# LIMITING BEHAVIOR OF A SEQUENCE OF DENSITY RATIOS

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## ICHTISAR

*Misalkan* $X_1, X_2, \ldots$ *ialah barisan variabel random dan* $\mathbf{P} = \{ P_\theta \ \theta \in \Theta \}$ *famili distribusi dari barisan tersebut.* $\mathbf{A}_n$ *adalah lapangan* $\sigma$ *terketjil terhadap mana* $X_1, \ldots, X_n$ *terukur. Djika* $\theta_1$ *dan* $\theta_2$ *dari* $\Theta$, *kita tentukan* $R_n(\theta_1, \theta_2)$ *sebagai rasio fungsi kepadatan dari* $P_{\theta_2}$ *dan* $P_{\theta_1}$ *pada* $\mathbf{A}_n$.

*Maksud dari karangan ini menjelidiki sifat² limit dari barisan* $\{ R_n \}$ *terhadap setiap* $P_\theta$. *Hal ini mempunjai aplikasi dalam sequential analysis, dimana kita ingin mengetahui apakah sequential probability ratio test berhenti dengan berkemungkinan satu. Djika barisan* $R_n$ *konvergen ke 0 (atau* $\lim\inf R_n = 0$*) atau konvergen ke* $\infty$ *(atau* $\lim\sup R_n = \infty$*) hampir tentu (h.t.) terhadap* $P_\theta$, *maka untuk* $\theta$ *ini setiap sequential probability ratio test berhenti dengan kemungkinan satu. Konklusi jang sama dapat diambil untuk generalized sequential probability ratio test dengan batas pemberhentiannja jang bergantung dari n.*

*Djika* $X_i$ *saling bebas dan mempunjai distribusi jang identik, maka* $\ln R_n$ *dapat dituliskan dengan* $\sum_{i=1}^{n} Y_i$ *dimana barisan variabel random* $Y_i$ *saling bebas dan berdistribusi identik. Sehingga* $\sum_{i=1}^{n} Y_i$ *konvergen h.t. ke* $\infty$ *atau ke* $-\infty$ *tergantung dari* $E_\theta \{ Y_i \} > 0$ *atau* $< 0$. *Untuk* $\theta = \theta_o$ *dimana* $E_{\theta_o} \{ Y_i \} = 0$ *kita dapat* $\lim\inf R_n = 0$ *dan* $\lim\sup R_n = \infty$ *h.t.* $P_{\theta_o}$.

*Pada barisan dari variabel random* $\{ X_n \}$ *jang tidak saling bebas maupun tidak berdistribusi identik, seringkali timbul dalam tes hipotesa komposit dengan adanja parameter „nuisance". Sebagai tjontoh adalah sequential t — test atau jang djuga disebut WAGR test. Dalam hal mana kesimpulan kwalitatip sama dengan halnja dalam* $X_i$ *jang saling bebas dan berdistribusi identik.*

*Tjontoh jang terachir ini memberikan saran untuk problem jang lebih umum sbb.: Dengan Assumptions A dan B kita dapat menghasilkan sbb.: Djika* $\theta_1 < \theta_2$ *maka* $R_n$ *konvergen h.t. ke 0 djika* $\theta \leq \theta_1$ *dan ke* $\infty$ *djika* $\theta \geq \theta_2$. *Untuk* $\theta$ *antara* $\theta_1$ *dan* $\theta_2$, *ketjuali barangkali untuk satu* $\theta_o$, *maka lim inf adalah 0 atau lim sup adalah* $\infty$ *h.t. Sehingga sequential probability ratio test berhenti dengan berkemungkinan satu ketjuali barangkali untuk satu harga* $\theta$. *Apakah betul ada* $\theta$ *untuk mana* $R_n$ *mempunjai lim inf positip dan lim sup jang terhingga, belumlah ada tjontoh jang dapat dipertundjukkan.*

## ABSTRACT

*Let* $X_1, X_2, \ldots$ *be a sequence of random variables and* $\mathbf{P} = \{ P_\theta, \theta \in \Theta \}$ *be a familiy of distributions of the sequence. For each n,* $\mathbf{A}_n$ *is the* $\sigma$*-field generated by* $X_1, \ldots, X_n$. *If* $\theta_1, \theta_2 \in \Theta$, *we define* $R_n(\theta_1, \theta_2)$ *as the density ratio of* $P_{\theta_2}$ *and* $P_{\theta_1}$ *on* $\mathbf{A}_n$.

*The main purpose of the paper is to investigate the limiting behavior of the sequence* $R_n$ *with respect to any* $P_\theta$. *This has applications in sequential analysis, where it is desired to know whether a sequential probability ratio test terminates with probability one. If the sequence* $R_n$ *converges to 0 (or the lim inf is 0) or converges to* $\infty$ *(or the lim sup is* $\infty$*) a.e. with respect to* $P_\theta$, *then for this* $\theta$ *any sequential probability ratio test terminates with probability one. The same conclusion can be drawn in the case of a generalized sequential probability ratio test, under sone restrictions as to how the stopping bounds vary with n.*

*If the* $X_i$ *are independent and identically distributed, then we can write* $\ln R_n$ *as* $\sum_{i=1}^{n} Y_i$ *where the* $Y_i$ *are independent and identically distributed. We have then that* $\sum_{i=1}^{n} Y_i$ *converges to* $\infty$ *or to* $-\infty$ *a.e. according as* $E_\theta \{ Y_i \} > O$ *or* $< O$. *For any* $\theta$, *say* $\theta_o$, *for which* $E_{\theta_o} \{ Y_i \} = 0$ *we have lim inf* $R_n = 0$ *and lim sup* $R_n = \infty$ *a.e.* $P_{\theta_o}$.

*A sequence of non-independent nor identically distributed random variables* $\{ X_i \}$ *may arise in tests of composite hypotheses in the presence of nuisance parameters. An example of the situation is the sequential t-test, by some authors called the WAGR test. In this example we have the same qualitative result as if the* $X_i$ *are independent and identically distributed.*

*The foregoing example suggested the more general problem with the Assumptions A and B (see Chapters 2 and 3). The result can be described as follows: If* $\theta_1 < \theta_2$ *then* $R_n$ *converges a.e. to 0 if* $\theta \leq \theta_1$ *and to* $\infty$ *if* $\theta \geq \theta_2$. *For* $\theta$ *between* $\theta_1$ *and* $\theta_2$, *except perhaps for one* $\theta_o$, *then lim inf is 0 or lim sup is* $\infty$ *a.e. So that a sequential probability ratio test terminates with probability one, except perhaps for one value of* $\theta$. *There is no example known to show that there may exist a* $\theta_o$ *for which the sequence of density ratios has a positive lim inf and a finite lim sup.*

## 1. DEFINITIONS AND PRELIMINARY RESULTS.

Let $(\Omega, \mathbf{A}, \mathbf{P})$ be a probability space, where $\Omega$ is a space of points $\omega$, $\mathbf{A}$ is a $\sigma$-field of subsets of $\Omega$, and $\mathbf{P}$ is a family of probability measures on $\mathbf{A}$ indexed by $\theta$, which is a member of an indexed set $\Theta$ : $\mathbf{P} = \{P_\theta, \theta \in \theta\}$. $\Theta$ will sometimes be called a parameter. If a statement holds except possibly on a set of $P_\theta$ measure 0, we shall follow the statement by : a.e. $P_\theta$. If $\mathbf{A}_o$ is a sub $\sigma$-field of $\mathbf{A}$ we shall write $\mathbf{A}_o \subset \mathbf{A}$ and, for short, call $\mathbf{A}_o$ a subfield of $\mathbf{A}$. If $\mathbf{A}_o \subset \mathbf{A}$ and $u$ some probability measure on $\mathbf{A}_o$ that dominates $P_\theta$, for some $\theta \in \Theta$, we define the density of $P_\theta$ on $\mathbf{A}_o$ with respect to $u$, written $p_\theta^{\mathbf{A}_o}$, as a non-negative $\mathbf{A}_o - u -$ integrable function such that for any set $A \in \mathbf{A}_o$.

$$P_\theta (A) = \int_A p_\theta^{\mathbf{A}_o} \, du \tag{1.1}$$

Note that if u is a probability measure on $\mathbf{A}$ dominating $P_\theta$ on $\mathbf{A}$, it is also a probability measure on $\mathbf{A}_o$ dominating $P_\theta$ on $\mathbf{A}_o$. The converse is not true, i.e. a probability measure on u on $\mathbf{A}$ may dominate $P_\theta$ on $\mathbf{A}_o$ without dominating it on $\mathbf{A}$. If $\mathbf{A}_o \subset \mathbf{A}$ and $f$ is an $\mathbf{A} - P_\theta -$ integrable function, shall denote the conditional expectation of $f$ given $\mathbf{A}_o$ with respect to $P_\theta$ by $E_\theta \{f \mid \mathbf{A}_o\}$. Sometimes the conditional expectation will be taken with respect to some probability measure u that is not necessarily a member of $\mathbf{P}$, and will then be written $E_u \{f \mid \mathbf{A}_o\}$.

**Definition** 1.1: *Let* $\{\mathbf{B}_n, n \geqslant 1\}$ *be a nondecreasing sequence of subfields of* $\mathbf{A}$, *and let* $\{f_n, n \geqslant 1\}$ *be a sequence of functions on* $\Omega$ *such that for some* $\theta \in \Theta$ *and every* $n$, $f_n$ *is* $\mathbf{B}_n - P_\theta -$ *integrable. The stochastic process* $\{f_n, \mathbf{B}_n, n \geqslant 1\}$ *will be called*:

(i) *a martingale with respect to* $P_\theta$ *if for every* $n$,
$E_\theta \{f_{n+1} \mid \mathbf{B}_n\} = f_n$ a.e. $P_\theta$

(ii) *an upper martingale with respect to* $P_\theta$ *if for every* $n$,
$E_\theta \{f_{n+1} \mid \mathbf{B}_n\} \geqslant f_n$ a.e. $P_\theta$

(iii) *a lower martingale with respect to* $P_\theta$ *if for every* $n$,
$E_\theta \{f_{n+1} \mid \mathbf{B}_n\} \leqslant f_n$ a.e. $P_\theta$

Let $X_1, X_2, \ldots.$ be a sequence of random variables on $\Omega$. Denote by $\mathbf{A}_n$ the subfield generated by $X_1, \ldots, X_n$, $n = 1, 2, \ldots.$ written $\mathbf{A}_n = \mathbf{B}(X_1, \ldots, X_n)$, and $\mathbf{A}_\infty$ as the smallest subfield of $\mathbf{A}$ containing $\mathbf{A}_n$, $n = 1, 2, \ldots\ldots$

Let $\mathbf{A}_o \subset \mathbf{A}^* \subset \mathbf{A}$ and let u be a probability measure on $\mathbf{A}$ that dominates $P_\theta$ on $\mathbf{A}$, then we have the following relation:

$$p_\theta^{\mathbf{A}_o} = E_u \left\{ p_\theta^{\mathbf{A}^*} \mid \mathbf{A}_o \right\} \qquad \text{a.e. } u \tag{1.2}$$

Indeed, if $A \in \mathbf{A}_o$, and therefore $A \in \mathbf{A}^*$, we have

$$\int_A p_\theta^{\mathbf{A}_o} \, du = \int_A p_\theta^{\mathbf{A}^*} \, du \tag{1.3}$$

because both sides are equal to $P_\theta(A)$. By taking in particular $\mathbf{A}^* = \mathbf{A}$ we have:

$$p_\theta^{\mathbf{A}_o} = E_u \left\{ p_\theta^{\mathbf{A}} \mid \mathbf{A}_o \right\} \qquad \text{a.e. u} \tag{1.4}$$

Consider the stochastic process $\left\{ p_\theta^{\mathbf{A}_n}, \mathbf{A}_n, n \geqslant 1 \right\}$ where $p_\theta^{\mathbf{A}_n}$ is the density of $P_\theta$ on $\mathbf{A}_n$ with respect to $u$. By (1.2) it is a martingale and $p_\theta^{\mathbf{A}_\infty}$ is its last element. Since for every $n$, $E_u \left\{ p_\theta^{\mathbf{A}_n} \right\} = 1$ and $E_u \left\{ p_\theta^{\mathbf{A}_\infty} \right\} = 1$, by a well known martingale convergence theorem [2], Chap. VII, we have $\lim_{n \to \infty} p_\theta^{\mathbf{A}_n} = p_\theta^{\mathbf{A}_\infty}$ a.e. $u$

Let $\theta_1, \theta_2 \in \Theta$. There is always a probability measure u that dominates both $P_{\theta_1}$ and $P_{\theta_2}$ on $\mathbf{A}$, and therefore on $\mathbf{A}_n$ for every $n$, including $n = \infty$. For instance, we can take $u = \frac{1}{2}(P_{\theta_1} + P_{\theta_2})$. For any choice of $u$, let $p_{\theta_1}^{\mathbf{A}_n}$ be the density of $P_{\theta_1}$ on $\mathbf{A}_n$, $i = 1, 2$. The ratio

$$R_n(\theta_1, \theta_2) = p_{\theta_2}^{\mathbf{A}_n} / p_{\theta_1}^{\mathbf{A}_n}, \; 1 \leqslant n \leqslant \infty \tag{1.5}$$

will be called the density ratio of $P_{\theta_2}$ and $P_{\theta_1}$ on $\mathbf{A}_n$. It is defined only up to a set of $P_{\theta_2}$ and $P_{\theta_1}$ measure 0. We shall sometimes suppress the dependence of $R_n$ on $\theta_1$ and $\theta_2$. It is easy to see that $R_n$ does not depend on the particular choice of $u$. We may even let $u$ depends on $n$.

**Lemma** 1.1 : *Let $\mathbf{A}_o \subset \mathbf{A}$ and u be a probability measure on $\mathbf{A}$. Let f be a non-negative $\mathbf{A} - u -$ integrable function and define $f_o = E_u\{f \mid \mathbf{A}_o\}$. Then $\{f > 0\} \subset \{f_o > 0\}$* a.e. $u$

**Proof:** Let $N_o = \{f_o = 0\}$. Since $f_o$ is defined as an a.e. $u$ $\mathbf{A}_o$ — measurable function, there is $N_o^* \in \mathbf{A}_o$ such that $u\{N_o \triangle N_o^*\} = 0$, where $\triangle$ denotes the symmetric difference, i.e. $N_o \triangle N_o^* = (N_o - N_o^*) \cap (N_o^* - N_o)$. Let $I_A$ denote the indicator of a set $A \subset \Omega$, i.e. it is a function which has a value 1 on $A$ and 0 otherwise. We compute:

$$\begin{aligned} E_u\{f I_{N_o} \mid \mathbf{A}_o\} &= E_u\{f I_{N_o^*} \mid \mathbf{A}_o\} && \text{a.e. } u \\ &= f_o I_{N_o^*} && \text{a.e. } u \\ &= f_o I_{N_o} && \text{a.e. } u \\ &= 0 && \text{a.e. } u \end{aligned}$$

Since $f I_{N_o} \geqslant 0$ we must have $f I_{N_o} = 0$ a.e. $u$, which means that except for a set of $u$ — measure zero $f_o(\omega) = 0$ implies $f(\omega) = 0$ or $\{f_o = 0\} \subset \{f = 0\}$ a.e. $u$.

**Theorem** 1.1: *Let $\theta_1$ and $\theta_2$ be in $\Theta$, and let u be a probability measure on $\mathbf{A}_\infty$ that dominates $P_{\theta_1}$ and $P_{\theta_2}$ on $\mathbf{A}_\infty$. The stochastic process $\{R_n(\theta_1, \theta_2), \mathbf{A}_n, n \geqslant 1\}$ is*

(i) *a lower martingale with respect to $P_{\theta_1}$.*

*If, for every n, $P_{\theta_2}$ is absolutely continuous with respect to $P_{\theta_1}$ on $\mathbf{A}_n$ then it is*

(ii) *a martingale with respect to $P_{\theta_1}$.*

(iii) *an upper martingale with respect to $P_{\theta_2}$.*

*Furthermore,* $\lim_{n \to \infty} R_n(\theta_1, \theta_2) = R_\infty(\theta_1, \theta_2)$ a.e. $P_{\theta_1}$ *and* a.e. $P_{\theta_2}$.

**Proof:** To show (i), let $A \in \mathbf{A}_{n-1}$ and denote $B_n = A \cap \{ p_{\theta_1}^{\mathbf{A}_n} > 0 \}$.

By Lemma 1.1 we have $\{ p_{\theta_1}^{\mathbf{A}_n} > 0 \} \subset \{ p_{\theta_1}^{\mathbf{A}_{n-1}} > 0 \}$ a.e. $u$, so that $B_n \subset B_{n-1}$ a.e. $u$. Since $P_{\theta_2}(B_n) = \int_A R_n(\theta_1,\theta_2)\, dP_{\theta_1}$ and $P_{\theta_2}(B_{n-1}) = \int_A R_{n-1}(\theta_1,\theta_2)\, dP_{\theta_1}$ we have $\int_A R_n(\theta_1,\theta_2)\, dP_{\theta_1} \leqslant \int_A R_{n-1}(\theta_1,\theta_2)\, dP_{\theta_1}$ which means:

$$E_{\theta_1}\{R_n(\theta_1,\theta_2) \mid \mathbf{A}_{n-1}\} \leqslant R_{n-1}(\theta_1,\theta_2) \qquad \text{a.e. } P_{\theta_1}$$

To show (ii), if $P_{\theta_2}$ is absolutely continuous with respect to $P_{\theta_1}$ on $\mathbf{A}_n$, we have: $\int_A R_n(\theta_1,\theta_2)\, dP_{\theta_1} = P_{\theta_2}(A) = \int_A R_{n-1}(\theta_1,\theta_2)\, dP_{\theta_1}$ if $A \in \mathbf{A}_{n-1}$ so that

$$E_{\theta_1}\{R_n(\theta_1,\theta_2) \mid \mathbf{A}_{n-1}\} = R_{n-1}(\theta_1,\theta_2) \qquad \text{a.e. } P_{\theta_1}$$

To show (iii), apply Jensen's inequality applied to the convex function $1/x$ for $x > 0$, we have

$$\begin{aligned} E_{\theta_2}\{R_n(\theta_1,\theta_2) \mid \mathbf{A}_{n-1}\} &\geqslant 1/E_{\theta_2}\{R_n(\theta_2,\theta_1) \mid \mathbf{A}_{n-1}\} && \text{a.e. } P_{\theta_2} \\ &= 1/R_{n-1}(\theta_2,\theta_1) && \text{a.e. } P_{\theta_2} \\ &= R_{n-1}(\theta_1,\theta_2) \end{aligned}$$

To show the assertion about the limit, let $M = \{ p_{\theta_1}^{\mathbf{A}_\infty} > 0 \}$,

$$\begin{aligned} \lim_{n\to\infty} R_n(\theta_1,\theta_2) &= \lim_{n\to\infty} p_{\theta_2}^{\mathbf{A}_n} I_M / p_{\theta_1}^{\mathbf{A}_n} && \text{a.e. } P_{\theta_1} \\ &= \lim_{n\to\infty} p_{\theta_2}^{\mathbf{A}_n} \lim_{n\to\infty} I_M / p_{\theta_1}^{\mathbf{A}_n} && \text{a.e. } P_{\theta_1} \\ &= p_{\theta_2}^{\mathbf{A}_\infty} I_M / p\theta_1^{\mathbf{A}_\infty} && \text{a.e. } P_{\theta_1} \\ &= R_\infty(\theta_1,\theta_2) && \text{a.e. } P_{\theta_1} \end{aligned}$$

We have then also $\lim_{n\to\infty} R_n(\theta_2,\theta_1) = R_\infty(\theta_2,\theta_1)$ a.e. $P_{\theta_1}$. By interchanging $\theta_1$ and $\theta_2$ we have the limit with respect to $P_{\theta_2}$.

**Definition** 1.1: *Two probability measures $P_{\theta_1}$ and $P_{\theta_2}$ are called orthogonal on $\mathbf{A}_\infty$, if there is a set $A \in \mathbf{A}_\infty$ such that $P_{\theta_1}(A) = 1$ and $P_{\theta_2}(A) = 0$.*

The following theorem is an immediate consequence of Theorem 1.1, since $R_\infty(\theta_1,\theta_2) = 0$ a.e. $P_{\theta_1}$ if and only if $P_{\theta_1}$ and $P_{\theta_2}$ are orthogonal on $\mathbf{A}_\infty$.

**Theorem** 1.2 : *The following three conditions are equivalent:*

(i) *$P_{\theta_1}$ and $P_{\theta_2}$ are orthogonal on $\mathbf{A}_\infty$.*

(ii) $\lim_{n\to\infty} R_n(\theta_1,\theta_2) = 0$ a.e. $P_{\theta_1}$

(iii) $\lim_{n\to\infty} R_n(\theta_1,\theta_2) = \infty$ a.e. $P_{\theta_2}$

In other words, the convergence $R_n \to 0$ a.e. $P_{\theta_1}$ and $R_n \to \infty$ a.e. $P_{\theta_2}$ happens if and only if the measures $P_{\theta_1}$ and $P_{\theta_2}$ are orthogonal on $\mathbf{A}_\infty$. We would like to find conditions under which $R_n \to 0$ or $R_n \to \infty$ a.e. $P_\theta$, for $\theta$ not necessarily equal to $\theta_1$ or $\theta_2$, and for any choice of $\theta_1$, $\theta_2$. We see then that these conditions should at least imply that any two members of $\mathbf{P}$ are orthogonal on $\mathbf{A}_\infty$.

## 2. MONOTONICITY PROPERTIES IN A MONOTONE LIKELIHOOD RATIO FAMILY

Let $\mathbf{P} = \{P_\theta, \theta \in \Theta\}$ with $\Theta$ an ordered set with ordering "$<$". If $\mathbf{A}_o \subset \mathbf{A}$, the notation $p_\theta^{\mathbf{A}_o}$ was introduced in Chapter 1, meaning the density of $P_\theta$ on $\mathbf{A}_o$ with respect to some probability measure $u$ that dominates $P_\theta$ on $\mathbf{A}_o$. In the following, the measure $u$ will usually not be mentioned explicitly. If $\mathbf{A}_o$ is a subfield generated by a random variable $X$, we shall denote the density of $P_\theta$ on $\mathbf{A}_o$ by $p_\theta^X$.

The density ratio $R_n(\theta_1, \theta_2)$, introduced in Chapter 1, is $\mathbf{A}_n$ —measurable, where $\mathbf{A}_n = \mathbf{B}(X_1, \ldots, X_n)$. Therefore, there exists a Baire function $r_n^*$ mapping Euclidean $n$-space $E^n$ into the real line $R$, where $r_n^*$ satisfies

$$r_n^*(X_1, \ldots, X_n; \theta_1, \theta_2) = R_n(\theta_1, \theta_2) \tag{2.1}$$

If $f$ is a real valued function on $E^n$, we shall call it a nondecreasing function if it is nondecreasing in each argument separately. We shall call $f$ on $E^n$ nonincreasing if $-f$ is nondecreasing.

The following definitions are taken from [6] and [3]:

**Definition** 2.1: *A dominated family* $\mathbf{P}$ *is called a monotone likelihood ratio (MLR) family on* $\mathbf{A}_n$ *if for every* $\theta_1 < \theta_2$ *there exist versions of the densities such that* $r_n^*(x_1, \ldots, x_n; \theta_1, \theta_2)$ *is a nondecreasing function of* $x_1, \ldots, x_n$.

**Definition** 2.2: *Let* $\mathbf{A}_o \subset \mathbf{A}^* \subset \mathbf{A}$. *The subfield* $\mathbf{A}_o$ *of* $\mathbf{A}^*$ *is called sufficient for the family* $\mathbf{P}$ *on* $\mathbf{A}^*$, *if for any bounded* $\mathbf{A}^*$*-measurable function* $f$, $E_\theta\{f \mid \mathbf{A}_o\}$ *can be chosen free of* $\theta$. *A random variable* $X$ *is a sufficient statistic for* $\mathbf{P}$ *on* $\mathbf{A}^*$ *if* $X$ *is* $\mathbf{A}^*$*-measurable and the* $\sigma$*-field generated by* $X$ *is a sufficient* $\sigma$*-field for* $\mathbf{P}$ *on* $\mathbf{A}^*$.

**Definition** 2.3: *The family of probability measures* $\mathbf{P}$ *is homogeneous on* $\mathbf{A}_o$ *if for every* $\theta'$ *and* $\theta''$ *in* $\Theta$, $P_{\theta'}$ *is absolutely continous with respect to* $P_{\theta''}$ *on* $\mathbf{A}_o$.

If $\mathbf{P}$ is homogeneous and a statement holds a.e. with respect to $P_{\theta_o}$ for some $\theta_o \in \Theta$, then it holds a.e. with respect to $P_\theta$ for all $\theta \in \Theta$. We shall then write "a.e. $\mathbf{P}$".

Since the family $\mathbf{P}$ will remain fixed, for simplicity we shall say that $X$ is sufficient on $\mathbf{A}^*$ rather than $X$ is sufficient for $\mathbf{P}$ on $\mathbf{A}^*$.

From a well known factorization theorem [3], if $\mathbf{A}_o$ is sufficient on $\mathbf{A}^*$ and a σ-finite measure $u$ dominates $\mathbf{P}$ on $\mathbf{A}^*$, the density $p_\theta^{\mathbf{A}^*}$ of $P_\theta$ with respect to $u$ can be factorized as follows:

$$p_\theta^{\mathbf{A}^*} = g_\theta^{\mathbf{A}_o} h^{\mathbf{A}^*} \tag{2.2}$$

where, for every $\theta \in \Theta$, $g_\theta^{\mathbf{A}_o}$ is an $\mathbf{A}_o$-measurable function and $h^{\mathbf{A}^*}$ is an $\mathbf{A}^*$-measurable function that does not involve $\theta$.

**Lemma 2.1**: *If, for some n, $X_n$ is sufficient on $\mathbf{A}_n$ and $P_{\theta_2}$ is absolutely continous with respect to $P_{\theta_1}$ on $\mathbf{A}_n$, then*

$$p_{\theta_2}^{\mathbf{A}_n} / p_{\theta_1}^{\mathbf{A}_n} = p_{\theta_2}^{X_n} / p_{\theta_1}^{X_n} \qquad \text{a.e. } P_{\theta_1} \tag{2.3}$$

**Proof:** Let $\mathbf{A}_{x_n}$ be the subfield generated by $X_n$. Applying (2.2) with $\mathbf{A}^* = \mathbf{A}_n$, $\mathbf{A}_o = \mathbf{A}_{x_n}$, we find that $p_{\theta_2}^{\mathbf{A}_n} / p_{\theta_1}^{\mathbf{A}_n}$ is $\mathbf{A}_{x_n}$-measurable. Furthermore, for any $A \in \mathbf{A}_{x_n}$:

$$\int_A \left( p_{\theta_2}^{\mathbf{A}_n} / p_{\theta_1}^{\mathbf{A}_n} \right) dP_{\theta_1} = \int_A \left( p_{\theta_2}^{X_n} / p_{\theta_1}^{X_n} \right) dP_{\theta_1},$$

the common value being $P_{\theta_2}(A)$. Thus (2.3) follows.

**Remark.** By redefining the various densities on a set of $P_{\theta_1}$ measure 0, if necessary, we can make the two sides of (2.3) equal everywhere. We shall assume throughout that this has been done. Furthermore, since the right hand side of (2.3) is $\mathbf{A}_{x_n}$-measurable, there is a Baire function $r_n(\,.\,; \theta_1,\theta_2)$ mapping $R \rightarrow R$, such that

$$p_{\theta_2}^{X_n} / p_{\theta_1}^{X_n} = r_n(X_n\,; \theta_1,\theta_2) \tag{2.4}$$

With the notation (2.4) we can express (2.3) as

$$R_n(\theta_1,\theta_2) = r_n(X_n\,; \theta_1,\theta_2) \tag{2.5}$$

Suppose $X$ is a random variable and $P$ a probability measure on $\mathbf{A}$. If $f$ and $g$ are Baire functions of a real variable, either both nondecreasing or both nonincreasing such that $f(X)$ and $g(X)$ are $P$-integrable, then:

$$E_P\{f(X)\,g(X)\} \geqslant E_P\{f(X)\}\,E_P\{g(X)\} \tag{2.6}$$

To show (2.6), let $X_1$ and $X_2$ be random variables defined on a probability space ($\Omega'$, $\mathbf{A}'$, $P'$), such that $X_1$ and $X_2$ are independent and have the same distribution as $X$. By the monotonicity assumptions on $f$ and $g$ we have

$$\{f(X_1)-f(X_2)\}\,\{g(X_1)-g(X_2)\} \geqslant 0$$

Taking on both sides the expectation with respect to $P'$ we obtain

$$E_{P'}\{f(X_1)\,g(X_1)\} + E_{P'}\{f(X_2)\,g(X_2)\} \geqslant E_{P'}\{f(X_1)\,g(X_2)\} + E_{P'}\{f(X_2)\,g(X_1)\} \tag{2.7}$$

Since each term on the left hand side is equal to $E_P\{f(X)\,g(X)\}$ and each term on the right hand side is equal to $E_P\{f(X)\}\,E_P\{g(X)\}$, after dividing both sides of (2.7) by 2 we have (2.6).

Inequality (2.6) can be generalized as follows

**Lemma** 2.2: *Let $X$ be a random variable on $\mathbf{A}$, $P$ a probability measure on $\mathbf{A}$, and let $\mathbf{A}_o \subset \mathbf{A}$.*

(i) *If $f$ and $g$ are Baire functions of a real variable, either both nondecreasing or both nonincreasing, such that $f(X)$ and $g(X)$ are $P$-integrable, then*:

$$E_P\{f(X)\,g(X) \mid \mathbf{A}_o\} \geqslant E_P\{f(X) \mid \mathbf{A}_o\}\ E_P\{g(X) \mid \mathbf{A}_o\} \qquad \text{a.e. } P \tag{2.8}$$

(ii) *If, on the other hand, $f$ and $g$ are monotonic in opposite directions, then*:

$$E_P\{f(X)\,g(X) \mid \mathbf{A}_o\} \leqslant E_P\{f(X) \mid \mathbf{A}_o\}\,E_P\{g(X) \mid \mathbf{A}_o\} \qquad \text{a.e. } P \tag{2.9}$$

**Proof:** We only need to prove (i), since (ii) follows by applying (i) to $-f$ and $g$. In the following the sets $B$ are understood to be Borel subsets of the real line. Let $p(B,\omega)$ be a conditional probability distribution of $X$ in the wide sense, relative to $\mathbf{A}_o$ (see [2], p.29), i.e.

(a) for each linear Borel set $B$, $p(B,.)$ is a version of $P(X^{-1}(B) \mid \mathbf{A}_o)$;

(b) for each $\omega \in \Omega$, $p(\,.\,,\omega)$ is a probability measure on the $\sigma$-field of linear Borel sets.

The existence of such a conditional distribution in the wide sense was shown by Doob [2], Chap. I, sect. 9, and also that for any real valued function $h$ such that $h(X)$ is integrable we have

$$E_P\{h(X) \mid \mathbf{A}_o\} = \int_{-\infty}^{\infty} h(x)\,p(dx,\,.\,) \qquad \text{a.e. } P \tag{2.10}$$

Now apply (2.10) to $h = fg$, $h = f$ and $h = g$, successively, then use (2.6) with $P$ replaced by $p(\,.\,.\,,\omega)$ for every fixed $\omega$. This leads immediately to (2.8).

**Corollary** 2.1: *Under the same conditions as in Lemma* 2.2 (i),

$$E_P\{f(X)\,g(X)\,I_A\}\,P\{A\} \geqslant E_P\{f(X)\,I_A\}\,E_P\{g(X)\,I_A\} \text{ for any } A \in \mathbf{A} \tag{2.11}$$

**Proof:** (2.11) follows from (2.8) by taking $\mathbf{A}_o = \{\Omega, A, A^c, \varnothing\}$ where $A^c$ is the complement of $A$. We can also derive (2.11) from (2.6) immediately by applying (2.6) to the space $A$ with probability measure $P/P(A)$.

In order to avoid repetition we make the following assumption:

**Assumption** $A$:

(i) *For every finite n,* $\mathbf{P}$ *is a MLR family and homogeneous on* $\mathbf{A}_n$.

(ii) *For every finite n,* $X_n$ *is sufficient on* $\mathbf{A}_n$.

**Lemma** 2.3: *Let Assumption A be satisfied, let* $\mathbf{A}_o \subset \mathbf{A}_n$ *for some n, and let* $\theta', \theta'' \in \Theta$, *with* $\theta' < \theta''$. *If f is a nondecreasing function of a real variable such that* $f(X_n)$ *is integrable with respect to* $P_\theta'$ *and* $P_\theta''$, *then:*

$$E_{\theta'}\{f(X_n)\,|\,\mathbf{A}_o\} \leqslant E_{\theta''}\{f(X_n)\,|\,\mathbf{A}_o\} \qquad \text{a.e. } \mathbf{P} \tag{2.12}$$

**Proof:** We shall need the following equation:

$$E_{\theta'}\{R_n(\theta',\theta'')\,|\,\mathbf{A}_o\} = p_{\theta''}^{\mathbf{A}_o} \;/\; p_{\theta'}^{\mathbf{A}_o} \qquad \text{a.e. } P_{\theta'} \tag{2.13}$$

This was shown in Theorem 1.1 (ii) for the case $\mathbf{A}_o = \mathbf{A}_{n-1}$. The proof of (2.13) goes in exactly the same way, and will not be repeated here.
Now let $A \in \mathbf{A}_o$ and, for short write $f$ instead of $f(X_n)$. From (2.5): $R_n(\theta',\theta'') = r_n(X_n;\theta',\theta'')$ a.e. $P_{\theta'}$. Using Lemma 2.2 (i) and (2.13), and noting that $r_n$ is nondecreasing, we have:

$$E_{\theta'}\{f\,R_n(\theta',\theta'')\,|\,\mathbf{A}_o\} \geqslant E_{\theta'}\{f\,|\,\mathbf{A}_o\}\,(p_{\theta''}^{\mathbf{A}_o}/p_{\theta'}^{\mathbf{A}_o}) \qquad \text{a.e. } P_{\theta'}$$

Since $\int_A E_{\theta''}\{f\,|\,\mathbf{A}_o\}\,dP_{\theta''} = \int_A f\,dP_{\theta''} = \int_A f\,R_n(\theta',\theta'')\,dP_{\theta'}$

and $\int_A E_{\theta'}\{f\,|\,\mathbf{A}_o\}\,dP_{\theta''} = \int_A E_{\theta'}\{f\,|\,\mathbf{A}_o\}\,(p_{\theta''}^{\mathbf{A}_o}/p_{\theta'}^{\mathbf{A}_o})\,dP_{\theta'}$,

we have $\int_A E_{\theta''}\{f\,|\,\mathbf{A}_o\}\,dP_{\theta''} \geqslant \int_A E_{\theta'}\{f\,|\,\mathbf{A}_o\}\,dP_{\theta''}$ for every $A \in \mathbf{A}_o$

or:

$$E_{\theta''}\{f\,|\,\mathbf{A}_o\} \geqslant E_{\theta'}\{f\,|\,\mathbf{A}_o\} \qquad \text{a.e. } P_{\theta''} \tag{2.14}$$

Since $\mathbf{P}$ is homogeneous on $\mathbf{A}_n$, (2.14) is true a.e. $\mathbf{P}$. This concludes the proof of Lemma 2.3.

The following theorem follows from Theorem 1.1 and Lemma 2.3 by taking $f(x) = r_n(x;\theta_1,\theta_2)$ and $\mathbf{A}_o = \mathbf{A}_{n-1}$.

**Theorem** 2.1: *Let Assumption A be satisfied. If* $\theta_1,\theta_2 \in \Theta$ *with* $\theta_1 < \theta_2$, *then the stochastic process* $\{R_n(\theta_1,\theta_2),\ \mathbf{A}_n,\ n \geqslant 1\}$ *is:*

(i) *a lower martingale with respect to* $P_\theta$ *for* $\theta < \theta_1$

(ii) *a martingale with respect to* $P_{\theta_1}$

(iii) *an upper martingale with respect to* $P_\theta$ for $\theta > \theta_1$

The following two definitions are due to Lehmann [6]:

**Definition** 2.4: *A set* $S \in \mathbf{A}_n$ *is called an increasing set if for any two n-tuples of real numbers* $(a_1, \ldots, a_n)$ *and* $(b_1, \ldots, b_n)$ *with* $a_k \leqslant b_k$, $k=1,\ldots, n$,

$$\bigcap_{k=1}^{n} X_k^{-1}(a_k) \subset S \quad \textit{implies} \quad \bigcap_{k=1}^{n} X_k^{-1}(b_k) \subset S$$

**Definition** 2.5: *A dominated family of probability measures* $\mathbf{P}$ *whose index set* $\Theta$ *is ordered, is said to have the increasing property on* $\mathbf{A}_n$ *if for every increasing set* $S \in \mathbf{A}_n : P_{\theta'}(S) \leqslant P_{\theta''}(S)$ *whenever* $\theta' < \theta''$.

It is easy to see that Definition 2.5 is equivalent to: every nondecreasing Baire function $f$ on $E^n$ has the property $E_{\theta'}\{f(X_1, \ldots, X_n)\} \leqslant E_{\theta''}\{f(X_1, \ldots, X_n)\}$ whenever $\theta' < \theta''$, provided the expectations exist.

It was shown [6], if $X_1, \ldots, X_n$ are mutually independent with respect to every member of $\mathbf{P}$ and if $\mathbf{P}$ is a MLR family, then it has the increasing property. There were examples in [6] that in general a MLR family does not have the increasing property. We are going to show that under Assumption $A$, the increasing property is true.

**Lemma** 2.4: *Under Assumption* $A$, $\mathbf{P}$ *has the increasing property on* $\mathbf{A}_n$.

**Proof :** We know from Lemma 2.3 that $\mathbf{P}$ has the increasing property on $\mathbf{A}_1$. Suppose the increasing property is true on $\mathbf{A}_{n-1}$, we are going to show that it is true on $\mathbf{A}_n$. Let $f_n$ be a nondecreasing function of $x_1, \ldots, x_n$ such that $f_n(X_1, \ldots, X_n)$ is $P_{\theta'}$ and $P_{\theta''}$-integrable. Remembering $r_n(x_n; \theta',\theta'')$ is nondecreasing in $x_n$, let a be a number such that $r_n(a; \theta',\theta'') \geqslant 1$ and for every $x_n < a : r_n(x_n; \theta'\theta'') \leqslant 1$. Since $f_n$ is also nondecreasing in $x_n$, we have: $f_n(x_1, \ldots, x_n)\{r_n(x_n; \theta',\theta'') - 1\} \geqslant f_n(x_1, \ldots, a)\{r_n(x_n; \theta',\theta'') - 1\}$
Wed define $f_{n-1}(x_1, \ldots, x_{n-1}) = f_n(x_1, \ldots, x_{n-1}, a)$. If is easy to see that $f_{n-1}$ is a nondecreasing function of $x_1, \ldots, x_{n-1}$. So we have the following:

$$\begin{aligned}
&E_{\theta''}\{f_n(X_1, \ldots, X_n)\} - E_{\theta'}\{f_n(X_1, \ldots, X_n)\} \\
&\quad = E_{\theta'}\{f_n(X_1, \ldots, X_n)(R_n(\theta',\theta'') - 1)\} \\
&\quad \geqslant E_{\theta'}\{f_{n-1}(X_1, \ldots, X_{n-1})(R_n(\theta',\theta'') - 1)\} \\
&\quad = E_{\theta''}\{f_{n-1}(X_1, \ldots, X_{n-1})\} - E_{\theta'}\{f_{n-1}(X_1, \ldots, X_{n-1})\}
\end{aligned}$$

which is $\geqslant 0$ because we suppose $\mathbf{P}$ has the increasing property on $\mathbf{A}_{n-1}$.

**Lemma** 2.5: *Let, for every finite n,* **P** *be homogeneous and having the increasing property on* $\mathbf{A}_n$. *Let* $f_n$ *be a non-negative and nondecreasing Baire function on* $E^n$ *such that* $Y_n = f_n(X_1, \ldots, X_n)$ *is* **P**-*integrable.*
*Then for any* $\theta' < \theta''$:

(2.15) $$E_{\theta'} \{\lim \inf_{n \to \infty} Y_n\} \leqslant E_{\theta''} \{\lim \inf_{n \to \theta} Y_n\}$$

(2.16) $$E_{\theta'} \{\lim \sup_{n \to \infty} Y_n\} \leqslant E_{\theta''} \{\lim \sup_{n \to \infty} Y_n\}$$

**Proof:** Define $Y_{km} = \inf_{k \leq n \leq m} Y_n$. For fixed $k$, we have

(2.17) $$E_{\theta'} \{ Y_{km}\} \leqslant E_{\theta''} \{ Y_{km}\}$$

which follows from the increasing property hypothesis. By letting $m \to \infty$ and using Lebesque's monotone convergence theorem, we have:

(2.18) $$E_{\theta'} \{\inf_{k \leq n} Y_n\} \leqslant E_{\theta''}\{\inf_{k \leq n} Y_n\}$$

Now let $k \to \infty$ and use once more the Lebesque's monotone convergence theorem, we have (2.15). Note that the inequality (2.15) is always true, whether $E_{\theta''} \{\lim \inf_{n \to \infty} Y_n\}$ is finite or infinite, because of the Lebesque's monotone convergence theorem. For (2.16) the proof proceeds in the same way, by considering $Y'_{km} = \sup_{k \leq n \leq m} Y_n$.

We are going to state two martingale convergence theorem from [2] Chap. VII, Theorem, 4.1. (i) and 4.ls. (i).

**Statement** 2.1: *Let* $\{f_n, \mathbf{B}_n, n \geqslant 1\}$ *be a martingale. If* $\lim_{n \to \infty} E\{|f_n|\} < \infty$ *then* $\lim_{n \to \infty} f_n$ *exists with probability one and is finite.*

**Statement** 2.2: Let $\{f_n, \mathbf{B}_n, n \geqslant 1\}$ *be an upper martingale. If* $\sup_n E\{|f_n|\} < \infty$, *then* $\lim_{n \to \infty} f_n$ *exists with probability one and is finite.*
*In particular, if the* $f_n$*' s are non positive, the condition is always satisfied.*
By considering $-f_n$ from Statement 2.2, we have:

**Statement** 2.3: *Let* $\{f_n, \mathbf{B}_n, n \geqslant 1\}$ *be a lower martingale. If* $\sup_n E\{|f|\} < \infty$, *then* $\lim_{n \to \infty} f_n$ *exists with probability one and is finite.*
*In particular, if the* $f_n$*'s are non negative, the condition is always satisfied.*

**Theorem** 2.2: *Let Assumption A be satisfied and let* $\theta_1 < \theta_2$. *Then for every* $\theta \leqslant \theta_1$ : $\lim_{n \to \infty} R(\theta_1, \theta_2)$ *exists a.e.* $P_\theta$ *and is finite. Furthermore:*
$E_\theta \{\lim_{n \to \infty} R_n(\theta_1, \theta_2)\} \leqslant E_{\theta_1}\{\lim_{n \to \infty} R_n(\theta_1, \theta_2)\}$

**Proof:** By Theorem 2.1 the stochastic process $\{R_n(\theta_1, \theta_2), \mathbf{A}_n, n \geqslant 1\}$ is a martingale with respect to $P_{\theta_1}$. Since $\lim_{n \to \infty} E_{\theta_1} \{ | R_n(\theta_1, \theta_2) | \} = 1$, by Statement 2.1 we have that $\lim_{n \to \infty} R_n(\theta_1,\theta_2)$ exists a.e. $P_{\theta_1}$ and is finite. With respect to $P_\theta$, $\theta < \theta_1$, the sequence is a lower martingale by Theorem 2.1. Applying Statement 2.3, it has limit a.e. $P_\theta$ and is finite. Finally, (2.5) and Lemma 2.5 give the second conclusion of the theorem.

**Theorem** 2.3: *Let Assumption A be satisfied, and let* $\theta_1 < \theta_2$, *then the following three conditions are equivalent*:

(i) $P_{\theta_1}$ and $P_{\theta_2}$ are orthogonal on $\mathbf{A}_\infty$

(ii) $\lim_{n \to \infty} R_n(\theta_1,\theta_2) = 0$ a.e. $P_\theta$ for $\theta \leqslant \theta_1$

(iii) $\lim_{n \to \infty} R_n(\theta_1,\theta_2) = \infty$ a.e. $P_\theta$ for $\theta \geqslant \theta_2$

**Proof:** By Theorem 1.2, $P_{\theta_1}$ and $P_{\theta_2}$ are orthogonal if and only if $\lim_{n \to \infty} R_n(\theta_1,\theta_2) = 0$ a.e. $P_{\theta_1}$. Applying Theorem 2.2, the latter condition is equivalent to $\lim_{n \to \infty} R_n(\theta_1,\theta_2) = 0$ a.e. $P_\theta$ for $\theta \leqslant \theta_1$. (iii) is equivalent to (ii) by interchanging $\theta_1$ and $\theta_2$.

So the orthogonality is a necessary and sufficient condition for the sequence of density ratio going to 0 or to $\infty$ with respect to $P_\theta$ for $\theta \leqslant \theta_1$ and $\theta \leqslant \theta_2$ respectively. However, with respect to $P_\theta$ for $\theta_1 < \theta < \theta_2$, it is not known whether in general the limit exists or whether at least the limit infimum is 0 or limit supremum is $\infty$, even though Assumption A and orthogonality of $P_{\theta_1}$ and $P_{\theta_2}$ are satisfied.

## 3. LIMITING BEHAVIOR OF THE SEQUENCE OF DENSITY RATIOS ON A SYMMETRIC SPACE

Let $\mathbf{P}'$ be a family of probability measures on $\mathbf{A}$, where $\mathbf{P}' = \{P_\delta, \delta \in \triangle\}$, $\triangle$ an abstract set. Consider a sequence of random variables $Z_1, Z_2, \ldots$ such that with respect to every member $P_\delta$ of $\mathbf{P}'$, the $Z_i$'s are mutually independent and identically distributed. For each $n \geqslant 1$, let there be given a Baire function $f_n$ on $E^n$, such that $f_n(z_1, z_2, \ldots, z_n)$ is invariant under all permutation of $z_1, \ldots, z_n$. Define $X_n = f_n(Z_1, \ldots, Z_n)$, and suppose that the distribution of $X_n$ depends on $\delta$ only through a certain function of $\delta$, say $\theta = \theta(\delta)$, where $\theta$ lies in an ordered set $\theta$. The family of distributions of $X_1, \ldots, X_n$ is denoted by $\mathbf{P} = \{P_\theta, \theta \in \Theta\}$ as in the preceding chapters. For example, let the $Z_i$'s be normally distributed with mean $\xi$ and variance $\sigma^2$. If we let:

$$X_1 = 0 , \qquad X_n = \frac{U_n}{\sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Z_1 - U_n)^2}} \qquad , n \geqslant 2$$

(the choise $f_1(z) = 0$ for all $z$ is purely arbitrary), where $U_n = \frac{1}{n} \sum_{i=1}^{n} Z_1$ we have a family of noncentral $t$-distributions, with parameter $\theta = \xi / \sigma$. In our example $\triangle$ could be considered as a set of pairs $(\xi, \sigma)$ and $\Theta$ as the real line. Note that in our example $X_n$ is sufficient on $\mathbf{B}(X_1, \ldots, X_n)$.

Let us make a new assumption which will be used in this chapter.

**Assumption** B:

(i) $X_n = f_n(Z_1, \ldots, Z_n)$, *where* $Z_1, Z_2, \ldots$ *are independent and identically distributed, and* $f_n$ *is a Baire function of n real variables that is invariant under all permutations of these variables*

(ii) $X_n$ is sufficient on $\mathbf{B}(X_1, \ldots, X_n)$

The reason we make the sufficiency assumption of $X_n$ twice, once in $A$ and once in $B$, is that Assumption $A$ and $B$ are not always used at the same time. As usual we write $\mathbf{A}_n = \mathbf{B}(X_1, \ldots, X_n)$ etc.

In [4] Hewitt and Savage have shown that if $Z_1, Z_2, \ldots$ are independent and identically distributed, and if $f$ is a Baire function of the real variables $z_1, z_2, \ldots$ that is invariant under every finite permutation of the $z$'s then $f(Z_1, Z_2, \ldots)$ is constant a.e. This theorem is sometimes called the Hewitt-Savage 0—1 law, because it implies that a set, invariant in the sense described above, has probability 0 or 1.

Let us assume throughout the dicussion that corresponding to different parameters, the probability measures are different.

**Theorem** 3.1: *Let Assumption B be satisfied and let* $\theta', \theta'' \in \Theta$ *where* $\theta' \neq \theta''$. *Then*

$$\lim_{n \to \infty} R_n(\theta', \theta'') = 0 \qquad \text{a.e. } P_{\theta'}$$

*and*

$$\lim_{n \to \infty} R_n(\theta', \theta'') = \infty \qquad \text{a.e. } P_{\theta''}$$

**Proof:** We need to show only the convergence with respect to $P_{\theta'}$, since the convergence with respect to $P_{\theta''}$ follows then by interchanging $\theta'$ and $\theta''$. The stochastic process $\{R_n, \mathbf{A}_n, n \geq 1\}$ is a lower martingale with respect to $P_{\theta'}$ (Theorem 1.1). From Statement 2.3 the limit exists and is finite a.e. $P_{\theta'}$. By sufficiency we have (2.5), and then it follows from the Hewitt and Savage 0—1 law that the limit is a constant a.e. $P_{\theta'}$, say $c$. We have $0 \leq c \leq 1$ which follows from Fatou's lemma:

$$E_{\theta'} \{\lim_{n \to \infty} R_n(\theta', \theta'')\} \leq \lim_{n \to \infty} E_{\theta'} \{R_n(\theta', \theta'')\} \leq 1$$

Furthermore, we know from Theorem 1.1 that

$$\lim_{n\to\infty} R_n(\theta',\theta'') = R_\infty(\theta',\theta'') \qquad \text{a.e. } P_{\theta'}\text{, or}$$

(3.1) $$R_\infty(\theta',\theta'') = c \qquad \text{a.e. } P_{\theta'}$$

We shall show now that $0 < c \leqslant 1$ leads to a contradiction. Suppose first $c = 1$, then (3.1) is the same as $P_{\theta'}\{R_\infty(\theta',\theta'') = 1\} = 1$, so that $P_{\theta'}$ and $P_{\theta''}$ agree on a set of $P_{\theta'}$ measure 1, which is therefore also of $P_{\theta''}$ measure 1. Hence $P_{\theta'} = P_{\theta''}$, but this is excluded since $\theta' \neq \theta''$. Suppose now $0 < c < 1$, then

(3.2) $$P_{\theta'}\{R_\infty(\theta',\theta'') = c\} = 1$$

Let $u$ be the probability measure on $\mathbf{A}$ dominating $P_{\theta'}$ and $P_{\theta''}$, with respect to which the densities have been defined. Then

$$P_{\theta''}\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0\right\} = \int_{\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0\right\}} p_{\theta''}^{\mathbf{A}_\infty}\, du = \int_{\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0\right\}} c\, p_{\theta'}^{\mathbf{A}_\infty}\, du = c < 1$$

which implies

(3.3) $$P_{\theta''}\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0, p_{\theta''}^{\mathbf{A}_\infty} > 0\right\} = c > 0$$

From (3.2) we have

(3.4) $$P_{\theta'}\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0, p_{\theta''}^{\mathbf{A}_\infty} > 0\right\} = 1$$

On the other hand, by interchanging $\theta'$ and $\theta''$ we have $R_\infty(\theta'',\theta') = c'$ a.e. $P_{\theta''}$ where $0 \leqslant c' \leqslant 1$. Since we exclude $P_{\theta'} = P_{\theta''}$ which is equivalent to $c' = 1$, we must have $c' < 1$. If $c' = 0$ we have

(3.5) $$P_{\theta''}\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0\right\} = 0$$

and if $0 < c' < 1$ we have

(3.6) $$P_{\theta'}\left\{p_{\theta'}^{\mathbf{A}_\infty} > 0, p_{\theta''}^{\mathbf{A}_\infty} > 0\right\} = c' < 1$$

Since (3.5) contradicts (3.3) and (3.6) contradicts (3.4), $0 < c < 1$ is impossible. The only remaining conclusion is $c = 0$, as was to be proved.

Consider two fixed parameters $\theta_1$ and $\theta_2$ with $\theta_1 < \theta_2$. Let $1 < a < \infty$ and define:

(3.7) $$M_n(a) = \{\omega : 1/a \leqslant r_n(X_n(\omega); \theta_1,\theta_2) \leqslant a\}$$

(3.8) $m_n(a) = \{x : 1/a \leqslant r_n(x; \theta_1,\theta_2) \leqslant a\}$

(3.9) $\beta_n(a;\theta) = P_\theta\{M_n(a)\}$

(3.10) $\gamma_n(a; \theta',\theta'') = \beta_n(a; \theta'') / \beta_n(a; \theta')$

Note that $M_n(a)$ is a subset of $\Omega$, whereas $m_n(a)$ is a subset of the real line. One sees immediately from the definitions that $M_n(a)$ is the inverse image of $m_n(a)$ under the mapping $X_n$. In the applications in this chapter, $r_n$ will be a monotonic function of $x$, and consequently $m_n(a)$ will be an interval. In the remainder of this chapter $\mathbf{P}$ will be assumed to be homogeneous on $\mathbf{A}_n$, for every finite $n$. If for some $S \in \mathbf{A}_n$ we write $P(S) > 0$, we mean this to be true for some $P \in \mathbf{P}$, and therefore, by homogeneity, for all $P \in \mathbf{P}$. If $S$ and $S'$ are two sets, we shall often write $P\{S,S'\}$ instead of $P\{S \cap S'\}$.

We shall frequently make use of the following two statements: If $\mathbf{P}$ is homogeneous on $\mathbf{A}_n$, if $S \in \mathbf{A}_n$ with $P(S) > 0$, then for any $\theta,\theta_1 \in \Theta$:

(3.11) $$P\{S,\ R_n(\theta,\theta_1) \geqslant P_{\theta_1}(S)/P_\theta(S)\} > 0$$

and

(3.12) $$P\{S,\ R_n(\theta,\theta_1) \leqslant P_{\theta_1}(S)/P_\theta(S)\} > 0$$

For suppose (3.11) to be false, then

$$S \subset \{R_n(\theta,\theta_1) < P_{\theta_1}(S)/P_\theta(S)\} \qquad \text{a.e. } \mathbf{P}$$

which implies

$P_{\theta_1}(S) = \int_S R(\theta,\theta_1)\, dP_\theta < \int_S P_{\theta_1}(S)/P_\theta(S)\, dP_\theta = P_{\theta_1}(S)$, leading to a contradiction. The proof (3.12) is analogous.

**Lemma** 3.1: *Let $\theta_1 < \theta,\theta' < \theta_2$ and let Assumption A be satisfied. If $1 < a < \infty$ and for some $n$, $\beta_n(a; \theta_1) > 0$, then*

(i) $x \in m_n(a) \Longrightarrow \dfrac{1}{a}\gamma_n(a; \theta,\theta_1) \leqslant r_n(x; \theta,\theta_2) \leqslant a\,\gamma_n(a; \theta,\theta_1)$

(ii) $x \in m_n(a) \Longrightarrow \dfrac{1}{a}\gamma_n(a; \theta,\theta') \leqslant r_n(x; \theta,\theta') \leqslant a^2\,\gamma_n(a; \theta,\theta')$

**Proof:** To show (i), take $S = M_n(a)$ in (3.11) and (3.12). We get

(3.13) $$P\{M_n(a),\ R_n(\theta,\theta_1) \geqslant \gamma_n(a; \theta,\theta_1)\} > 0$$

(3.14) $$P\{M_n(a),\ R_n(\theta,\theta_1) \leqslant \gamma_n(a; \theta,\theta_1)\} > 0$$

From (3.13) and (3.14) it follows that there are real numbers $x_n'$ and $x_n''$, both in $m_n(a)$, such that

(3.15) $r_n(x_n'; \theta,\theta_1) \geqslant \gamma_n(a; \theta,\theta_1)$ , $x_n' \in m_n(a)$ ,

(3.16) $r_n(x_n''; \theta,\theta_1) \leqslant \gamma_n(a; \theta,\theta_1)$ , $x_n'' \in m_n(a)$ , Now $\theta > \theta_1$ so that $r_n(x; \theta,\theta_1)$ is a nonincreasing function of $x$, or $r_n(x; \theta,\theta_1) \geqslant r_n(x'_n; \theta,\theta_1)$ if $x \leqslant x'_n$. From (3.15) it follows then that

(3.17) $r_n(x; \theta,\theta_1) \geqslant \gamma_n(a; \theta,\theta_1)$ if $x \leqslant x'_n$

Now suppose $x \in m_n(a)$ so that $r_n(x; \theta_1,\theta_2) \geqslant 1/a$. We have then

$r_n(x; \theta,\theta_2) = r_n(x; \theta_1,\theta_2)\, r_n(x; \theta,\theta_1) \geqslant \frac{1}{a} r_n(x; \theta,\theta_1)$. Using (3.17) we get

(3.18) $r_n(x; \theta,\theta_2) \geqslant \frac{1}{a} \gamma_n(a; \theta,\theta_1)$ if $x \in m_n(a)$

provided $x \leqslant x'_n$. However, it also holds for $x > x'_n$, because $r_n(x; \theta,\theta_2)$ is a nondecreasing function of $x$ since $\theta < \theta_2$. Since $r_n(x; \theta,\theta_1)$ is a nonincreasing function of $x$, $r_n(x; \theta,\theta_1) \leqslant r_n(x; \theta,\theta_1)$ if $x \geqslant x''_n$. From (3.16) it follows that

(3.19) $r_n(x; \theta,\theta_1) \leqslant \gamma_n(a; \theta,\theta_1)$ if $x \geqslant x''_n$

Supposing $x \in m_n(a)$ so that $r_n(x; \theta_1,\theta_2) \leqslant a$, we have
$r_n(x; \theta,\theta_2) = r_n(x; \theta_1,\theta_2)\ r_n(x; \theta,\theta_1) \leqslant a\, r_n(x; \theta,\theta_1)$
Using (3.19) we get

(3.20) $r_n(x; \theta,\theta_2) \leqslant a\, \gamma_n(a; \theta,\theta_1)$ if $x \in m_n(a)$

provided that $x \geqslant x''_n$. Since $r_n(x; \theta,\theta_2)$ is nondecreasing in $x$, it also holds for $x < x''_n$. From (3.18) and (3.20) we have (i). To show (ii) we apply (i) to $\theta$ and $\theta'$; we have

(3.21) $x \in m_n(a) \Longrightarrow \frac{1}{a} \gamma_n(a; \theta,\theta_1) \leqslant r_n(x; \theta,\theta_2) \leqslant a\, \gamma_n(a; \theta,\theta_1)$

and

(3.22) $x \in m\ (a) \Longrightarrow \frac{1}{a} \gamma_n\ (a; \theta_1,\theta') \leqslant r_n(x; \theta_2,\theta') \leqslant a\, \gamma_n\ (a; \theta_1,\theta')$

Remembering $r_n(x; \theta,\theta_2)\, r_n(x; \theta_2,\theta') = r_n(x; \theta,\theta')$ and
$\gamma_n(a; \theta,\theta_1)\ \gamma_n(a; \theta_1,\theta') = \gamma_n(a; \theta,\theta')$, (3.21) and (3.22) give (ii).

**Lemma** 3.2: *Let Assumption A be satisfied and let* $1 < a < \infty$. *If there is an integer N such that for all finite n with* $n \geqslant N$, $\beta_n(a; \theta_1) < 0$ *and if* $\theta_1 < \theta < \theta_2$ *then the following two conditions are equivalent*:

(i) $\lim\inf_{n \to \infty} \beta_n(a; \theta) > 0$

(ii) there is a finite number $d$ such that for all $n \geqslant N$,

$$x \in m_n(a) \Longrightarrow \frac{1}{a}\, 1\, \beta_n(a; \theta_1) \leqslant r_n(x; \theta,\theta_2) \leqslant d\, \beta_n(a; \theta_1)$$

**Proof:** We shall first show (i) implies (ii). From (3.10) we have $\beta_n(a; \theta_1) \leqslant \gamma_n(a; \theta,\theta_1)$; then applying Lemma 3.1 (i)

(3.23) $x \in m_n(a) \Longrightarrow \frac{1}{a} \beta_n(a; \theta_1) \leqslant r_n(x; \theta,\theta_2)$

Since for all $n \geqslant N$, $\beta_n(a; \theta) > \theta$, and since $\lim \inf_{n\to\infty} \beta_n(a; \theta) > 0$ there is a number $c > 0$ with $\inf_{n \geq N} \beta_n(a; \theta) = c > 0$. As a result, we have by

(3.10), $\gamma_n(a; \theta,\theta_1) \leqslant \frac{1}{c} \beta_n(a; \theta_1)$. Using Lemma 3.1 (i) we have:

$$x \in m_n(a) \Longrightarrow r_n(x; \theta,\theta_2) \leqslant \frac{a}{c} \beta_n(a; \theta_1) \tag{3.24}$$

Taking $d = a/c$, (3.23) and (3.24) give condition (ii) of the lemma. To show (ii) implies (i), let $d$ satisfy:

$$x \in m_n(a) \Longrightarrow \frac{1}{a} \beta_n(a; \theta_1) \leqslant r_n(x; \theta,\theta_2) \leqslant d\, \beta_n(a; \theta_1) \tag{3.25}$$

for all $n \geqslant N$. Now $R_n(\theta,\theta_2) = R_n(\theta,\theta_1)\, R_n(\theta_1,\theta_2)$ and remembering the definition (3.7) of $M_n(a)$ we derive from (3.13):

$$P\{M_n(a),\ R_n(\theta,\theta_2) \geqslant \frac{1}{a} \gamma_n(a; \theta,\theta_1)\} > 0 \tag{3.26}$$

so that there exists a number $x_n^* \in m_n(a)$ such that

$$r_n^*(x_n^*; \theta,\theta_2) \geqslant \frac{1}{a} \gamma_n(a; \theta,\theta_1) \quad , \ x_n^* \in m_n(a) \tag{3.27}$$

Combining (3.25) and (3.27) we have

$$\frac{1}{a} \gamma_n(a; \theta,\theta_1) \leqslant r_n(x_n^*; \theta,\theta_2) \leqslant d\, \beta_n(a; \theta_1)$$

for all $n \geqslant N$, so that

$$\gamma_n(a; \theta,\theta_1) \leqslant ad\ \beta_n(a; \theta_1) \qquad \text{for all } n \geqslant N$$

Dividing both sides by $\beta_n(a; \theta_1)$ and using (3.10) gives the desired result $\beta_n(a; \theta) \geqslant 1/ad$ for all $n \geqslant N$.

Suppose we have $\lim \sup_{n\to\infty} \beta_n(a; \theta) > 0$, then there is a subsequence such that $\lim_{n\to\infty} \beta_{n_k}(a; \theta) > 0$, so we have the following corollary which follows immediately from Lemma 3.2.

**Corollary** 3.1: *Let Assumption A be satisfied and let* $1 < a < \infty$. *If for all* $n \geqslant N$, $\beta_n(a; \theta_1) > 0$ *and if* $\theta_1 < \theta < \theta_2$ *then the following conditions are equivalent.*

(i) $\lim \sup_{n\to\infty} \beta_n(a; \theta) > 0$

(ii) *there is a finite number d and a subsequence* $\{n_k\}$ *of positive integers such that for all k*

$$x \in m_{n_k}(a) \Longrightarrow \frac{1}{a} \beta_{n_k}(a; \theta_1) \leqslant r_{n_k}(x; \theta,\theta_2) \leqslant d\beta_{n_k}(a; \theta_1)$$

**Theorem** 3.2: *Let Assumptions A and B be satisfied. If there is* $\theta_o$, $\theta_1 < \theta_o < \theta_2$, *and* $a_o > 1$ *with* $\lim\inf_{n\to\infty} \beta_n(a_o, \theta_o) > 0$, *then for every* $\theta \neq \theta_o$, $\theta_1 < \theta < \theta_2$, *and all* $a < \infty$, *we have*: $\lim_{n\to\infty} \beta_n(a; \theta) = 0$.

**Proof:** Let $N$ be an integer such that for all $n \geqslant N$, $\beta_n(a; \theta_1) > 0$. From Lemma 3.2, there is a finite number $d$ such that

(3.28) $x \in m_n(a_o) \Longrightarrow \frac{1}{a_o} \beta_n(a_o; \theta_1) \leqslant r_n(x; \theta_o, \theta_2) \leqslant d\, \beta_n(a_o; \theta_1)$ for all $n \geqslant N$.

Since $r_n(x; \theta_o, \theta_2) = r_n(x; \theta, \theta_2)/r_n(x; \theta, \theta_o)$, (3.28) becomes:

(3.29) $$x \in m_n(a_o) \Longrightarrow \frac{1}{a_o} \beta_n(a_o; \theta_1)\, r_n(x; \theta, \theta_o) \leqslant r_n(x; \theta, \theta_2) \leqslant d\, \beta_n(a_o; \theta_1)\, r_n(x; \theta, \theta_o)$$

Suppose for some $\theta \neq \theta_o$ and $\theta_1 < \theta < \theta_2$:

(3.30) $$\lim\sup_{n\to\infty} \beta_n(a_o; \theta) > 0.$$

On the other hand, from Lemma 3.1 (i) and remembering $\gamma_n(a_o; \theta, \theta_1) \geqslant \beta_n(a_o; \theta_1)$ we have:

(3.31) $$x \in m_n(a_o) \Longrightarrow \frac{1}{a_o} \beta_n(a_o; \theta_1) \leqslant r_n(x; \theta, \theta_2)$$

From (3.29) and (3.31) we have:

(3.32) $$x \in m_n(a_o) \Longrightarrow \frac{1}{a_o} \beta_n(a_o; \theta_1) \leqslant r_n(x; \theta, \theta_2) \leqslant d\, \beta_n(a_o, \theta_1)\, r_n(x; \theta, \theta_o)$$

so that

(3.33) $$x \in m_n(a_o) \Longrightarrow \frac{1}{a_o} \beta_n(a_o; \theta_1) \leqslant d\, \beta_n(a_o; \theta_1)\, r_n(x; \theta, \theta_o)$$

and after dividing by $\beta_n(a_o; \theta_1)$ on both sides we have:

(3.34) $$x \in m_n(a_o) \Longrightarrow 1/a_o d \leqslant r_n(x; \theta, \theta_o)$$

which is equivalent to

(3.35) $$M_n(a_o) \subset \{1/a_o d \leqslant R_n(\theta, \theta_o)\}$$

so that by (3.30):

(3.36) $$\lim\sup_{n\to\infty} P_\theta\{1/a_o d \leqslant r_n(x_n; \theta, \theta_o)\} > 0$$

which contradicts Theorem 3.1 that states $\lim_{n\to\infty} R_n(\theta, \theta_o) = 0$ a.e. $P_\theta$. Therefore (3.30) is impossible, which means that for all $\theta \neq \theta_o$, $\lim_{n\to\infty} \beta_n(a_o; \theta) = 0$, which implies also

(3.37) $$\lim_{n\to\infty} \beta_n(a; \theta) = 0 \qquad \text{for } a \leqslant a_o.$$

On the other hand $\lim\inf_{n\to\infty} \beta_n(a_o; \theta_o) > 0$ implies $\lim\inf_{n\to\infty} \beta_n(a; \theta_o) > 0$ for all $a \geqslant a_o$. Thus we have (3.37) for all $a < \infty$.

**Theorem** 3.3: *Let assumptions A and B be satisfied.*

(i) *If there is* $\theta_o$, $\theta_1 < \theta_o < \theta_2$, *with* $\limsup_{n\to\infty} R_n(\theta_1,\theta_2) = b$ a.e. $P_{\theta_o}$, $0 < b < \infty$, *then* :

$\lim_{n\to\infty} R_n(\theta_1,\theta_2) = 0$ a.e. $P_\theta$ if $0 < \theta_o$ and

$\limsup_{n\to\infty} R_n(\theta_1,\theta_2) = \infty$ a.e. $P_\theta$ if $\theta > \theta_o$.

(ii) *If there is* $\theta_o$, $\theta_1 < \theta_o < \theta_2$, *with* $\liminf_{n\to\infty} R_n(\theta_1,\theta_2) = b$ a.e. $P_\theta$, $0 < b < \infty$, *then*:

$\liminf_{n\to\infty} R_n(\theta_1,\theta_2) = 0$ a.e. $P_\theta$ if $\theta < \theta_o$ and

$\lim_{n\to\infty} R_n(\theta_1,\theta_2) = \infty$ a.e. $P_\theta$ if $\theta > \theta_o$.

**Proof:** We need to show only (i), since (ii) is obtained from (i) by interchanging $\theta_1$ and $\theta_2$. By (2.5) we have $R_n(\theta_1,\theta_2) = r_n(X_n; \theta_1,\theta_2)$ and it follows from the Hewitt and Savage 0—1 law that $\limsup_{n\to\infty} R_n(\theta_1,\theta_2)$ is constant a.e. with respect to any member of **P**. Suppose $\theta < \theta_o$, applying Lemma, 2.5 we have:

$$E_\theta \{\limsup_{n\to\infty} R_n(\theta_1,\theta_2)\} \leqslant E_{\theta_o}\{ \limsup_{n\to\infty} R_n(\theta_1,\theta_2)\}.$$

We have then: $\limsup_{n\to\infty} R_n(\theta_1,\theta_2) = c$, say, a.e. $P_\theta$, where $c \leqslant b$. Suppose $c > 0$, we are going to show that it leads to a contradiction.

Choose $a_o > 1$ with $1/a_o < c \leqslant b < a_o$. Applying Lemma 3.1 (ii) with $\theta,\theta'$ replaced by $\theta_o$, $\theta$, we have

$$(3.38) \qquad M_n(a_o) \subset \{ \frac{1}{a_o^2} \gamma_n(a_o; \theta_o,\theta) \leqslant R_n(\theta_o,\theta) \leqslant a_o^2 \gamma_n(a_o; \theta_o,\theta)\}$$

for all finite $n$. Since by assumption $1/a_o < \limsup_{n\to\infty} R_n(\theta_1,\theta_2) < a_o$ a.e. $P_\theta$ we have

$$(3.39) \qquad P_\theta \{\limsup_{n\to\infty} M_n(a_o)\} = 1.$$

We know that $\lim_{n\to\infty} R_n(\theta_o,\theta) = \infty$ a.e. $P_\theta$, so (3.38) and (3.39) give

$$(3.40) \qquad \limsup_{n\to\infty} \gamma_n(a_o; \theta_o,\theta) = \infty$$

Choose any positive number $d$, and let $\{n_k\}$ be the subsequence of integers such that

$$(3.41) \qquad \gamma_{n_k}(a_o; \theta_o,\theta) \geqslant d > 0$$

Note that by (3.40), this subsequence is not empty. Let $\{n_m\}$ be the set of positive integers such that $\{n_k\} + \{n_m\} = \{n\}$. We have then

$$(3.42) \qquad P_\theta\{\limsup_{n\to\infty} M_n(a_o)\} \leqslant P_\theta\{\limsup_{k\to\infty} M_{n_k}(a_o)\} + P_\theta\{\limsup_{m\to\infty} M_{n_m}(a_o)\}.$$

In view of (3.41) we have $\gamma_{n_m}(a_o; \theta_o,\theta) < d$. Furthermore, $\lim_{n\to\infty} R_n(\theta_o,\theta) = \infty$ a.e. $P_\theta$. It follows, applying (3.38) to the subsequence $\{n_m\}$, that

(3.43) $$P_\theta\{\lim\ \sup_{m\to\infty} M_{n_m}(a_o)\} = 0$$

Substitution of (3.39) and (3.43) into (3.42) yields

(3.44) $$P_\theta\{\lim\ \sup_{k\to\infty} M_{n_k}(a_o)\} = 1$$

As a consequence we have

(3.45) $$1/a_o \leqslant \lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2) < a_o \qquad \text{a.e. } P_\theta$$

for $\lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2)$ is a constant a.e. $P_\theta$ according to the Hewitt and Savage 0—1 law, and is bounded above by $\lim\ \sup_{n\to\infty} R_n(\theta_1,\theta_2)$, which establishes the right inequality of (3.45). If the left inequality were false, we would have $\lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2) < 1/a_o$ a.e. $P_\theta$, and the left hand side of (3.44) would be 0 instead of 1.

On the other hand, using (3.41) and since $\lim_{n\to\infty} R_n(\theta_o,\theta) = 0$ a.e. $P_{\theta_o}$, (3.38) gives

(3.46) $$P_{\theta_o}\{\lim\ \sup_{k\to\infty} M_{n_k}(a_o)\} = 0$$

Applying Lemma 2.5 to the subsequence $\{n_k\}$ we have

(3.47) $$E_{\theta_o}\{\lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2)\} \geqslant E_\theta\{\lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2)\}.$$

By the Hewitt and Savage 0—1 law $\lim\ \sup_{n\to\infty} R_{n_k}(\theta_1,\theta_2)$ is a constant, say $b'$, a.e. $P_{\theta_o}$. From (3.47) and (3.45) it follows then that $b' \geqslant 1/a_o$. If $1/a_o \leqslant b' \leqslant a_o$ the left hand side of (3.46) would be 1 instead of 0. Hence we have

$$\lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2) > a_o \qquad \text{a.e. } P_{\theta_o}$$

which contradicts the fact that

$$\lim\ \sup_{k\to\infty} R_{n_k}(\theta_1,\theta_2) \leqslant \lim\ \sup_{n\to\infty} R_n(\theta_1,\theta_2) < a_o \quad \text{a.e. } P_\theta.$$

Thus we conclude $c = 0$. This proves the first part of (i).

The second part of (i) is proved analogously, by putting $\lim\ \sup_{n\to\infty} R_n(\theta_1,\theta_2) = c'$, say, a.e. $P_\theta$, where $b \leqslant c'$. Supposing $c' < \infty$ will lead to a contradiction.

**Theorem** 3.4: *Under Assumptions A and B, we have* $\lim\ \inf_{n\to\infty} \beta_n(a;\theta) = 0$ *for all* $a > 1$ *and all* $\theta$, *except perhaps for one parameter* $\theta_o$, *where* $\theta_1 < \theta_o < \theta_2$.

(i) *In case there is* $a_o > 1$ *and* $\theta_o$, $\theta_1 < \theta_o < \theta_2$, *such that*

(3.48) $$0 < \lim\ \inf_{n\to\infty} \beta_n(a_o;\theta_o) < 1$$

*we have*

(3.49) $$\lim\ \inf_{n\to\infty} R_n(\theta_1,\theta_2) = 0 \qquad \text{a.e. } P_\theta \quad \text{if } \theta < \theta_o$$

and

(3.50) $$\lim\ \sup_{n\to\infty} R_n(\theta_1,\theta_2) = \infty \quad \text{a.e. } P_\theta \text{ if } \theta > \theta_o.$$

(ii) *In case there is* $a_o > 1$ *and* $\theta_o$, $\theta_1 < \theta_o < \theta_2$, *such that*

(3.51) $$\lim_{n \to \infty} \beta_n(a_o; \theta_o) = 1$$

*we have*

$$\lim_{n \to \infty} R_n(\theta_1,\theta_2) = 0 \qquad \text{a.e. } P_\theta \quad \text{if } \theta < \theta_o$$

and

$$\lim_{n \to \infty} R_n(\theta_1,\theta_2) = \infty \qquad \text{a.e. } P_\theta \quad \text{if } \theta > \theta_o.$$

**Proof:** The first statement of the theorem follows immediately from Theorem 3.2. To Show (i), the left inequality in (3.48) implies

(3.52) $$\lim\inf_{n \to \infty} R_n(\theta_1,\theta_2) \leqslant a_o \qquad \text{a.e. } P_{\theta_o}$$

because it is constant a.e. $P_{\theta_o}$. If the constant is 0, applying Lemma 2.5 we get (3.49) and if the constant is $> 0$, we apply Theorem 3.3 (ii) to get (3.49). The conclusion (3.50) is proved analogously by interchanging $\theta_1$ and $\theta_2$. To show (ii), note that (3.51) implies

$$1/a_o \leqslant \lim\inf_{n \to \infty} R_n(\theta_1,\theta_2) \leqslant \lim\sup_{n \to \infty} R_n(\theta_1,\theta_2) \leqslant a_o \quad \text{a.e. } P_{\theta_o}$$

Using Theorem 3.3 we have the desired result.

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