Coefficient Estimates for Bi-univalent Functions Defined By (P, Q) Analogue of the Salagean Differential Operator Related to the Chebyshev Polynomials

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Abstract. In the present investigation, we use the Jackson (p,q)-differential operator to introduce the extended Salagean operator denoted by $R_{p,q}^k$. Certain bi-univalent function classes based on operator $R_{p,q}^k$ related to the Chebyshev polynomials are introduced. First two coefficient bounds and Fekete-Szego inequalities for the function classes are established. A number of corollaries are developed by varying parameters involved.

Keywords: analytic function; bi-univalent function; Chebyshev polynomial; Fekete-Szego inequalities; (p,q)-differential operator; Salagean operator.

1 Introduction

The q-calculus has great applications in the space of geometric functions theory because of their usefulness in the area of ordinary fractional calculus and optimal control problems. Jackson (see [1,2]) developed the concept of q-integral and q-derivative and much later its geometrical interpretation was identified through studies of quantum groups. This has attracted the attention of several researchers. Researchers all over the globe have applied it to construct and investigate several classes of analytic and bi-univalent functions. For recent expository work on so called post-quantum calculus or (p,q) calculus, see [3,4]. We here recall the definition of fractional q-calculus operators of complex valued function $f(z)$.

Definition 1.1. (see [3]) The (p,q)-derivative of $f$ is defined as:

$$ (D_{p,q}f)(z) = \frac{f(pz) - f(qz)}{(p-q)z} \left( z \neq 0 \right) $$

provided that $f$ is differentiable at 0. Now $D_{p,q}z^n = [n]_{p,q}z^{n-1}$, where
\[ [n]_{p,q} = \frac{p^n - q^n}{p-q} (0 < q < p \leq 1) \]  \hfill (2)

refers to a twin-basic number. For \( p=1 \), the Jackson \((p,q)\)-derivative reduces to the Jackson q-derivative given by:

\[ (D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0). \]

The class of all analytic functions \( f \) normalized by \( f(0) = f'(0) - 1 = 0 \) is given by:

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U) \]  \hfill (3)

where \( U := \{ z \in C : |z| < 1 \} \) represents the open unit disk. We denote such class by \( A \). Let \( S \) represent the class of all analytic univalent functions of the form (3) in \( U \). Let \( f, g \in A \). Then \( f \) is subordinate to \( g \), written as \( f \prec g \), if there is an analytic function \( w \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \) \( (z \in U) \) (see [5, 6]). “The Koebe One-Quarter-Theorem asserts that the image of \( U \) under every function \( f \in S \) contains a disk of radius \( \frac{1}{4} \). Therefore, the inverse of \( f \in S \) is a univalent analytic function on the disk \( U_\rho = \{ z : z \in C \text{ and } |z| < \rho, \rho \geq \frac{1}{4} \} \)”, see [7]. For each \( f \in S \), \( f(z) = w \) has an inverse function \( f^{-1}(w) \) of \( f(z) \) defined as:

\[ g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \]  \hfill (4)

If both \( f, f^{-1} \in S \) then \( f \) is said to be bi-univalent in \( U \). The class of all functions \( f \) given by (3) is denoted by \( S \). For a detailed history and other related properties of functions in the class \( S \), see recent works in [8-13].

For a function \( f \) given by (3), a simple calculation shows that

\[ \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1} = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}. \]  \hfill (5)

The \((p,q)\)-analogue of Salagean differential operator \( R^k_{p,q} : A \to A (k \in N_0 = N \cup \{0\}) \) is defined by:

\[ R^0_{p,q} f(z) = f(z) \]
\[ R^1_{p,q} f(z) = z(D_{p,q} f(z)), \]
\[ \ldots \]
\[ R^k_{p,q} f(z) = R^1_{p,q} (R^{k-1}_{p,q} f(z)) \]  \hfill (6)

Thus, for a function \( f(z) \) of the form (3), we have:

\[ R^k_{p,q} f(z) = z + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^n. \]  \hfill (7)
Similarly, for a function \( g \) of the form (4), we have:
\[
R_{p,q}^k g(w) = w - [2]_{p,q}^k a_2 w^2 + (2a_2^2 - a_3)[3]_{p,q}^k w^3 - (5a_2^3 - 5a_3 a_2 + a_4)[4]_{p,q}^k w^4 + \cdots \quad (8)
\]
From above, we observe that:
\[
\lim_{p\to 1, q\to 1-} R_{p,q}^k f(z) = z + \lim_{p\to 1, q\to 1-} \sum_{n=2}^{\infty} [n]_{p,q}^k a_n z^n = z + \sum_{n=2}^{\infty} n^k a_n z^n = D^k f(z), \quad (9)
\]
where \( D^k \) is the Salagean differential operator which was defined in [14] and has been studied by several authors.

Chebyshev polynomials of the first and second kind and their properties have been studied by several researchers (see, for details [15,16]). We consider
\[
L(z, t) = \frac{1}{1-2t z + t^2} (z \in U)
\]
as its generating function. Taking \( t = \cos \alpha, \alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \), we have:
\[
L(z, t) = \frac{1}{1-2 \cos \alpha z + z^2} = 1 + 2 \cos \alpha z + (3 \cos^2 \alpha - \sin^2 \alpha) z^2 + \cdots
\]
\[
= 1 + U_1(t) z + U_2(t) z^2 + \cdots (z \in U, t \in (-1,1)), \quad (10)
\]
where \( U_{n-1}(t) = \frac{\sin(n \cos^{-1} t)}{\sqrt{1-t^2}} \). Thus we have
\[
U_1(t) = 2t, \quad U_3(t) = 8t^3 - 4t, \quad U_2(t) = 4t^2 - 1, \quad U_4(t) = 16t^4 - 12t^2 + 1, \quad \cdots \quad (11)
\]
Recently, several researchers, Altinkaya and Yalcın [17-19], Bulut et al. [20,21] Guney et al. [22] and Caglar [23] (also see [24]) to mention a few, have obtained Fekete-Szegö inequalities and some coefficient bounds for different subclasses of bi-univalent functions. Motivated by the above researchers, we consider two subclasses of bi-univalent functions that are obtained by using the \( D_{p,q} \) operator of the Salagean type associated with the Chebyshev polynomial.

**Definition 1.2.** A function \( f \in \Sigma \) defined as (3) belongs to the function class \( R_{\Sigma, p,q}^k(y, t)(0 \leq y \leq 1) \) if the conditions
\[
(1 - y) \frac{h_{q+1}^{k+1}/f(x)}{h_{p,q}^{k+1}/f(x)} + y \frac{h_{q+2}^{k+2}/f(x)}{h_{p,q}^{k+1}/f(x)} < L(z, t) \left( \frac{1}{2} < t < 1; z \in U \right), \quad (12)
\]
and
\[(1 - \gamma) \frac{R^k_{p,q} g(w)}{R^k_{p,q} g(w)} + \gamma \frac{R^k_{p,q} g(w)}{R^k_{p,q} g(w)} < L(w, t) \left( \frac{1}{2} < t < 1; w \in U \right), \quad (13)\]

are satisfied, where \( g \) is stated in (4).

By specializing the parameters \( \gamma, p, q \) and \( k \) in the above definition, we obtain the various subclasses of \( \Sigma \).

**Definition 1.3.** A function \( f \in \Sigma \) belongs to the function class \( T^k_{\Sigma, p,q}(\beta, t) \) if

\[(1 - \beta) \frac{R^k_{p,q} f(z)}{z} + \beta (R^k_{p,q} f(z))' \prec L(z, t), \quad (14)\]

and

\[(1 - \beta) \frac{R^k_{p,q} g(w)}{w} + \beta (R^k_{p,q} g(w))' \prec L(w, t) \quad (15)\]

\( \left( 0 \leq \beta \leq 1, \frac{1}{2} < t < 1; z, w \in U \right) \), hold where \( R^k_{p,q} f (z) \) and \( R^k_{p,q} g (w) \) are given by (7) and (8) respectively.

**Remark 1.4.** For \( p \to 1, q \to 1^- \), we get the class \( T^k_{\Sigma, 1,1^-}(\beta, t) = R^k_{\Sigma}(\beta, L(z, t)) \) consists of function \( f \in \Sigma \) and satisfying

\[(1 - \beta) \frac{D^k f(z)}{z} + \beta (D^k f(z))' \prec L(z, t)\]

and

\[(1 - \beta) \frac{D^k g(w)}{w} + \beta (D^k g(w))' \prec L(w, t).\]

This class is due to Guney et al.[22].

**Remark 1.5.** For \( p \to 1, q \to 1^- \) and \( k=0 \), we obtain the class \( T^0_{\Sigma, 1,1^-}(\beta, t) = B_{\Sigma}(\beta, t) \) (see[20, 21]) where \( f \in \Sigma \) satisfying

\[(1 - \beta) \frac{f(z)}{z} + \beta (f (z))' \prec L(z, t)\]

and

\[(1 - \beta) \frac{g(w)}{w} + \beta (g (w))' \prec L(w, t).\]

In this work, we investigate the first two coefficient bounds and Fekete-Szego inequalities in the above newly constructed function classes by using the Chebyshev polynomial.
2 Coefficient Bounds

In the following theorems, we establish Chebyshev polynomial bounds \(|a_2|\) and \(|a_3|\) for the function classes \(R_k^{\Sigma p,q}(\gamma, t)\) and \(T_k^{\Sigma p,q}(\beta, t)\).

Theorem 2.1. Assume that \(f \in \Sigma\) defined as (3) is in the class \(R_k^{\Sigma p,q}(\gamma, t)\) for \(t < \frac{1}{2}\). Then

\[
|a_2| \leq \frac{2\sqrt{2t}}{\sqrt{|(A_1-A_2)4t^2+A_2|}} \tag{16}
\]

and

\[
|a_3| \leq \frac{4t^2}{[2]_{p,q}([2]_{p,q}-1)^2(1+y([2]_{p,q}-1))^2} \\
+ \frac{2t}{[3]_{p,q}([3]_{p,q}-1)(1+y([3]_{p,q}-1))} \tag{17}
\]

where

\[
A_1 = [3]_{p,q}^k(1+y([3]_{p,q}-1))([3]_{p,q}-1) - [2]_{p,q}^{2k}(1+y([2]_{p,q}-1))([2]_{p,q}-1), \tag{18}
\]

and

\[
A_2 = [2]_{p,q}^{2k}(1+y([2]_{p,q}-1))^2([2]_{p,q}-1)^2. \tag{19}
\]

Proof: Assume that \(f \in R_k^{\Sigma p,q}(\gamma, t)\). Definition 1.2 yields:

\[
(1 - \gamma) \frac{R_k^{\Sigma p+1,q}(x)}{R_k^{\Sigma p,q}(x)} + \gamma \frac{R_k^{\Sigma p+2,q}(x)}{R_k^{\Sigma p,q}(x)} = 1 + U_1(t)r(z) + U_2(t)r^2(z) + \cdots \tag{20}
\]

and

\[
(1 - \gamma) \frac{R_k^{\Sigma p+1,q}(w)}{R_k^{\Sigma p,q}(w)} + \gamma \frac{R_k^{\Sigma p+2,q}(w)}{R_k^{\Sigma p,q}(w)} = 1 + U_1(t)s(w) + U_2(t)s^2(w) + \cdots \tag{21}
\]

where \(r(z)\) and \(s(w)\) are analytic functions given by

\[
r(z) = c_1z + c_2z^2 + c_3z^3 + \cdots, \tag{22}
\]

\[
s(w) = d_1w + d_2w^2 + d_3w^3 + \cdots, \tag{23}
\]

where \(r(0) = s(0) = 0, |r(z)| < 1, |s(w)| < 1\) \((z, w \in U)\). If \(|r(z)| < 1\) and \(|s(w)| < 1\), then

\[
|c_i| \leq 1 \text{ and } |d_i| < 1 \text{ for all } i \in N. \tag{24}
\]

Making use of (22) in (20) and (23) in (21), we get
Using (7) and (8) that

\[
(1 - \gamma) \frac{R_{p,q}^{k+1} f(z)}{R_{p,q}^k f(z)} + \gamma \frac{R_{p,q}^{k+2} f(z)}{R_{p,q}^{k+1} f(z)} = 1 + U_1(t) c_1 z + [U_1(t) c_2 + U_2(t) c_1^2] z^2 + \cdots
\]  

(25)

and

\[
(1 - \gamma) \frac{R_{p,q}^{k+1} g(w)}{R_{p,q}^k g(w)} + \gamma \frac{R_{p,q}^{k+2} g(w)}{R_{p,q}^{k+1} g(w)} = 1 + U_1(t) d_1 w + [U_1(t) d_2 + U_2(t) d_1^2] w^2 + \cdots
\]  

(26)

It follows from (7) and (8) that

\[
(1 - \gamma) \frac{R_{p,q}^{k+1} f(z)}{R_{p,q}^k f(z)} + \gamma \frac{R_{p,q}^{k+2} f(z)}{R_{p,q}^{k+1} f(z)} = 1 + [2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) ([2]_{p,q} - 1) a_2 z + ([3]_{p,q} - 1)([3]_{p,q} - 1) a_3 z^2 + \cdots
\]  

(27)

and

\[
(1 - \gamma) \frac{R_{p,q}^{k+1} g(w)}{R_{p,q}^k g(w)} + \gamma \frac{R_{p,q}^{k+2} g(w)}{R_{p,q}^{k+1} g(w)} = 1 - [2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) ([2]_{p,q} - 1) a_2 w + [[2]_{p,q} - 1]([3]_{p,q} - 1)([3]_{p,q} - 1) a_3 w^2 + \cdots
\]  

(28)

Using (27) in (25) and (28) in (26), we obtain:

\[
1 + [2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) ([2]_{p,q} - 1) a_2 z + ([3]_{p,q} - 1)([3]_{p,q} - 1) a_3 z^2 + \cdots
\]  

(29)
\[ = 1 + U_1(t)d_1 + [U_1(t)d_2 + U_2(t)d_2^2] \omega^2 + \ldots \]  

Evaluating the coefficients in (29) and (30), we get:

\[ ([2]_{p,q} - 1)[2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) a_2 = U_1(t)c_1, \]

and

\[ -([2]_{p,q} - 1)[2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) a_2^2 + ([3]_{p,q} - 1)[3]_{p,q}^k (1 + \gamma([3]_{p,q} - 1)) a_3 = U_1(t)c_2 + U_2(t)c_2^2, \]

\[ -([2]_{p,q} - 1)[2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) a_2 = U_1(t)d_1, \]

and

\[ \left\{2([3]_{p,q} - 1)[3]_{p,q}^k (1 + \gamma([3]_{p,q} - 1)) - ([2]_{p,q} - 1)[2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1))\right\} a_2^2 - ([3]_{p,q} - 1)[3]_{p,q}^k (1 + \gamma([3]_{p,q} - 1)) a_3 = U_1(t)d_2 + U_2(t)d_2^2. \]

From (31) and (33), we obtain:

\[ c_1 = -d_1, \]

and

\[ 2([2]_{p,q} - 1)[2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1)) a_2^2 = U_1(t)(c_2^2 + d_2^2). \]

Adding (32) and (34) in the resulting equation, we obtain:

\[ \left[2([3]_{p,q} - 1)[3]_{p,q}^k (1 + \gamma([3]_{p,q} - 1)) - 2([2]_{p,q} - 1)[2]_{p,q}^k (1 + \gamma([2]_{p,q} - 1))\right] a_2^2 - \frac{U_1(t)c_2 + U_2(t)d_2}{\omega^2} \]

which gives:

\[ a_2^2 = \frac{(c_2 + d_2)U_1^2(t)}{2[A_1U_1^2(t) - A_2U_2(t)]}, \]

where \( A_1 \) and \( A_2 \) are given in (18) and (19) respectively. Applying (24) to the coefficients \( c_2 \) and \( d_2 \) and using (11) in (38), we get the desire estimate for \(|a_2|\).

Subtracting (34) from (32) and using (35) and (36) in the resulting equation yields:
\[ a_3 = \frac{(c_1^2 + d_1^2)u_2(t)}{2^{[2]_p,q}([2]_{p,q}^{-1})^2 + [2]^{[3]_p,q}([3]_{p,q}^{-1})^2 \left[ 1 + \gamma([2]_{p,q}^{-1}) \right]^2} + \frac{(c_2 - d_2)u_1(t)}{2^{[3]_p,q}([3]_{p,q}^{-1})^2 \left[ 1 + \gamma([3]_{p,q}^{-1}) \right]^2}. \]  

(39)

Taking the coefficient inequalities for \( c_1, c_2, d_1 \) and \( d_2 \) from (24) and making use of (11) in (39) we get the estimate for \( |a_3| \) as stated in (17). This proves the Theorem 2.1.

Letting \( p \to 1 \) and \( q \to 1^- \) in Theorem 2.1, we get the result for the class \( R_{\Sigma,1,1}^k(y, t) \equiv M_\Sigma^k(y, L(z, t)) \) due to Guney et al. [22] as follows:

**Corollary 2.2** (see [22]): Let \( f \in M_\Sigma^k(y, L(z, t)) \). Then

\[ |a_2| \leq \frac{2t\sqrt{27\tilde{t}}}{\sqrt{[\mathbb{N}([3]_{p,q}^{-1})][\mathbb{N}([3]_{p,q}^{-1})^{-1}] \left[ 1 + \gamma([\mathbb{N}([3]_{p,q}^{-1})])^2 \right]}} \]

and

\[ |a_3| \leq \frac{4t^2}{(1 + \gamma)^{z/2}} + \frac{t}{(1 + \gamma)^{z/2}}. \]

Letting \( \gamma = 0 \) in Theorem 2.1, the following result for the function class \( R_{\Sigma,p,q}^k(0, t) \equiv N_{\Sigma,p,q}^k(t) \) is obtained.

**Corollary 2.3.** If \( f \in N_{\Sigma,p,q}^k(t) \), then

\[ |a_2| \leq \frac{2t\sqrt{27\tilde{t}}}{\sqrt{[\mathbb{N}([3]_{p,q}^{-1})][\mathbb{N}([3]_{p,q}^{-1})^{-1}] \left[ 1 + \gamma([\mathbb{N}([3]_{p,q}^{-1})])^2 \right]}} \]

and

\[ |a_3| \leq \frac{4t^2}{[\mathbb{N}([3]_{p,q}^{-1})^2]. \]

Letting \( p \to 1 \) and \( q \to 1^- \) in the above corollary, we get the following result for the class \( R_{\Sigma,1,1}^k(0, t) \equiv N_{\Sigma}^k(t) \).

**Corollary 2.4.** Let \( f \in N_{\Sigma}^k(t) \). Then

\[ |a_2| \leq \frac{2t\sqrt{27\tilde{t}}}{\sqrt{[3^{k-2k}\tilde{t}^2 + 2^{2k}]}} \]

and

\[ |a_3| \leq \frac{4t^2}{2^{2k}}. \]

Putting \( \gamma = 1 \) in Theorem 2.1, the result for the class \( R_{\Sigma,p,q}^k(1, t) \equiv U_{\Sigma,p,q}^k(t) \) is as follows:

**Corollary 2.5.** Let \( f \in U_{\Sigma,p,q}^k(t) \). Then
\[ |a_2| \leq \frac{2t \sqrt{\pi}}{\sqrt{\left[ ((2)_{p,q}^{-1})(3)_{p,q}^{k+1} - (2)_{p,q} - 1\right)(2)_{p,q}^{2k+3} + 2(2)_{p,q}^{2k+2} + (3)_{p,q}^{k+1}}} \]

and

\[ |a_3| \leq \frac{4t^2}{2(2)_{p,q}^{2k+2} + (3)_{p,q}^{k+1}} + \frac{2t}{(3)_{p,q}^{k+1}}. \]

Taking \( p \to 1, q \to 1^- \) and \( \gamma = 1 \) in the above theorem, the result for the class \( R_{\Sigma,1,1}^k(L(z, t)) \) is obtained.

**Corollary 2.6** (see [22]): If \( f \in K_{\Sigma}^k(L(z, t)) \), then

\[ |a_2| \leq \frac{t \sqrt{\pi}}{\sqrt{[3]_{p,q}^{k+1} - 2(2)_{p,q}^{k+1} + 2(2)_{p,q}^{2k+2}}} \]

and

\[ |a_3| \leq \frac{t^2}{2 + \frac{t}{3k+1}}. \]

Theorem 2.1 for \( k=0 \) gives

**Corollary 2.7.** Let the function \( f \in V_{\Sigma,p,q}(y, t)(\equiv R_{\Sigma,p,q}^0(y, t)) \). Then

\[ |a_2| \leq \frac{2t \sqrt{\pi}}{\sqrt{4M_1 t^2 + M_2}} \]

and

\[ |a_3| \leq \frac{4t^2}{(1 + \gamma)(2)_{p,q}^{-1})(3)_{p,q}^{2k+3} + 2t}{(1 + \gamma)([2]_{p,q}^{-1} - 1)(3)_{p,q}^{2k+3} + 2t} \]

where

\[ M_1 = (1 + \gamma)([3]_{p,q} - 1)([3]_{p,q} - 1) - (2)_{p,q} - 1\right)(1 + \gamma([2]_{p,q} - 1) + ([2]_{p,q} - 1)(1 + \gamma([2]_{p,q} - 1))^2, \]

and

\[ M_2 = (1 + \gamma([2]_{p,q} - 1))^2 ([2]_{p,q} - 1)^2. \]

Letting \( p \to 1 \) and \( q \to 1^- \) in the above result, we get

**Corollary 2.8.** Let \( f \in V_{\Sigma}(y, t)(\equiv R_{\Sigma,1,1}^0(y, t)) \). Then

\[ |a_2| \leq \frac{2t \sqrt{\pi}}{\sqrt{(1 + \gamma)^2 - (\gamma + \gamma^2)t^2}} \]

and
\[ |a_3| \leq \frac{4t^2}{(1+\gamma)^2} + \frac{t}{1+2\gamma} \]

Putting \( \gamma = 0 \) in the above corollary gives the following.

**Corollary 2.9.** Let \( f \in V_\Sigma(t)(\equiv V_\Sigma(0, t)) \). Then
\[ |a_2| \leq 2t\sqrt{2t}, \]
and
\[ |a_3| \leq t + 4t^2. \]

Putting \( \gamma = 1 \) in Corollary 2.8 gives:

**Corollary 2.10.** Let \( f \in Q_\Sigma(t)(\equiv V_\Sigma(1, t)) \). For \( t \neq \frac{1}{\sqrt{2}} \), we have
\[ |a_2| \leq \frac{\sqrt{2t}}{\sqrt{|1-2t^2|}}, \]
and
\[ |a_3| \leq t^2 + \frac{t}{3}. \]

**Theorem 2.11.** Let \( f \in T^k_{\Sigma,p,q}(\beta, t) \). Then
\[ |a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{[3]_{p,q}^k(1+2\beta) - [2]_{p,q}^k(1+\beta)^2 |4t^2 + [2]_{p,q}^k(1+\beta)^2|}}, \] (40)
and
\[ |a_3| \leq \frac{2t}{(1+2\beta)[3]_{p,q}^k} + \frac{4t^2}{(1+\beta)^2[2]_{p,q}^k}. \] (41)

**Proof:** Let \( f \in T^k_{\Sigma,p,q}(\beta, t) \). Proceeding as before, we have
\[ (1 + \beta)[2]_{p,q}^k a_2 = U_1(t)c_1, \] (42)
\[ (1 + 2\beta)[3]_{p,q}^k a_3 = U_1(t)c_2 + U_2(t)c_1^2 \] (43)
and
\[ -(1 + \beta)[2]_{p,q}^k a_2 = U_1(t)d_1, \] (44)
\[ (1 + 2\beta)(2a_2^2 - a_3)[3]_{p,q}^k = U_1(t)d_2 + U_2(t)d_1^2. \] (45)

It follows from (42) and (44) that
\[ c_1 = -d_1, \] (46)
\[ 2(1 + \beta)^2[2]_{p,q}^k a_2^2 = U_1^2(t)(c_1^2 + d_1^2). \] (47)
Similarly, from (43) and (45) we have:

\[ 2(1 + 2\beta)[3^k]^{p,q}a_2^2 = U_1(t)(c_2 + d_2) + U_2(t)(c_1^2 + d_1^2). \]  

(48)

Using (47) in (48) and simplifying we get:

\[ a_2^2 = \frac{(c_2 + d_2)U_1^2(t)}{2[(1+2\beta)[3^k]^{p,q}U_2^2(t) - (1+\beta)2[2^k]^{p,q}U_2(t)]}. \]  

(49)

Putting the values of \( U_1(t), U_2(t) \) from (11) and using (24) in (49) we get the desire estimate for \(|a_2|\) as given by (40).

Subtracting (45) from (43) and making use of (46), (47) in the resulting equation and simplifying, we get:

\[ a_3 = \frac{(c_2 - d_2)U_1(t)}{2[1+2\beta][3^k]^{p,q}} + \frac{(c_2^2 - d_2^2)U_1^2(t)}{2(1+2\beta)^2[3^k]^{p,q}} \]  

(50)

Using (11) and (24) in (50) we get the bounds for \(|a_3|\). The proof of Theorem 2.11 is completed.

Taking \( q \to 1^- \), \( p \to 1 \) in Theorem 2.11, the result for the class \( T_{\Sigma,1,1}^k(\beta, t) \equiv F_{\Sigma}^k(\beta, L(z, t)) \) is obtained.

**Corollary 2.12.** Let \( f \in \Sigma \) given by (3) be in the class \( F_{\Sigma}^k(\beta, L(z, t)) \). Then

\[ |a_2| \leq \frac{2t\sqrt{2\pi}}{\sqrt{[(1+2\beta)3^k-(1+\beta)^22^{2k}]4t^2+(1+\beta)^22^{2k}}} \]

and

\[ |a_3| \leq \frac{2t}{(1+2\beta)3^k} + \frac{4t^2}{(1+\beta)^22^{2k}}. \]

Putting \( \beta = 0 \) in Corollary 2.12 we get the result for the function class \( T_{\Sigma,1,1}^k(0, t) \equiv F_{\Sigma}^k(L(z, t)) \) as follows:

**Corollary 2.13** (see [22]): Let \( f \in F_{\Sigma}^k(L(z, t)) \). Then

\[ |a_2| \leq \frac{2t\sqrt{2\pi}}{\sqrt{[3^k-2^{2k}]4t^2+2^{2k}}} \]

and

\[ |a_3| \leq \frac{2t}{3^k} + \frac{4t^2}{2^{2k}}. \]

Corollary 2.13 for \( \beta = 1 \) yields the result for the class \( T_{\Sigma,1,1}^k(1, t) \equiv H_{\Sigma}^k(L(z, t)) \) as below.
Corollary 2.14. Let \( f \in H^k_{\Sigma}(L(z,t)) \). Then
\[
|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|3k+1-2t(2k+1)|}}
\]
and
\[
|a_3| \leq \frac{2t}{3k+1} + \frac{t^2}{2t}
\]
Corollary 2.12 for \( k = 0 \) gives the result for the class \( T^0_{\Sigma,1,1}(\beta, t) \equiv F^0_{\Sigma}(\beta, L(z,t)) \) as below.

Corollary 2.15. Let \( f \in F^k_{\Sigma}(\beta, L(z,t)) \). Then
\[
|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(\beta+1)^2-4\beta^2t^2|}}
\]
and
\[
|a_3| \leq \frac{2t}{1+2\beta} + \frac{4t^2}{(1+\beta)^2}
\]
Putting \( \beta = 0 \) in Corollary 2.15 gives the result for the function class \( T^0_{\Sigma,1,1}(0, t) \equiv T_{\Sigma}(t) \).

Corollary 2.16. Let \( f \in T_{\Sigma}(t) \). Then
\[
|a_2| \leq 2t\sqrt{2t}
\]
and
\[
|a_3| \leq 2t + 4t^2.
\]
Letting \( \beta = 1 \) in Corollary 2.15, we get the result for the class \( F^{1}_{\Sigma}(1, L(z,t)) \equiv F_{\Sigma}(L(z,t)) \).

Corollary 2.17. Let \( f \in F_{\Sigma}(L(z,t)) \). Then
\[
|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{1-t^2}}
\]
and
\[
|a_3| \leq t^2 + \frac{2}{3}t.
\]

3 Fekete-Szego Inequalities
In the following section, we obtain the Fekete-Szego problems for the function class \( R^k_{\Sigma, p, q}(y, t) \) and \( T^k_{\Sigma, p, q}(\beta, t) \) as follows:
Theorem 3.1. Let \( f \in R^5_{\Sigma,p,q}(y,t) \). Then

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
2t & |\eta - 1| \leq \frac{A_3 + M_3}{M_5} \\
\frac{B_t^3|1-\eta|}{|M_5|4t^2+A_2} & |\eta - 1| \geq \frac{A_3 + M_3}{M_5},
\end{cases}
\]

(51)

where

\[
M_3 = M_5 - M_4 - A_2, \\
M_4 = ([2]_{p,q} - 1)[2]_{p,q} (1 + \gamma([2]_{p,q} - 1)) \\
M_5 = ([3]_{p,q} - 1)[3]_{p,q} (1 + \gamma([3]_{p,q} - 1))
\]

and \( A_2 \) is defined in (19).

Proof: It follows from (32) and (34) that

\[
a_3 - \eta a_2^2 = (1 - \eta) \frac{(c_2 + d_2)U_1(t)}{2([M_5 - M_4]U_1(t) - A_2U_2(t))} + \frac{(c_2 - d_2)U_1(t)}{2M_5} U_2(t) \left\{ g(\eta) + \frac{1}{2M_5} c_2 + (g(\eta) - \frac{1}{2M_5}) d_2 \right\},
\]

(55)

where

\[
g(\eta) = \frac{(1 - \eta)U_1(t)}{2([M_5 - M_4]U_1(t) - A_2U_2(t))}
\]

Taking the values of \( U_1(t) \) and \( U_2(t) \) from (11) and substituting it in (56) we conclude that

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{2t}{M_5} & 0 \leq |g(\eta)| \leq \frac{1}{2M_5} \\
\frac{4t|g(\eta)|}{M_5} & |g(\eta)| \geq \frac{1}{2M_5}.
\end{cases}
\]

(57)

The estimate (51) follows from (57). The proof of Theorem 3.1 is thus completed.

Taking \( p \to 1 \) and \( q \to 1^- \) in Theorem 3.1 yields:

Corollary 3.2. Let \( f \in M^5_{\Sigma}(y,L(z,t))(\equiv R^5_{\Sigma,1,1}(y,t)) \). Then

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{1}{3^k(1+2\gamma)}, |\eta - 1| \leq \frac{[1+(1+\gamma)^22^{2k}]4t^2(1+2\gamma)3^{2k}-(y^2+5y+2)^22^{2k}}{2(1+2\gamma)3^{2k}} \\
\frac{8|1-\eta|}{3^k} & |\eta - 1| \leq \frac{[2(1+2\gamma)3^{2k}-(y^2+5y+2)^22^{2k}]4t^2(1+\gamma)^22^{2k}}{2(1+2\gamma)3^{2k}} \\
2t & |\eta - 1| \geq \frac{[1+(1+\gamma)^22^{2k}]4t^2(1+2\gamma)3^{2k}-(y^2+5y+2)^22^{2k}}{2(1+2\gamma)3^{2k}}
\end{cases}
\]
Theorem 3.1 for \( \eta = 1 \) gives the following:

**Corollary 3.3.** Let \( f \in R^k_{\sum_p,q}(y, t) \). We have
\[
|a_3 - a_2^2| \leq \frac{2t}{M_3}.
\]

Letting \( \eta = 1 \) in Corollary 3.2 we have:

**Corollary 3.4** (see [22]): Let \( f \in M_3^k(\gamma, L(z, t)) \). We have
\[
|a_3 - a_2^2| \leq \frac{t}{(1 + 2\gamma)^3}. 
\]

Theorem 3.1 for \( \gamma = 0 \) gives

**Corollary 3.5.** Let \( f \in N^k_{\sum_p,q}(t) \equiv R^k_{\sum_p,q}(0, t) \). Then
\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{2t}{[3]_{p,q}([3]_{p,q}-1)}|\eta - 1| \leq \frac{[2]_{p,q}([2]_{p,q}-1)^2}{4t^2} + \frac{[3]_{p,q}([3]_{p,q}-1)-[2]_{p,q}^2([2]_{p,q}-1)}{[3]_{p,q}([3]_{p,q}-1)} \quad & \mbox{if } \eta > 1 \\
|\eta - 1| \geq \frac{[2]_{p,q}([2]_{p,q}-1)^2}{4t^2} - \frac{[3]_{p,q}([3]_{p,q}-1)-[2]_{p,q}^2([2]_{p,q}-1)}{[3]_{p,q}([3]_{p,q}-1)} \quad & \mbox{if } \eta < 1
\end{cases}
\]

Taking \( p \to 1 \) and \( q \to 1^- \) in Corollary 3.5, the result for the class \( N^k_{\sum}(t) \equiv N^k_{\sum,1^-}(t) \) is obtained.

**Corollary 3.6.** Let \( f \in N^k_{\sum}(t) \). Then for any real number \( \eta \),
\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{t}{3^k} \quad & \mbox{if } |\eta - 1| \leq \frac{2^{2k}3^{k-2}2^{2k}}{3^k} \\
\frac{8t^3|1-\eta|}{|[3^{k-2}2^k]_{p,q}2^k+2^{2k}|} \quad & \mbox{if } |\eta - 1| \geq \frac{2^{2k}3^{k-2}2^{2k}}{3^k}
\end{cases}
\]

Taking \( \eta = 1 \) and \( k = 0 \) in Corollary 3.6 we get the estimate for the class \( N^0_{\sum}(t) \).

**Corollary 3.7.** Let \( f \in \sum \) given by (3) be in the class \( N^0_{\sum}(t) \). Then
\[
|a_3 - a_2^2| \leq t.
\]

**Theorem 3.8.** Let \( f \in \sum \) given by (3) be in the class \( T^k_{\sum, p,q}(\beta, t) \). Then for any \( \eta \in R \), we have:
Coefficient Estimates for Bi-Univalent Functions

\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{2t}{(1+2\beta)[3]_{p,q}^k}, & |\eta - 1| \leq \frac{(1+2\beta)[3]_{p,q}^k-(1+\beta)^2[2]_{p,q}^k}{(1+2\beta)[3]_{p,q}^k} + \frac{(1+\beta)^2[2]_{p,q}^k}{4t^2} \frac{1}{\xi} \\
\frac{8|1-\eta|t^3}{|((1+2\beta)[3]_{p,q}^k-(1+\beta)^2[2]_{p,q}^k)4t^2+(1+\beta)^2[2]_{p,q}^k|}, & |\eta - 1| \geq \frac{(1+2\beta)[3]_{p,q}^k-(1+\beta)^2[2]_{p,q}^k}{(1+2\beta)[3]_{p,q}^k} + \frac{(1+\beta)^2[2]_{p,q}^k}{4t^2} \frac{1}{\xi}. 
\end{cases} \]

**Proof:** From (43) and (45) we have:

\[ a_3 - \eta a_2^2 = (1 - \eta) \frac{u_2^2(t)(c_2+d_2)}{2[(1+2\beta)[3]_{p,q}^k u_2^2(t)-(1+\beta)^2[2]_{p,q}^k u_2(t)]} + \frac{u_3(t)(c_2-d_2)}{2(1+2\beta)[3]_{p,q}^k} \]

\[ = U_1(t) \left\{ \left[ s(\eta) + \frac{1}{2(1+2\beta)[3]_{p,q}^k} \right] c_2 + \left[ s(\eta) - \frac{1}{2(1+2\beta)[3]_{p,q}^k} \right] d_2 \right\}, \]

where

\[ s(\eta) = \frac{(1-\eta)u_2^2(t)}{2[(1+2\beta)[3]_{p,q}^k u_2^2(t)-(1+\beta)^2[2]_{p,q}^k u_2(t)]}. \]

In view of (11), we obtain:

\[ |a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{2t}{(1+2\beta)[3]_{p,q}^k}, & 0 \leq |s(\eta)| \leq \frac{1}{2(1+2\beta)[3]_{p,q}^k} \\
4t |s(\eta)|, & |s(\eta)| \geq \frac{1}{2(1+2\beta)[3]_{p,q}^k}. 
\end{cases} \]

The estimates of Theorem 3.8 follow from (60). This completes the proof.

**Remark 3.9.** Many corollaries will be generated by varying parameters involved in Theorem 3.8.

### 4 Conclusion

A good amount of literature is available for the first few coefficients and the Fekete-Szego problem for different subclasses of univalent and bi-univalent analytic functions by making use of the class of Caratheodory functions. In the present investigation, the authors have introduced newly constructed bi-univalent analytic function classes \( R^\beta_{\Sigma,p,q}(y, t) \) and \( T^\beta_{\Sigma,p,q}(\beta, t) \) associated with the Chebyshev polynomials by using the Salagean (p,q)-differential operator and obtained initial coefficients and Fekete-Szego problems for the above mentioned classes. The generalization of some of the previous results studied by various researchers was obtained. The sigmoid function and Faber polynomial can be used to derive similar results for the classes studied.
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