) = \begin{cases} 3\sum_{s=1}^{w-1}(n+s) - 2k + 2 + 2n + 2w + i, & if \quad j = w \\ 3\sum_{s=1}^{w-1}(n+s) - 2k + i + 5n + j + 2w + 1, & if \quad w+1 \le j \le m \end{cases}$$

Under the labeling ψ_2 the total weights of the edges are described as follows:

(i) The edges $x_i^1 y_i^1$, $x_i^1 x_{i+1}^1$ and $c^1 x_i^1$ for $1 \le i \le n+1$ receive consecutive integers from the interval [3,3+n], [4+n,2n+4] and [2n+5,3n+5], respectively.

(ii) The edges $x_i^j y_i^j$ for $1 \le i \le n+j, 2 \le j \le m$ receive consecutive integers from the interval $\left[3\sum_{s=1}^{j-1}(n+s)+3,\ 3\sum_{s=1}^{j-1}(n+s)+n+j+2\right]$,

(iii) The edges $x_i^j x_{i+1}^j$ for $1 \le i \le n+j, 2 \le j \le m$ receive consecutive integers from the interval $\left[3\sum_{s=1}^{j-1}(n+s)+n+j+3,\ 3\sum_{s=1}^{j-1}(n+s)+2n+2j+2\right]$,

(iv) The edges $c^j x_i^j$ for $1 \le i \le n+j, 2 \le j \le m$ receive consecutive integers from the interval $\left[3\sum_{s=1}^{j-1}(n+s)+2n+2j+3,\ 3\sum_{s=1}^{j}(n+s)+2\right]$,

It is not difficult to see that all vertex and edge labels are at most k and the edge-weights of the edges $c^j x_i^j, x_i^j y_i^j$ and $x_i^j x_{i+1}^j$ are pairwise distinct. Thus, the resulting labeling is a total edge irregular k-labeling. This concludes the proof.

3 Conclusion

In this paper, we have determined the exact value of the total edge irregularity strength of the disjoint union of m copies of a helm graph as well as the disjoint union of non-isomorphic helm graphs $\bigcup_{j=1}^{m} H_{n+j}$. We conclude by stating the following open problem:

Open Problem. For $m \ge 2$ find the exact value of the total edge irregularity strength of a disjoint union of m arbitrary helm graphs.

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References

[1] Baca, M., Jendrol, S., Miller, M. & Ryan, J., *On Irregular Total Labellings,* Discrete Math., **307**, pp**.** 1378-1388, 2007.

[2] Chartrand, G., Jacobson, M.S., Lehel, J., Oellermann, O.R., Ruiz, S. & Saba, F., *Irregular Networks,* Congr. Numer., **64**, pp. 187-192, 1988.

[3] Bohman, T. & Kravitz, D., *On the Irregularity Strength of Trees,* J. Graph Theory, **45**, pp. 241-254, 2004.

[4] Frieze, A., Gould, R.J., Karonski, M. & Pfender, F., *On Graph Irregularity Strength*, J. Graph Theory, **41**, pp. 120-137, 2002.

[5] Jendrol, S., Tkac, M., & Tuza, Z., *The Irregularity Strength and Cost of The Union of Cliques*, Discrete Math., **150**, pp. 179-186, 1996.

[6] Nierhoff, T., *A Tight Bound on The Irregularity Strength of Graphs,* SIAM J. Discrete Math., **13**, pp. 313-323, 2000.

[7] Karonski, M., Luczak, T. & Thomason, A., *Edge Weights and Vertex Colours,* J. Combin. Theory B, **91**, pp. 151-157, 2004.

[8] Ivanco, J. & Jendrol, S., *Total Edge Irregularity Strength of Trees,* Discussiones Math. Graph Theory, **26**, pp. 449-456, 2006.

[9] Jendrol, S., Miskuf, J. & Sotak, R., *Total Edge Irregularity Strength of Complete and Complete Bipartite Graphs*, Electron. Notes Discrete Math., **28**, pp. 281-285, 2007.

[10] Jendrol, S., Miskuf, J. & Sotak, R., *Total Edge Irregularity Strength of Complete Graphs and Complete Bipartite Graphs*, Discrete Math., **310**, pp. 400-407, 2010.

[11] Miskuf, J. & Jendrol, S., *On Total Edge Irregularity Strength of The Grids*, Tatra Mt. Math. Publ., **36**, pp. 147-151, 2007.

[12] Nurdin, Salman, A.N.M. & Baskoro, E.T., *The Total Edge-Irregular Strengths of The Corona Product of Paths with Some Graphs*, J. Combin., Math. Combin. Comput., **65**, pp. 163-175, 2008.

[13] Brandt, S., Miskuf, J. & Rautenbach, D., *On A Conjecture about Edge Irregular Total Labellings*, J. Graph Theory, **57**, pp. 333-343, 2008.

[14] Al-Mushayt, O., Ahmad, A. & Siddiqui, M.K., *On the Total Edge Irregularity Strength of Hexagonal Grid Graphs*, Australas. J. Combin. **53**, pp. 263-271, 2012.

[15] Ahmad, A., Siddiqui, M.K. & Afzal, D., *On the Total Edge Irregularity Strength of Zigzag Graphs*, Australasian. J. Combin., **54**, pp. 141-149, 2012.

[16] Ahmad, A. & Baca, M., *Total Edge Irregularity Strength of A Categorical Product of Two Paths*, Ars. Combin, in press.

[17] Ahmad, A. & Baca, M., *Edge Irregular Total Labeling of Certain Family of Graphs*, AKCE J. Graphs. Combin., **6**(1), pp. 21-29, 2009.

[18] Siddiqui, M.K., *On Total Edge Irregularity Strength of A Categorical Product of Cycle And Path*, AKCE J. Graphs. Combin., **9**(1), pp. 43-52, 2012.

[19] Siddiqui, M.K., *On Tes of Subdivision of Star*, Int. J. of Math and Soft Compu., **2**(1), pp. 75-82, 2012.

[20] Ahmad, A., Baca, M. & Siddiqui, M.K., *On Edge Irregular Total Labeling of Categorical Product of Two Cycles*, Theory of Computing systems, doi: 10.1007/s00224-013-9470-3.

[21] Ahmad, A., Baca, M., Bashir, Y. & Siddiqui, M.K., *Total Edge Irregularity Strength of Strong Product of Two Paths*, Ars Combin., **106**, pp. 449-459, 2012.

[22] Ahmad, A., Baca, M. & Numan, M., *On Irregularity Strength of Disjoint Union of Friendship Graphs*, Electronic J. Graph Theory and Appl., in press (accepted on 19 November 2012).