THE LAPLACE TRANSFORM OF VECTOR-VALUED FUNCTIONS

BY

MOEDOMO SOEDIGDOMARTO

ICHTISAR. *Teori klasik dari transformasi Laplace-Stieltjes*

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t)$$

telah diperluas oleh E. Hille [4] *untuk fungsi² pada* $[0, \infty)$ *kedalam ruang Banach* X *jang mempunjai sifat bahwa untuk setiap bilangan positip* R,

$$\sup \sum_{j=1}^{n} \| \alpha(t_j) - \alpha(r_j) \| < \infty$$

dengan supermum jang diambil terhadap semua koleksi² hingga $\{[r_j, t_j]\}_{j=1}$ *dari sub-interval² lepas dari* $[0, R]$. *Setelah itu S. Zaidman* [8] *menjelidiki hal jang serupa untuk fungsi²* α *jang bersifat bahwa kumpulan²-variasinja* $V([0, R], \alpha)$ *jaitu koleksi semua elemen dari* X *jang berbentuk*

$$\sum_{J=1}^{n} [\alpha(t_j) - \alpha(r_j)] \text{ dengan } 0 \leq r_1 < t_1 < \ldots < r_n < t_n \leq R$$

merupakan sub-kumpulan jang kompak di X.

Karangan ini membitjarakan kemungkinan untuk membangun teori transformasi Laplace-Stieltjes untuk fungsi² jang sifat²nja lebih umum dari pada fungsi² jang dipeladjari oleh Hille dan Zaidman. Ternjata bahwa banjak hukum² jang berlaku untuk tranformasi fungsi dengan harga skalar berlaku djuga disini.

Perhitungan absis konvergensi $\sigma_c(\alpha)$ *dapat dikerdjakan serupa dengan perhitungan untuk fungsi² berharga skalar (lihat* [6]*), dan begitu pula hukum² jang berhubungan dengan analistas berlaku disini.*

Suatu hal jang menarik ialah bahwa rumus inversi

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} e^{st} \frac{f(s)}{s} ds = \begin{cases} \theta & t < 0 \\ \frac{1}{2}\alpha(0+) & t = 0 \\ \frac{1}{2}[\alpha(t-0) + \alpha(t+0)] & t > 0 \end{cases}$$

dengan $a > \max[0, \sigma_c(\alpha)]$, *tidak berlaku tanpa sjarat tambahan jang harus dipenuhi oleh fungsi* α.

Demikian Hille telah membuktikan bahwa rumus inversi diatas berlaku bila α *mempunjai variasi jang terbatas absolut pada setiap interval jang hingga, sedangkan Zaidman mempergunakan sjarat tambahan bahwa variasi* α *pada setiap interval hingga adalah kompak.*

Ternjata bahwa kedua sjarat diatas dapat diperlunak mendjadi sjarat bahwa variasi α *pada setiap interval jang hingga harus kompak dalam topologi lemah dari X. Lagi pula, rumus inversi akan berlaku tanpa sjarat untuk* α*, asal ruang X adalah lengkap-lemah, chususnja bila X refleksif. Hasil ini kami dasarkan atas hasil karya Bartle, Dunford dan Schawrtz* [1].

1. **Introduction.** E Hille [4] has developed the theory of the Laplace-Stieltjes transform

$$f(s) = \int_0^{\infty} e^{-st} \, d\alpha(t) \tag{1.1}$$

for functions α on $[0, \infty)$ into a Banach space X and having the property that for every positive R,

$$\sup \sum_{j=1}^{n} \| \alpha(t_j) - \alpha(r_j) \| < \infty, \tag{1.2}$$

where the supremum is taken over all finite collections $\{[r_j, t_j]\}_{j=1}^{j=n}$ of disjoint sub-intervals of $[0,R]$. Recently, S. Zaidman [6] developed a similar theory for the case where the function α satisfies the condition that for each positive R, the collection of all elements of X which are of the form

$$\sum_{j=1}^{n} [\alpha(t_j) - \alpha(r_j)]$$

with $0 \leq r_1 < t_1 < \ldots < r_n < t_n \leq R$, constitute a conditionally compact subset of X.

This paper is concerned with giving an account of the possibility of extending the theory of the Laplace-Stieltjes transform to include functions α of more general types. In Section 2 we give a generalization of the concept of a function of bounded variation. The variation of a function α, defined on an interval I to an arbitrary Banach space X, is defined as the set $V(I,\alpha)$ consisting of all elements of X of the form

$$\sum_{j=1}^{n} [\alpha(t_j) - \alpha(r_j)]$$

where $[r_j,t_j]$ $(1 \leq j \leq n)$ are disjoint sub-intervals of I. A classification, based on the properties of the set $V(I,\alpha)$, gives rise to a number of different classes of functions. Thus, we say that α is of bounded (respectively, weakly compact, compact) variation on I if the set $V(I,\alpha)$ is bounded (respectively, conditionally weakly compact, conditionally compact) in X. If there exists a constant M such that

$$\sum_{j=1}^{n} \| \alpha(t_j) - \alpha(r_j) \| < M$$

for all finite collections of disjoint sub-intervals $\{[r_j,t_j]\}_{j=1}^{n}$ of I, then we say that α is of absolute finite variation on I.

The Riemann-Stieltjes integral over a bounded closed interval of a continuous complex-valued function with respect to a vector-valued function of bounded variation is defined in exactly the same way as in the scalar case. If α is a vector-valued function of bounded variation on $[p,q]$, then, the operator T, defined by

$$Th = \int_p^q h(t)\, d\alpha(t),$$

is a bounded operator from the space $C[p,q]$ of all complex-valued continuous functions on $[p,q]$ into the Banach space X. Furthermore, the operator defined by a function of (weakly) compact variation is (weakly) completely continuous.

The continuity properties of vector-valued functions have been studied by D. G. Kendall and J. E. Moyal [5], and also by D. E. Edwards [3]. In general, a vector-valued function of bounded variation does not behave as nicely as a scalar-valued function of bounded variation. For example there are vector-valued functions which are discontinuous everywhere. A function of weakly compact variation however, can have at most a countable number of discontinuities. Moreover, such a function takes its values in a separable subspace of X and it can be normalized.

Zaidman has shown that a function of compact variation need not be of absolute finite variation or vice versa [8]. However, a function of absolute finite variation, if normalized, is necessarily of weakly compact variation.

It is shown in Section 3, that the computation of the abscissa of convergence $\sigma_c(\alpha)$ of the integral (1.1), where α is a function on $[0, \infty)$ to X which is of bounded variation on every bounded interval, is similar to the scalar case (cf. [6]). The Laplace-Stieltjes transform of α is then analytic in the half-plane $Re(s) > \sigma_c(\alpha)$.

The complex inversion formula does not hold, in general. Hille has shown that if α is of absolute finite variation on every finite interval, then, for $a > \max[0,\sigma_c(\alpha)]$,

(1.3)
$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{a-iT}^{a+iT} e^{st}\frac{f(s)}{s}\,ds = \begin{cases} 0 & (t<0), \\ \frac{1}{2}\,\alpha(0+) & (t=0), \\ \frac{1}{2}[\alpha(t-0)+\alpha(t+0)] & (t>0). \end{cases}$$

Zaidman has proved that formula (1.3) also holds, provided α is of compact variation on every finite interval. Theorem 3.5 states that the complex inversion formula holds for $t > 0$ if α is of weakly compact variation on some neighbourhood of t, thereby generalizing Zaidman's result. The proof of this is based on the fact, proved in [1], that if $C(S)$ is a compact Hausdorff space and $C(S)$ demotes the Banach space of all complex-valued continuous functions on S, then any weakly completely continuous operator on $C(S)$ into another Banach space X maps weakly fundamental sequences into strongly convergent sequences. Furthermore, if the basic space is weakly complete, then the complex inversion formula holds without any restriction on α.

2. **Preliminary notions.** Let α be a function defined on a bounded closed interval $[p,q]$ to a complex Banach space X. We will say that α is of

(*i*) *bounded variation on* $[p,q]$ if the set $V([p,q],\alpha)$ consisting of all elements of X of the form

$$\sum_{j=1}^{n} [\alpha(t_j) - \alpha(r_j)], \text{ where } p \leq r_1 < t_1 < \ldots < r_n < t_n \leq q,$$

is bounded in X,

(*ii*) *weakly compact variation on* $[p,q]$ if $V([p,q],\alpha)$ is conditionally weakly compact in X,

(*iii*) *compact variation on* $[p,q]$ if $V([p,q],\alpha)$ is conditionally compact in X,

(*iv*) *absolute finite variation on* $[p,q]$ if there exists a constant $M > 0$ such that

$$\sum_{j=1}^{n} \| \alpha(t_j) - \alpha(r_j) \| < M$$

if $p \leq r_1 < t_1 < \ldots < r_n < t_n \leq q$.

For practical purposes we are going to modify the above definition by replacing the set $V([p,q],\alpha)$ by a larger set without changing its boundedness or (weak) compactness. We need the following lemma.

LEMMA. 2.1. *Let x_j $(1 \leq j \leq n)$ be elements of X and let K consist of all elements of the form $\sum_{s=1}^{n} x_{j_s}$, where $1 \leq j_s \leq n$ and $j_s \neq j_{s'}$ if $s \neq s'$. If ε_j $(1 \leq j \leq n)$ are complex numbers such that $|\varepsilon_j| \leq 1$, then $\sum_{j=1}^{n} \varepsilon_j x_j$ belongs to the convex hull of the set $\sum_{r=0}^{3} i^r K$.*

Proof. Suppose first that $0 \leq \varepsilon_j \leq 1$ $(1 \leq j \leq n)$. Without loss of generality we may assume that $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_{n-1} \leq \varepsilon_n \leq 1$. Then
$$\sum_{j=1}^{n} \varepsilon_j x_j = \varepsilon_1 \sum_{j=1}^{n} x_j + \sum_{k=2}^{n} (\varepsilon_k - \varepsilon_{k-1}) \left(\sum_{j=k}^{n} x_j \right).$$
Hence $\sum_{j=1}^{n} \varepsilon_j x_j = \sum_{k=1}^{n} \delta_k z_k$, where $z_k \in K$, $0 \leq \delta_k \leq 1$, and $\sum_{k=1}^{n} \delta_k = \varepsilon_n \leq 1$. Therefore $\sum_{j=1}^{n} \varepsilon_j x_j$ is in the convex hull of $K \cup \{\theta\}$. Now if the numbers ε_j are complex and if $|\varepsilon_j| \leq 1$, put $\varepsilon_j = \sum_{r=0}^{n} i^r \varepsilon_{j,r}$, where $0 \leq \varepsilon_{j,r} \leq 1$ $(1 \leq j \leq n, 0 \leq r \leq 3)$. Then
$$\sum_{j=1}^{n} \varepsilon_j x_j = \sum_{j=1}^{n} \sum_{j=0}^{n} i^r \varepsilon_{j,r} x_j = \sum_{r=0}^{n} \sum_{j=1}^{n} i^r \varepsilon_{j,r} x_j = \sum_{r=0}^{3} i^r z_r,$$
where each z_r is in the convex hull of $K \cup \{\theta\}$. Hence $\sum_{j=1}^{n} \varepsilon_j x_j$ is in the convex hull of $\sum_{r=0}^{n} i^r K$.

THEOREM 2.1. *A function α on $[p,q]$ to X is of bounded variation if and only if the set $W([p,q],\alpha)$, consisting of all elements of the form $\sum_{j=1}^{n} \varepsilon_j [\alpha(t_j) - \alpha(r_j)]$. where $p \leq r_1 \leq t_1 \leq \ldots \leq r_n \leq t_n \leq q$ and the ε_j are complex numbers such that $|\varepsilon_j| \leq 1$, is bounded.*

The function α is of (weakly) compact variation on $[p,q]$ if and only if $W([p,q],\alpha)$ is conditionally (weakly) compact.

Proof. Let $p \leq r_1 \leq t_1 \leq \ldots \leq r_n \leq t_n \leq q$ and for a subset A of X let $co(A)$ be its convex hull. The set K, determined by the elements $x_j = \alpha(t_j) - \alpha(r_j)$ $(1 \leq j \leq n)$ in the sense of Lemma 2.1, is a subset of $V([p,q],\alpha)$, so that if $|\varepsilon_j| \leq 1$ $(1 \leq j \leq n)$ then

$$(2.1) \qquad \sum_{j=1}^{n} \varepsilon_j \, [\alpha(t_j) - \alpha(r_j)] \ \varepsilon \ co(\sum_{r=0}^{n} i^r K) \subseteq co(\sum_{r=0}^{n} i^r \, V([p,q],\alpha)).$$

The conclusion follows from the relation

$$V([p,q],\alpha) \subseteq W([p,q],\alpha) \subseteq co(\sum_{r=0}^{n} i^r \, V([p,q],\alpha)).$$

If α is a function of bounded variation on $[p,q]$ to X, and h is a complex-valued continuous function on $[p,q]$, then one can define the Riemann-Stieltjes integral $\int_p^q h(t)\, d\alpha(t)$ as the strong limit of sums of the type

$$\sum_{j=1}^{n} h(\tau_j)\, [\alpha(t_j) - \alpha(t_{j-1})],$$

$p = t_0 < t_1 < \ldots . < t_{n-1} < t_n = q$, $t_{j-1} \leq \tau_j \leq t_j \quad (1 \leq j \leq n)$, when $\max_{1 \leq j \leq n} [(t_j - t_{j-1})]$ tends to 0 [4].

THEOREM 2.2. *Let α be a function of bounded variation on $[p,q]$ to X. Then the transformation T defined by*

$$(2.2) \qquad Th = \int_p^q h(t)\, d\alpha(t)$$

is a bounded linear operator from the Banach space $C[p,q]$ of all continuous complex-valued functions on $[p,q]$ into X. Moreover,

$$(2.3) \qquad \| T \| \leq 4 \, sup \{ \| x \| : x \ \varepsilon \ V([p,q], \alpha)\}.$$

Conversely, if T is a bounded linear operator from $C[p,q]$ into X, then there exists a function α of bounded variation on $[p,q]$ to X^{**}, the second conjugate of X, such that $Th = \int_p^q h(t) d\alpha(t)$ for all h in $C[p,q)]$.

Proof. If α is of bounded variation on $[p,q]$, $h \ \varepsilon \ C[p,q]$, and $|h(t)| \leq 1 \quad (p \leq t \leq q)$, then by (2.1)

$$\sum_{j=1}^{n} h(\tau_j)\, [\alpha(t_j) - \alpha(t_{j-1})] \ \varepsilon \ W([p,q],\alpha) \subseteq co(\sum_{r=0}^{3} i^r \, V([p,q], \alpha)),$$

if $p = t_0 < t_1 < \ldots . < t_{n-1} < t_n = q$ and $t_{j-1} \leq \tau_j \leq t_j \quad (1 \leq j \leq n)$. Therefore $\int_p^q h(t)\, d\alpha(t)$ is in the convex closure of $\sum_{r=0}^{3} i^r \, V([p,q],\alpha)$, so that

T is a bounded operator from $C[p,q]$ into X and

$$\|T\| \leq 4 \sup \{ \|x\| : x \varepsilon V([p,q],\alpha)\}.$$

To prove the second statement we first let X be the space $C[p,q]$ itself and T the identity operator of $C[p,q]$. If E is a Borel subset of $[p,q]$, then its characteristic funstion χ_E may be considered as an element of $C^{**}[p,q]$. Moreover, $\|\chi_E\| = 1$ if E is non-void. Now let α_0 be defined by

$$\alpha_0(t) = \chi_{[p,t]} \quad (p \leq t \leq q). \tag{2.4}$$

Clearly α_0 takes its values in $C^{**}[p,q]$ and is of bounded variation on $[p,q]$. Further, it can be verified easily that $\int_p^q h(t)d\alpha_0(t) = h$ for all $h \varepsilon C[p,q]$. Now let T be an arbitrary bounded from $C[p,q]$ into X. Let α be the function defined by

$$\alpha(t) = T^{**}\alpha_0(t) \quad (p \leq t \leq q), \tag{2.5}$$

where T^{**} is the second adjoint of T, mapping $C^{**}[p,q]$ into X^{**}. It is clear that α is of bounded variation on $[p,q]$. Moreover, if $h \varepsilon C[p,q]$, then

$$\int_p^q h(t)d\alpha(t) = \int_p^q h(t)\, dT^{**}\, \alpha_0(t) = T^{**} \int_p^q h(t)\, d\alpha(t) = T^{**}h = Th.$$

In general, if T is a bounded operator from $C[p,q]$ into X one can not expect to find a function α of bounded variation on $[p,q]$ to X, such that $Th = \int_p^q h(t)\, d\alpha(t)$ for all $h \varepsilon C[p,q]$.

THEOREM 2.3. *Let α be a function of (weakly) compact variation on $[p,q]$ to X. Then the operator T defined by (2.2) is (weakly) completely continuous.*

Conversely, if T is a (weakly) completely continuous operator from $C[p,q]$ into X, then there exists a function α of (weakly) compact variation on $[p,q]$ to X such that $Th = \int_p^q h(t)\, d\alpha(t)$ for all $h \varepsilon C[p,q]$.

Proof. The first statement follows from the fact that if $h \varepsilon C[p,q]$ and $|h(t)| \leq 1$ $(p \leq t \leq q)$, then $\int_p^q h(t)\, d\alpha(t)$ is in the convex closure of

$W([p,q],\alpha)$. For, if α is of (weakly) compact variation on $[p,q]$, then by theorem 2.1, the set $W([p,q],\alpha)$ is conditionally (weakly) compact, so that T is (weakly) completely continuous.

If T is (weakly) completely continuous, then so is T^{**} and T^{**} maps $C^{**}[p,q]$ into X, and the unit sphere of $C^{**}[p,q]$ into a conditionally (weakly) compact subset of X. Hence the function α defined by (2.5) is of (weakly) compact variation on $[p,q]$.

The definitions at the beginning of this section can be extended to include the case where the interval need not be bounded or closed. Let α be a funcition defined on the interval I, which may be half-open, finite or infinite, to X. Then α is said to be of

(*i*) *bounded variation on* I if the set union
$V(I,\alpha) = \cup \{ V([p,q],\alpha) : ([p.q] \subseteq I - \infty < p < q < \infty \}$
is bounded in X,

(*ii*) *weakly compact variation on* I if $(V(I,\alpha)$ is conditionally weakly compact in X,

(*iii*) *compact variation on* I if $V(I,\alpha)$ is conditionally compact in X,

(*iv*) *absolute finite variation on* I if there exists a constant $M > 0$ such that

$$\sum_{j=1}^{n} \| \alpha(t_j) - \alpha(r_j) \| < M$$

if $r_1 < t_1 < \ldots . < r_n < t_n$ and $[r_1,t_n] \subseteq I$.

Instead of $V(I,\alpha)$ one can also take the set $W(I,\alpha)$ consisting of all elements of X of the from $\sum_{j=1}^{n} \varepsilon_j [\alpha(t_j) - \alpha(r_j)]$, where the ε_j are complex numbers of absolute value less or equal one.

Let α be a function of bounded variation on $[b, \infty)$ to X. If h is a complex-valued continuous function on $[b, \infty)$, then the improper integral $\int_b^\infty h(t)d\alpha(t)$ will be defined as the strong limit of $\int_b^R h(t)\, d\alpha(t)$ as $R \to \infty$.

That this limit does not always exist can be seen from the following example. Let $\alpha(t) = \chi_{[b,t]} (b, \leq t < \infty)$. Then α is defined and of bounded variation on $[b, \infty)$ to the Banach $M([b, \infty))$ of all bounded complex-valued functions on $[b, \infty)$. In fact, the set $V([b, \infty), \alpha)$ is contained in the unit sphere of $M([b, \infty))$. Now let $h(t) = 1$ $(b \leq t < \infty)$. Then

$\int_R^{R'} h(t)\,d\alpha(t) = \chi_{(R,R']}$ if $R < R' < \infty$. But $\|\chi_{(R,R']}\| = 1$, so that the improper integral $\int_b^{\infty} h(t)d\alpha(t)$ does not exist in this case.

THEOREM 2.4. *Let α be a function of bounded variation on $[b,\infty)$ to X and let $C[b,\infty]$ denote the Banach space of all complex-valued continuous functions on $[b,\infty]$ then the improper integral $\int_b^{\infty} h(t)d\alpha(t)$ exists for all h in $C[b,\infty]$, which vanish at ∞. Further the integral $\int_b^{\infty} h(t)d\alpha(t)$ exists for all h in $C[b,\infty]$ if and only if $\lim_{t\to\infty} \alpha(t)$ exists.*

If $h \in C[b,\infty]$, $|h(t)| \leq 1$ $(b \leq t < \infty)$, and $\int_b^{\infty} h(t)\,d\alpha(t)$ exists, then $\int_b^{\infty} h(t)d\alpha(t)$ is in the closure of $W([b,\infty),\alpha)$.

Proof. Let $h \in C[b,\infty]$ and $h(\infty) = \lim_{t\to\infty} h(t) = 0$. If $b < R < R' < \infty$, then by (2.3)

$$\left\|\int_R^{R'} h(t)d\alpha(t)\right\| \leq 4\,[\sup_{R\leq t\leq R'} |h(t)|]\ [\sup\{\|x\| : x \in V([b,\infty),\alpha)\}].$$

From this it follows that $\lim_{R\to\infty} \left\|\int_R^{R'} h(t)d\alpha(t)\right\| = 0$ and hence the integral $\int_b^{\infty} h(t)d\alpha(t)$ exists. Now if h is an arbitrary function in $C[b,\infty]$, put $g = h - h(\infty)$. For every $R > 0$

$$\int_b^{R} h(t)d\alpha(t) = \int_b^{R} g(t)d\alpha(t) + h(\infty)[\alpha(R) - \alpha(b)].$$

Now $\lim_{R\to\infty} \int_b^{R} g(t)\,d\alpha(t)$ exists and therefore the integral $\int_b^{\infty} h(t)d\alpha(t)$ exists for all $h \in C[b,\infty]$ if and only if $\lim_{R\to\infty} \alpha(R)$ exists. The last statement is a consequence of the fact that for every $R > b$, $\int_b^{R} h(t)\,d\alpha(t)$ is in the closure of $W([b,\infty),\alpha)$ if $h \in C[b,\infty]$ and $|h(t)| \leq 1$ $(b \leq t)$.

The continuity properties of vector-valued functions have been discussed by Kendall and Moyal [5] and also by Edwards [3].

THEOREM 2.5. *Let* α *be a function defined on* $(-\infty, \infty)$ *to* X. *If* α *is of weakly compact variation on every bounded interval, then the strong limits* $\alpha(t \pm 0)$ *exists everywhere, and they satisfy*

$$\alpha(t-0) = \alpha(t) = \alpha(t+0)$$

for all except an (at most) countable set. Moreover, the values of α *lie in a separable subspace.*

For the proof we refer to [3]. Using Edwards' methods it can be shown that if α is of weakly compact variation on $(-\infty, \infty)$ to X, then the strong limits $\alpha(\pm\infty)$ exist.

A function α on $(-\infty, \infty)$ to X is said to be *normalized* if the strong limits $\alpha(t \pm 0)$ exist everywhere and satisfy

$$\alpha(t) = \tfrac{1}{2}[\alpha(t-0) + \alpha(t+0)], \tag{2.6}$$

and if $\lim_{t\to\infty} \alpha(t) = \theta$.

If α is defined on $[0, \infty)$ to X, then α is said to be *normalized*, if the strong limits $\alpha(t \pm 0)$ exist for every $t > 0$, $\alpha(0) = \theta$, and if α satisfies (2.6) for every $t > 0$.

Functions of weakly compact variation on $(-\infty, \infty)$ or $[0, \infty)$ to X can be normalized, and it can be shown that the same is true for functions of absolute finite variation. However, not every function of bounded variation has this property. For example, the function given just before theorem 2.4 can not be normalized.

THEOREM 2.6. *If* α *is a normalized function of weakly compact variation on* $[0, \infty)$ *to* X, *and if*

$$Th = \int_0^\infty h(t)d\alpha(t) \quad (h \in C[0, \infty]), \tag{2.7}$$

then T *is a weakly completely continuous operator from* $C[0, \infty]$ *into* X.

Conversely, if T *is a weakly completely continuous operator from* $C[0, \infty]$ *into* X, *then there exists a unique normalized function* α *of weakly compact variation on* $[0, \infty)$ *to* X *such that* (2.7) *holds.*

Proof. The first statement follows from theorem 2.4. Now let T be a weakly completely continuous operator from $C[0, \infty]$ into X. Let the function α_1 be defined by

$$\alpha_1(t) = \chi_{[o,t]} \quad (0 \leq t < \infty) \tag{2.8}$$

Then α_1 is of bounded variation on $[0, \infty)$ to $C^{**}[0,\infty]$, For each h in $C[0,\infty]$ and $R > 0$

$$\int_0^R h(t)d\alpha_1(t) = h.\ \chi_{[0,R]}\ \varepsilon\ C^{**}[0,\infty].$$

Moreover, $\int_0^R h(t)d\alpha_1(t)$ converges in the w^* — topology to h as $R \to \infty$.

Now consider the function β defined by

(2.9) $$\beta(t) = T^{**}\alpha_1(t) \quad (0 \leq t < \infty).$$

Then β is of weakly compact variation on $[0, \infty)$ to X. If $h\ \varepsilon\ C[0,\infty]$ and $x^*\ \varepsilon\ X^*$, then

$$x^*\int_0^R h(t)\,d\beta(t) = \int_0^R h(t)\,dx^*\beta(t) = \int_0^R h(t)\,dx^*[T^{**}\alpha_1(t)] =$$

$(T^*x^*)\int_0^R h(t)\,d\alpha_1(t)$, so that on letting $R \to \infty$, we have

$$x^*\int_0^R h(t)d\beta(t) = (T^*x^*)\ (h) = x^*(Th).$$

Therefore $Th = \int_0^\infty h(t)\,d\beta(t)$ for all $h\ \varepsilon\ C[0,\infty]$. Next define the function α by

$$\alpha(t) = \tfrac{1}{2}[\beta(t-0) + \beta(t+0)] - \beta(0)\ \ (t > 0),$$
$$\alpha(0) = \theta.$$

Then α is a normalized function of weakly compact variation on $[0, \infty)$ to X and $Th = \int_0^\infty h(t)\,d\alpha(t)$ for all $h\ \varepsilon\ C[0,\infty]$. Finally, let γ be a function of bounded variation on $[0,\infty)$ to X such that it is normalized and $\int_0^\infty h(t)\,d\gamma(t) = \int_0^\infty h(t)\,d\alpha(t)$ for all $h\ \varepsilon\ C[0,\infty]$. For each $x^*\ \varepsilon\ X^*$ and each $h\ \varepsilon\ C[0,\infty]$, we have $\int_0^\infty h(t)\,dx^*\gamma(t) = \int_0^\infty h(t)\,dx^*\alpha(t)$.

Hence $x^*\gamma(t) = x^*\alpha(t)$ for all $x^*\ \varepsilon\ X^*$ and $t \geq 0$ or $\alpha \equiv \gamma$, so that α is uniquely determined by T.

Zaidman [7] has given an example of a function of compact variation on [0,1] which is not of absolute finite variation, and an example of a function which is of absolute finite variation but not of compact variation. We will show that a function of absolute finite variation on $[0, \infty)$, if normalized, is necessarily of weakly compact variation. We need a lemma which is proved in [1].

LEMMA 2.2. *Let S be an abstract set and Σ be a σ-field of subsets of S. Let $ca\,(\Sigma)$ denote the Banach space of all countably additive complex-valued measures defined on Σ having finite variation. Then a subset K of $ca\,(\Sigma)$ is conditionally weakly compact if and only if*

(1) the set K is bounded, and

(2) there exists a positive μ in $ca\,(\Sigma)$ such that

$$\lim_{\mu(E)\to 0} \lambda(E) = 0$$

uniformly for λ in K.

For the proof we refer to [1].

COROLLARY. *If K is a subset of $ca(\Sigma)$ and if for some positive μ in $ca\,(\Sigma)$, $|\lambda(E)| \leq \mu\,(E)$ for all $E \,\varepsilon\, \Sigma$ and all $\lambda \,\varepsilon\, K$, then K is conditionally weakly compact.*

THEOREM 2.7. *Let α be a function of absolute finite variation on $[0, \infty)$ to X. Then $\int_0^\infty h(t)d\alpha(t)$ exists for all h in $C[0, \infty]$. Further, if*

$$Th = \int_0^\infty h(t)d\alpha(t) \quad (h \,\varepsilon\, C\,[0, \infty])$$

then T is a weakly completely continuous operator from $C[0, \infty]$ into X. Hence if α is also normalized, then it is also of weakly compact varation on $[0, \infty)$ to X.

Proof. The convergence of $\int_0^\infty h(t)d\alpha(t)$, $h \,\varepsilon\, C\,[0, \infty]$, follows from the fact that $\lim_{t\to\infty} \alpha(t)$ exists and by theorem 2.4. From the same theorem we see that T is a bounded operator from $C[0, \infty]$ into X. We will show that the adjoint T^* of T is a weakly completely continuous operator from X^* into $C^*[0, \infty]$, which, by the Riesz representation theorem, may be represented by $ca(\Sigma)$, where Σ is the Borel filed of $[0, \infty]$. Now let $\tilde{\alpha}$ be the increasing bounded function defined by

$$\tilde{\alpha}(t) = \sup \left\{ \sum_{j=l}^{n} \|\alpha(t_j) - \alpha(r_j)\| : 0 \leq r_1 < t_1 < \ldots . < r_n < t_n \leq t \right\} \quad (t > 0).$$

For each $x^* \,\varepsilon\, X^*$, let $\lambda(x^*) = T^*x^* \,\varepsilon\, ca(\Sigma)$, and let $\mu \,\varepsilon\, ca(\Sigma)$ be such that $\int_0^{\infty} h(t)\, d\tilde{\alpha}(t) = \int_{[0,\infty]} h(t)\mu(dt)$ for all $h \varepsilon C\ [0, \infty]$. If $\|x^*\| \leq 1$ and $0 \leq h(t) \leq 1 \quad (t > 0)$, then $\int_{[0,\infty]} h(t)\tilde{\lambda}(x^*)(dt) \leq \int_{[0,\infty]} h(t)\, \mu(dt)$, where $\tilde{\lambda}(x^*)$ is the variation of $\lambda(x^*)$. If follows that

$$|\lambda(x^*)(E)| \leq \tilde{\lambda}(x^*)(E) \leq \mu(E)$$

for all E in Σ. Therefore, by the corollary of lemma 2.2, the set

$$K = \{T^*x^* = \lambda(x^*) : \|x^*\| \leq 1\}$$

is conditionally weakly compact. It follows that T^* and hence T is weakly completely continuous. If α is normalized, then, by theorem 2,6, α must be of weakly compact variation on $[0, \infty)$.

Remark. In a similar fashion one can show the existence of a one-to-one correspondence between the weakly completely continuous operators from $C[-\infty, \infty]$ into X and the normalized functions of weakly compact variation on $(-\infty, \infty)$ to X, the correspondence being given by the relation.

$$Th = \int_{-\infty}^{\infty} h(t)\, d\alpha(t) = \lim_{R\to\infty} \int_{-\infty}^{R} h(t) d\alpha(t) \quad (h \varepsilon C[-\infty, \infty]).$$

A theorem similar to theorem 2.7 also holds in this case.

3. The complex inversion formula. Let α be a function defined on $[0, \infty)$ to X such that α is of bounded variation on every bounded closed interval $[0, R]$. Let

$$\beta(s,u) = \int_0^u e^{-st}\, d\alpha(t),\ s \text{ complex},\ u \geq 0. \tag{3.1}$$

It is clear that for each fixed complex number s, the function $\beta(s,.)$ is a strongly continuous function on $(0, \infty)$ to X.

THEOREM 3.1. (*i*) *If for some constant $M > 0$ and some complex number s_0,*

$$\sup_{u \geq 0} \|\beta(s_0,u)\| < M,$$

then the improper integral $\int_0^{\infty} e^{-st}\, d\alpha(t)$ converges for every complex number s with $Re(s) > Re(s_0)$. Moreover, we have

$$(3.2)\qquad \int_0^{\infty} e^{-st}\, d\alpha(t) = (s - s_0) \int_0^{\infty} e^{-(s-s_0)t}\, \beta(s_0,t)\, dt.$$

(ii) There exists a real number $\sigma_c(\alpha)$, *the abscissa of convergence of* α, *such that* $\int_0^{\infty} e^{-st}\, d\alpha(t)$ *converges if* $Re(s) > \sigma_c(\alpha)$ *and diverges if* $Re(s) < \sigma_c(\alpha)$. *We have the relation*

$$(3.3)\qquad \sigma_c(\alpha) = \sup\{\sigma_c(x^*\alpha) : x^* \varepsilon X^*\}.$$

(iii) If f is the Laplace-Stieltjes transform of α, *i.e.,*

$$(3.4)\qquad f(s) = \int_0^{\infty} e^{-st}\, d\alpha(t),\ Re(s) > \sigma_c(\alpha),$$

then f is analytic in the half-plane $Re(s) > \sigma_c(\alpha)$ *and*

$$(3.5)\qquad f^{(k)}(s) = \int_0^{\infty} (-t)^k e^{-st}\, d\alpha(t) \quad (k = 1,2,\ldots).$$

Proof. (*i*) For every positive number R,

$$\int_0^{R} e^{-st}\, d\alpha(t) = \int_0^{R} e^{-(s-s_0)t}\, d\beta(s_0,t).$$

Integration by parts gives

$$\int_0^{R} e^{-st}\, d\alpha(t) = e^{-(s-s_0)R}\, \beta(s_0,R) + (s - s_0) \int_0^{R} e^{-(s-s_0)t}\, \beta(s_0,t)\, dt.$$

Now if $\sigma = Re(s)$ and $\sigma_0 = Re(s_0)$, then

$$\limsup_{R\to\infty} \| e^{-(s-s_0)R}\, \beta(s_0,R) \| \le \limsup_{R\to\infty} M e^{-(\sigma-\sigma_0)R} = 0,$$

and

$$\limsup_{R\to\infty} \left\| \int_R^{2R} e^{-(s-s_0)t}\, \beta(s_0,t)\, dt \right\| \le \limsup_{R\to\infty} M \int_R^{2R} e^{-(\sigma-\sigma_0)t}\, dt = 0$$

(*ii*) If $\int_0^{\infty} e^{-s_0 t}\, d\alpha(t)$ converges, then $\int_0^{\infty} e^{-s_0 t}\, d\alpha(t)$ converges for all s with $Re(s) > Re(s_0)$. On the other hand, if $\int_0^{\infty} e^{-s_0 t}\, d\alpha(t)$ diverges, then $\int_0^{\infty} e^{-st}\, d\alpha(t)$ diverges for all s with $Re(s) < Re(s_0)$. Hence there exists a real number $\sigma_c(\alpha)$ such that $\int_0^{\infty} e^{-st}\, d\alpha(t)$ converges if $Re(s) > \sigma_c(\alpha)$ and diverges if $Re(s) < \sigma_c(\alpha)$. Now let τ be the right hand side of formula (3.3). If $Re(s) > \sigma_c(\alpha)$ then $\int_0^\infty e^{-st}\, d\alpha(t)$ converges and hence also $\int_0^\infty e^{-st}\, dx^* \alpha(t)$ converges for all $x^* \varepsilon X^*$. If follows that $Re(s) \geq \sigma_c(x^*\alpha)$ for all $x^* \varepsilon X^*$ and since s is arbitrary, $\sigma_c(\alpha) \geq \tau$. Conversely, if $Re(s) = \sigma > \tau$), then for each x^* in X^*,

$$\sup_{u>0} |x^* \beta(s,u)| = \sup_{u>0} \left| \int_0^u e^{-st}\, dx^* \alpha(r) \right| < \infty,$$

so that by the uniform boundedness principle, $\sup_{u>0} \|\beta(s,u)\| < \infty$. By (*i*), $Re(s) \geq \sigma_c(\alpha)$. Since s is arbitrary, $\tau \geq \sigma_c(\alpha)$.

(*iii*) For each x^* in X^*, the function x^*f is analytic in the half-plane $Re(s) > \sigma_c(\alpha)$. By a theorem of Dunford, f is analytic in the strong sense in the same region. The rest follows by applying linear functionals to the derivatives $f^{(k)}$ $(k = 1,2,3, \ldots)$.

THEOREM 3.2. *If for some real number* γ

(1) $\|\alpha(t)\| = 0(e^{\gamma t}) \quad (t \to \infty)$

or if

(2) $\alpha(\infty) = \lim_{t\to\infty} \alpha(t)$ *exists and* $\|\alpha(t) - \alpha(\infty)\| = 0(e^{\gamma t})(t \to \infty)$, then $\gamma \geq \sigma_c(\alpha)$.

Proof. Suppose first that $\|\alpha(t)\| = 0(e^{\gamma t}) \quad (t \to \infty)$. If s is a complex number such that $Re(s) = \sigma > \gamma$, then for some constant $M > 0$, $\|e^{-st}\alpha(t)\| \leq Me^{-(\sigma-\gamma)t}$ for all $t > 0$, so that if $0 < R < R' < \infty$, we have

$$\int_{R'}^{R'} e^{-st}\, d\alpha(t) = \alpha(R')\, e^{-sR'} - \alpha(R)e^{-sR} + s \int_R^{R'} e^{-st}\, d(t)dt.$$

Now $\lim_{R\to\infty} \alpha(R)\, e^{-sR} = \lim_{R'\to\infty} \alpha(R')\, e^{-sR'} = \theta$, and

$$\left\| \int_R^{R'} e^{-st}\, \alpha(t)\, dt \right\| \leq M \int_R^{R'} e^{-(\sigma-\gamma)t}\, dt \to 0 \text{ as } R \text{ and } R' \text{ tend to } \infty,$$

so that $\int_0^\infty e^{-st}\, d\alpha(t)$ converges. Thus, if $\sigma > \gamma$, then $\sigma \geq \sigma_c(\alpha)$ which implies that $\gamma \geq \sigma_c(\alpha)$. Now if $\lim_{t\to\infty} \alpha(t) = \alpha(\infty)$ exists put $\varphi = \alpha - \alpha(\infty)$. Then $\int_0^R e^{-st}\, d\varphi(t) = \int_0^R e^{-st}\, d\alpha(t)$ for all $R > 0$. Therefore if

$\|\alpha(t) - \alpha(\infty)\| = \|\varphi(t)\| = 0(e^{\gamma t})$ $(t \to \infty)$, then the integral $\int_0^\infty e^{-st}\, d\alpha(t)$ converges for all s with $Re(s) > \gamma$, or $\gamma \geq \sigma_c(\alpha)$.

THEOREM 3.3. *Suppose that* $\int_0^\infty e^{-s_0 t}\, d\alpha(t)$ *converges and let* $\sigma_0 = Re(s_0)$.

(*i*) *If* $\sigma_0 > 0$, *then* $\|\alpha(t)\| = 0(e^{\sigma_0 t})$ $(t \to \infty)$.

(*ii*) *If* $\sigma_0 < 0$, *then* $\lim_{t\to\infty} \alpha(t) = \alpha(\infty)$ *exists and*

$\|\alpha(t) - \alpha(\infty)\| = o(e^{\sigma_0 t})\,(t \to \infty)$.

Proof. (*i*) On integrating by parts, we have for $t > 0$,

$$\alpha(t) - \alpha(0) = \int_0^t e^{s_0 u}\, d\beta(s_0,u)$$

$$= \beta(s_0,t)e^{s_0 t} - s_0 \int_0^t e^{s_0 u}\, \beta(s_0,u)du.$$

Hence

$$[\alpha(t) - \alpha(0)]e^{-s_0 t} = \beta(s_0,t) - s_0 e^{-s_0 t} \int_0^t e^{s_0 u}\, \beta(s_0,u)du$$

$$= [\beta(s_0,t) - \beta(s_0,\infty)] + [\beta(s_0,\infty) - \beta(s_0,\infty)s_0 e^{-s_0 t}\int_0^t e^{s_0 u}du]$$

$$+ [s_0 e^{-s_0 t} \int_0^t \{\beta(s_0,\infty) - \beta(s_0,u)\}\, e^{s_0 u}\, du].$$

Now $\lim_{t\to\infty} \|\beta(s_0,t) - \beta(s_0,\infty)\| = 0$ by hypothesis, and

$$\lim_{t\to\infty} \|\beta(s_0,\infty) - \beta(s_0,\infty)s_0 e^{-s_0 t} \int_0^t e^{s_0 u}\, du\| = 0.$$

Finally, to show that

$$\lim_{t\to\infty} \|s_0 e^{-s_0 t} \int_0^t \{\beta(s_0,\infty) - \beta(s_0,u)\}e^{s_0 u}du\| = 0,$$

let ε be an arbitrary positive number and put $M = \sup_{u>0} \|\beta(s_0,\infty) - \beta(s_0,u)\|$.

Let $t_0 > 0$ be such that $\|\beta(s_0,\infty) - \beta(s_0,u)\| < \frac{\varepsilon\sigma_0}{2}$ for all $u > t_0$. Now there exists a number $t_1 > t_0$ such that $e^{-\sigma_0 (t_1 - t_0)} < \frac{\sigma_0\varepsilon}{2M}$. If $t > t_1$, then

$$\|e^{-s_0 t} \int_0^t e^{s_0 u}\{\beta(s_0,\infty) - \beta(s_0,u)\}\, du\|$$

$$\leq e^{-\sigma_0 t}\int_0^{t_0} e^{\sigma_0 u}\|\beta(s_0,\infty)-\beta(s_0,u)\|\,du + e^{-\sigma_0 t}\int_0^{t} e^{\sigma_0 u}\|\beta(s_0,\infty)-\beta(s_0,u)\|\,du$$

$$\leq M\frac{e^{\sigma_0 t_0}-1}{\sigma_0 e^{\sigma_0 t}} + \frac{\varepsilon\sigma_0}{2}\,\frac{e^{\sigma_0 t}-e^{\sigma_0 t_0}}{\sigma_0 e^{\sigma_0 t}} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

If follows that $\lim\limits_{t\to\infty} e^{-\sigma_0 t}\|\alpha(t)-\alpha(0)\| = 0$. Hence $\|\alpha(t)-\alpha(0)\| = o(e^{\sigma_0 t})$ and therefore $\|\alpha(t)\| = 0(e^{\sigma_0 t})\,(t\to\infty)$.

(*ii*) The existence of $\alpha(\infty) = \lim\limits_{t\to\infty}\alpha(t)$ follows from the convergence of $\int_0^{\infty} d\alpha(t)$ $(s=0)$. On integrating by parts we have

$$\alpha(\infty)-\alpha(t) = \int_t^{\infty} e^{s_0 u}\,d\beta(s_0,u)$$

$$= e^{s_0 t}\,\beta(s_0,t) - s_0\int_t^{\infty} e^{s_0 u}\,du.$$

Hence

$$e^{-s_0 t}[\alpha(\infty)-\alpha(t)]$$

$$= [\beta(s_0,\infty)-\beta(s_0,t)] + [s_0 e^{-s_0 t}\int_t^{\infty} e^{s_0 u}\{\beta(s_0,\infty)-\beta(s_0,u)\}\,du]$$

$$-[\beta(s_0,\infty)+s_0 e^{-s_0 t}\int_t^{\infty} e^{s_0 u}(\beta(s_0,\infty)\,du].$$

Now $s_0 e^{-s_0 t}\int_t^{t'} e^{s_0 u}\,du = e^{s_0(t'-t)}-1$ if $t' > t$. Since $Re(s_0)=\sigma_0<0$, it follows that $\lim\limits_{t\to\infty} s_0 e^{-s_0 t}\int_t^{\infty} e^{s_0 u}\,du = -1$ and the third term in the above inequality tends to θ as $t\to\infty$. Finally,

$$\|s_0 e^{-s_0 t}\int_t^{\infty} e^{s_0 u}\{\beta(s_0,\infty)-\beta(s_0,u)\}\,du\| \leq |s_0| e^{-\sigma_0 t}\int_t^{\infty} e^{\sigma_0 u}\|\beta(s_0,\infty)-\beta(s_0,u)\|\,du$$

$$\leq \frac{|s_0|}{-\sigma_0}\sup_{u\geq t}\|\beta(s_0,\infty)-\beta(s_0,u)\| \to 0 \text{ as } t\to\infty.$$

Hence $\lim\limits_{t\to\infty} e^{-\sigma_0 t}\|\alpha(t)-\alpha(\infty)\| = 0$ and the proof is complete.

The abscissa of convergence can be computed in a manner which is similar to the scalar case. Let α be a function on $[0,\infty)$ to X such that it is of bounded variation on every bounded interval. Let

$$(3.6) \qquad a(\alpha) = \limsup_{t\to\infty} \frac{\log \|\alpha(t)\|}{t},$$

and if $\alpha(\infty) = \lim_{t\to\infty} \alpha(t)$ exists, let

$$(3.7) \qquad b(\alpha) = \limsup_{t\to\infty} \frac{\log \|\alpha(t) - \alpha(\infty)\|}{t}.$$

THEOREM 3.4. (*i*) *If* $a(\alpha) \neq 0$, *then* $\sigma_c(\alpha) = a(\alpha)$; *if* $\alpha(\infty)$ *exists and* $b(\alpha) \leq 0$, *then* $\sigma_c(\alpha) = b(\alpha)$.

(*ii*) *If* $a(\alpha) = 0$ *and if* $\lim_{t\to\infty} \alpha(t)$ *does not exist then* $\sigma_c(\alpha) = a(\alpha)$; *if* $a(\alpha) = 0$ *and if* $\lim_{t\to\infty} \alpha(t)$ *exists, then* $\sigma_c(\alpha) = b(\alpha)$.

(*iii*) *If* $\sigma_c(\alpha) \geq 0$, *then* $\sigma_c(\alpha) = a(\alpha)$ *and if* $\sigma_c(\alpha) < 0$, *then* $\lim_{t\to\infty} \alpha(t)$ *exists and* $\sigma_c(\alpha) = b(\alpha)$.

Proof. (*i*) First suppose that $a(\alpha) > 0$. Then for every $\sigma > a(\alpha)$, $\|\alpha(t)\| = 0(e^{\sigma t})$ $(t \to \infty)$. By theorem 3.2, $\sigma \geq \sigma_c(\alpha)$ and since σ is arbitrary, it follows that $a(\alpha) \geq \sigma_c(\alpha)$. If $\sigma_c(\alpha) < a(\sigma)$, let γ be a positive number such that $\sigma_c(\alpha) < \gamma < a(\alpha)$. Then $\int_0^\infty e^{-\gamma t}\, d\alpha(t)$ converges, and by theorem 3.3 (*i*) $\|\alpha(t)\| = 0(e^{\gamma t})\,(t \to \infty)$. However, this means that $a(\alpha) \leq \gamma$ and we have a contradiction.

If $a(\alpha) < 0$ and $a(\alpha) < \tau < 0$, then $\|\alpha(t)\| = 0(e^{\tau t})$ $(t \to \infty)$. Hence $\lim_{t\to\infty} \alpha(t) = \theta$ and by theorem 3.2, $\sigma_c(\alpha) \leq \tau$. Since τ is chosen arbitrarily, it follows that $\sigma_c(\alpha) \leq a(\alpha) < 0$. Now suppose that the inequality $\sigma_c(\alpha) < a(\alpha)$ holds and let $\sigma_c(\alpha) < \gamma < a(\alpha)$. By theorem 3.3 (*ii*), $\|\alpha(t) - \alpha(\infty)\| = o(e^{\gamma t})$ $(t \to \infty)$. Since this implies that $a(\alpha) \leq \gamma$, we again have a contradiction.

Now suppose that $\lim_{t\to\infty} \alpha(t)$ exists and let $b(\alpha) < \sigma$. Then $\|\alpha(t) - \alpha(\infty)\| = 0(e^{\sigma t})\,(t \to \infty)$ and hence by theorem 3.2, $\sigma_c(\alpha) \leq \sigma$. Since σ is arbitrary, $\sigma_c(\alpha) \leq b(\alpha)$. On the other hand, if $\sigma < b(\alpha)$, then $\sigma < 0$, and the convergence of $\int_0^\infty e^{-\sigma t}\, d\alpha(t)$ implies that $\|\alpha(t) - \alpha(\infty)\| = o(e^{\sigma t})$ $(t \to \infty)$ by theorem 3.3 (*ii*). Since this means that $b(\alpha) \leq \sigma$, we arrive at a contradiction and hence $\sigma_c(\alpha) = b(\alpha)$.

(ii) If $a(\alpha) = 0$, then $\int_0^\infty e^{-st}\,d\alpha(t)$ converges whenever $Re(s) > 0$. On the other hand, if $\lim_{t\to\infty} \alpha(t)$ does not exist, then $\int_0^\infty d\alpha(t)$ diverges and therefore $\sigma_c(\alpha) = 0 = a(\alpha)$. Now suppose that $\lim_{t\to\infty} \alpha(t)$ exists and that $a(\alpha) = 0$. Since α in bounded on $[0, \infty)$, $\alpha(t) = 0(e^{\sigma t})$ and hence also $\|\alpha(t) - \alpha(\infty)\| = 0(e^{\sigma t})$ $(t \to \infty)$ for every positive number σ. If follows that $\limsup_{t\to\infty} \dfrac{\log\|\alpha(t) - \alpha(\infty)\|}{t} \leq \sigma$ for every positive σ or $b(\alpha) \leq 0$. Therefore we can apply (i) and we have $\sigma_c(\alpha) = b(\alpha)$.

(iii) Let $\sigma_c(\alpha) \geq 0$. If $a(\alpha) \neq 0$, then $\sigma_c(\alpha) = a(\alpha)$ by (i). On the other hand, if $a(\alpha) = 0$, then $\|\alpha(t)\| = 0(e^{\sigma t})(t \to \infty)$ for all $\sigma > 0$. By theorem 3.2, $\sigma_c(\alpha) \leq 0$ and hence $\sigma_c(\alpha) = 0 = a(\alpha)$.

Now let $\sigma_c(\alpha) < 0$, then $\lim_{t\to\infty} \alpha(t)$ exists and $\sup_{t\geq 0} \|\alpha(t) - \alpha(\infty)\| < \infty$. Hence $b(\alpha) \leq 0$ and by (i), $\sigma_c(\alpha) = b(\alpha)$

We shall now derive the complex inversion formula for the general Laplace-Stieltjes transform. We will need the following important lemma which is proved in [1].

LEMMA 3.1. *Let S be a compact Hausdorff space and let $C(S)$ be the Banach space of all complex-valued continuous functions on S. Then any weakly completely continuous operator from $C(S)$ into another Banach space X maps weakly fundamental sequences into strongly convergent sequences.*

For the proof we refer to [1].

LEMMA 3.2. *Let β be a function of bounded variation on $(-\infty, \infty)$ to X such that $\lim_{t\to-\infty} \beta(t) = 0$. Then*

$$B(t,T) = \lim_{R\to\infty} \frac{1}{\pi} \int_{-R}^{R} \frac{\sin Ty}{y}\, \beta(t+y)\,dy \tag{3.8}$$

exists for $T > 0$ and real t. Further, if $si(t) = \dfrac{1}{\pi}\displaystyle\int_t^\infty \frac{\sin y}{y}\,dy$, then

$$B(t,T) = \int_{-\infty}^{\infty} si(Ty)\,d_y\,\beta(t+y) = \int_{-\infty}^{\infty} si[T(z-t)]\,d\beta(z). \tag{3.9}$$

If β is of weakly compact variation on $[t_0 - \delta, t_0 + \delta]$, $\delta > 0$, then

$$\lim_{T\to\infty} B(t_0,T) = \tfrac{1}{2}[\beta(t_0 - 0) + \beta(t_0 + 0)]. \tag{3.10}$$

Proof. If $R < R'$, then integration by parts gives

$$\int_R^{R'} si\,[T(z-t)]\, d\beta(z)$$

$$= si\,[T(R'-t)]\,\beta(R') - si\,[T(R-t)]\,\beta(R) + \int_R^{R'} \frac{\sin T(z-t)}{\pi(z-t)}\,\beta(z)dz.$$

Now if t and T are fixed, and if $T > 0$, then $\lim_{z \to \infty} si\,[T(z-t)] = 0$. Moreover, the functions si and β are bounded On $(-\infty, \infty)$, so that we have $\lim_{R \to \pm\infty} si\,[T(R-t)]\,\beta(R) = \theta$, since $\lim_{R \to -\infty} \beta(R) = \theta$. On the other hand, for fixed R, $\int_R^{\infty} si\,[T(z-t)] d\beta(z)$ converges by theorem 2.4. Further, $si\,[T(.-t)]$ is in $C[-\infty, \infty]$ for all real t. This implies (see theorem 2.4) that for fixed R the integral $\int_{-\infty}^{R} si\,[T(z-t)]\, d\beta(z)$ converges. Hence

$$\int_{-\infty}^{\infty} si\,[T(z-t)]\, d\beta(z) = \int_{-\infty}^{\infty} \frac{\sin T(z-t)}{\pi(z-t)}\,\beta(z)\,dz$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Ty}{y}\,\beta(t+y)\,dy.$$

Now suppose first that β is of weakly compact variation on $(-\infty, \infty)$. We know that if γ is a scalar valued function of bounded variation on $(-\infty, \infty)$ such that $\lim_{t \to -\infty} \gamma(t) = 0$, then

$$\lim_{T \to \infty} \int_{-\infty}^{\infty} si\,[T(z-t)]\, d\gamma(z) = \tfrac{1}{2}[\gamma(t-0) + \gamma(t+0)]$$

for all real t. In other words, if t is a real number and $[T_n]_{n \geq 1}$ is a sequence of positive numbers such that $\lim_{n \to \infty} T_n = \infty$, then the functions $si[T_n(.-t)]$ form a weakly fundamental sequence in $C[-\infty, \infty]$. By lemma 3.1, the sequence $[\int_{-\infty}^{\infty} si\,[T_n(z-t)]\, d\beta(z)]_{n \geq 1}$ converges in norm for every real t, so that $\lim_{T \to \infty} B(t,T) = \frac{1}{2}[\beta(t-0) + \beta(t+0)]$ for each $t \,\varepsilon\, (-\infty, \infty)$.

Next assume that for some t_0 and $\delta > 0$, the function β assumes the value θ on the interval $[t_0 - \delta, t_0 + \delta]$. Consider the integral $\int si\,(Ty)\, d_y\, \beta(t_0 + y)$ $(T > 1)$ evaluated over the intervals

$(-\infty,-1/\sqrt{T}),(-1/\sqrt{T},-1/T^2),(-1/T^2,1/T^2),(1/T^2,1/\sqrt{T}),(1\sqrt{T},\infty)$. The corresponding values $B_k(t_0,T)\,(1\leq k\leq 5)$ will be considered separately.

Put $si(Ty)=1+u(Ty)$, where $\lim\limits_{y\to-\infty} u(Ty)=0$. Then

$$B_1(t_0,T)=\int_{-\infty}^{-1/\sqrt{T}} d_y\beta(t_0+y)+\int_{-\infty}^{-1/\sqrt{T}} u(Ty)\,d_y\,\beta(t_0+y)$$

$$=\beta(t_0-1/\sqrt{T})+\int_{-\infty}^{-\sqrt{T}} u(\tau)\,d_\tau\,\beta(t_0+\tau/T).$$

Since $\beta(t)=\theta$ *if* $t_0-\delta\leq t\leq t_0+\delta$, and

$$\left\|\int_{-\infty}^{-\sqrt{T}} u(\tau)\,d_\tau\,\beta(t_0+\tau/T)\right\|\leq 4\,[\sup_{x\varepsilon V((-\infty,\infty),\beta)}\|x\|].\,[\sup_{\tau\leq-\sqrt{T}}|u(\tau)|],$$

if follows that $\lim\limits_{T\to\infty} B_1(t_0,T)=\theta$. Similarly,

$$B_5(t_0,T)=\int_{1/\sqrt{T}}^{\infty} si\,(Ty)d_y\beta(t_0+y)=\int_{\sqrt{T}}^{\infty} si(\tau)\,d_\tau\beta(t_0+\tau/T).$$

Therefore,

$$\|B_5(t_0,T)\|\leq 4\,[\sup_{x\varepsilon V((-\infty\,\infty),\beta)}\|x\|].\,[\sup_{\tau\geq\sqrt{T}}|si(\tau)|],$$

and hence $\lim\limits_{T\to\infty} B_5(t_0,T)=\theta$. Now let x^* be an arbitrary element of X^*, and let $v=x^*\beta$. If $\tilde{v}$ is the variation of v, then $\tilde{v}$ is constant on $(t_0-\delta,t_0+\delta)$, so that if $T>\dfrac{4}{\delta^2}$, then

$$|x^*B_2(t_0,T)|\leq[\sup_y|si(y)|]\,[\tilde{v}(t_0-1/T^2)-\tilde{v}(t_0-1/\sqrt{T})]=0$$

and

$$|x^*B_4(t_0,T)|\leq[\sup_y|si(y)|]\,[\tilde{v}(t_0+1/\sqrt{T}-\tilde{v}(t_0+1/T^2)]=0.$$

Since x^* is arbitrary, it follows that

$$\lim_{T\to\infty} B_2(t_0,T)=\lim_{T\to\infty} B_4(t_0,T)=0.$$

Finally, on integrating by parts, we obtain

$$B_3(t_0,T) = si\,(1/T)\,\beta\,(t_0 + 1/T^2) - si\,(-1/T)\,\beta\,(t_0 - 1/T^2) + \int_{-1/T^2}^{1/T^2} \frac{\sin Ty}{y}\,\beta\,(t_0 + y)\,dy.$$

Hence $B_3(t_0,T) = \theta$ if $T > 1/\sqrt{\delta}$.

Now if β is of weakly compact variation on $[t_0 - \delta, t_0 + \delta]$, let $\beta_1(t) = \beta(t)$ if $t_0 - \delta \leq t \leq t_0 + \delta$, and $\beta_1(t) = \theta$ otherwise, and let $\beta_2 = \beta - \beta_1$. If $B^i(t_0,T)\,(i = 1,2)$ is the expression (3.9) corresponding with β_i, then $B(t_0,T) = B^1(t_0,T) + B^2(t_0,T)$. Hence

$$\lim_{T\to\infty} B(t_0,T) = \lim_{T\to\infty} B^1(t_0,T) + \lim_{T\to\infty} B^2(t_0,T) = \tfrac{1}{2}[\beta(t_0 - 0) + \beta(t_0 + 0)].$$

THEOREM 3.5. *Let α be a function on $[0, \infty)$ to X which is of bounded variation on every finite interval. If f is the Laplace-Stieltjes transform of α, $f(s) = \int_0^\infty e^{-st}\,d\alpha(t)$ $(Re(s) > \sigma_c(\alpha))$, let*

$$A(t,T) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} e^{st}\,\frac{f(s)}{s}\,ds, \tag{3.11}$$

where $T > 0$ and $a > \max[0,\sigma_c(\alpha)]$. Then

$$\lim_{T\to\infty} A(t,T) = \begin{cases} \tfrac{1}{2}[\alpha(t-0) + \alpha(t+0)] & (t > 0), \text{ if } \alpha \text{ is of weakly compact variation on } [t-\delta, t+\delta] \text{ for some } \delta > 0, \\ \tfrac{1}{2}\,\alpha(0+) & \text{if } \alpha \text{ is of weakly compact variation on } [0,\delta] \text{ for some } \delta > 0, \\ \theta & (t < 0). \end{cases} \tag{3.12}$$

Proof. For each $x^* \in X^*$

$$x^*A(t,T) = \frac{1}{\pi}\int_0^\infty x^*\alpha(y)\,e^{a(t-y)}\,\frac{\sin T(t-y)}{t-y}\,dy.$$

Setting

$$\beta_r(y) = \alpha(y)\,e^{-ay} \qquad \text{for } 0 \leq y \leq r,$$
$$= \theta \qquad \text{for } y < 0 \text{ and } y > r,$$

we see that β_r satisfies the conditions of lemma 3.2. Further, β_r is of weakly compact variation on $[p,q]$ if $p < q < 0$. Therefore, if for each $r > 0$,

$$B_r(t,T) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin Ty}{y}\,\beta_r(t+y)\,dy,$$

then $\lim\limits_{T\to\infty} e^{at} B_r(t,T) = \theta$ for $t < 0$. If α is of weakly compact variation on $[0,\delta]$ for some $\delta > 0$, then so is β_r. Also if α is of weakly compact variation on $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$, $t_0 - \delta > 0$, then the same is true for β_r. Now if $a > \sigma > \max[0,\sigma_c(\alpha)]$, then by theorem 3.3 there exists a constant $M > 0$ such that $\|\alpha(t)\| \leq Me^{\sigma t}$ $(0 \leq t)$, so that for $t < p < r$, we have

$$\|B_r(t,T)\,e^{at} - A(t,T)\| \leq \frac{M}{\pi}\int_r^{\infty} \frac{e^{\sigma y}\, e^{a(t-y)}}{t-y}\,dy$$
$$\leq \frac{Me^{-(a-\sigma)r}\,e^{ap}}{\pi(r-p)\,(a-\sigma)}.$$

It follows that $\lim\limits_{r\to\infty} B_r(t,T)\,e^{at} = A(t,T)$ uniformaly in T and uniformly for t belonging to any finite interval. Now suppose that α is of weakly compact variation on $[t_0 - \delta, t_0 + \delta)]$, and let ε be an arbitrary positive number. Let $r > t_0 + \delta$ be such that $\|B_r(t_0,T)\,e^{at_0} - A(t_0,T)\| < \frac{\varepsilon}{2}$ for all $T > 0$. Since β_r is of weakly compact variation on $[t_0 - \delta, t_0 + \delta]$ it follows that $\lim\limits_{T\to\infty} B_r(t_0,T) = \frac{1}{2}[\alpha(t_0 - 0) + \alpha(t_0 + 0)]$. Now choose T_0 in such a manner that

$$\|B_r(t_0,T)\,e^{at_0} - \tfrac{1}{2}[\alpha(t_0 - 0) + \alpha(t_0 + 0)]\| < \frac{\varepsilon}{2}$$

if $T > T_0$, then $\|A(t_0,T) - \frac{1}{2}[\alpha(t_0 - 0) + \alpha(t_0 + 0)]\| < \varepsilon$. If α is of weakly compact variation on $[0,\delta]$ for some $\delta > 0$, the same method can be applied to show that $\lim\limits_{T\to\infty} A(0,T) = \dfrac{\alpha(0+)}{2}$.

If we examine the proof of the preceeding theorem, we see that if α is a function of bounded variation on $[0,\infty)$ to X and f is its Laplace-Stieltjes transform, then the inversion formula holds at those points $t\,\varepsilon\,(0,\infty)$ for which the following are satisfied:

(1) the limits $\alpha(t \pm 0)$ exist,

(2) $\lim\limits_{T\to\infty}\displaystyle\int_{t-\delta}^{t+\delta} si\,[T(z-t)]\,d\alpha(z) = \lim\limits_{T\to\infty}\int_{-\delta}^{+\delta} si\,(T\tau)\,d\alpha(t+\tau)$

exists for some $\delta > 0$, where δ depends on t. As we have seen earlier, for each sequence T_n of positive numbers such that $\lim_{n \to \infty} T_n = \infty$, the sequence of functions $[h_n]$ where

$$h_n(\tau) = si(T_n \tau), \quad \tau \in (-\infty, \infty),$$

is weakly fundamental in $C[-\infty, \infty]$.

Now suppose that X is weakly complete. For each $t \in (0, \infty)$, the operator Q_t defined by

$$Q_t h = \int_{-\delta}^{+\delta} h(\tau)\, d\alpha(\tau)$$

is a bounded operator on $C[-\infty, \infty]$ into X. Now any bounded operator on $C[-\infty, \infty]$ into a weakly complete Banach space is weakly completely continuous and maps weakly fundamental sequences into strongly convergent sequences (cf. [1]). In particular, for each $t \in (0, \infty)$,

$$\lim_{T \to \infty} \int_{-\delta}^{+\delta} si(T\tau)\, d\alpha(t + \tau)$$

exists for any finite positive δ. On the other hand, Kendall and Moyal [5] have proved that any function α of bounded variation on any interval to a weakly complete Banach space has the property that $\alpha(t \pm 0)$ exists at every point. The following theorem is then a consequence of our above discussions.

THEOREM 3.6. *Let α be a function on $[0, \infty)$ to a weakly complete Banach space X which is of bounded variation on every finite interval. If $a > max\,[0, \sigma_c(\alpha)]$, then*

$$\lim_{T \to \infty} \int_{a-iT}^{a+iT} e^{st} \frac{f(s)}{s}\, ds = \begin{cases} 0 & (t < 0), \\ \frac{1}{2}\alpha(0+) & (t = 0), \\ \frac{1}{2}[\alpha(t-0) + \alpha(t+0)] & (t > 0). \end{cases}$$

The following example shows that the complex inversion formula does not hold, in general.

Let α be defined on $[0, \infty)$ to $L^\infty\,[0, \infty]$ by the formula:

$$\alpha(t) = \chi_{[0,t]} \quad (0 \leq t < \infty).$$

Clearly α is of bounded variation on $[0, \infty)$, and for each $s > 0$,

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t) = e_s,$$

where $e_s(t) = e^{-st}$, so that $f(s)$ belongs to the subspace $C[0, \infty]$ of $L^\infty\,[0, \infty]$. Furthermore, $\sigma_c(\alpha) = 0$. Now suppose that for some $t > 0$, the limit

$$\lim_{T\to\infty}\int_{1-iT}^{1+iT} e^{st}\,\frac{f(s)}{s}\,ds$$

exists, and call the limit x. Since $f(s)\ \varepsilon\ C[0,\infty]$ for all $s>0$, it follows that

$$\int_{1-T}^{1+iT} e^{st}\,\frac{f(s)}{s}\,ds$$

belongs to $C[0.\ \infty]$ for every $T>0$, and hence x belongs to $C[0,\infty]$. On the other hand, for each φ in $L^1[0,\infty]$ and t in $(0,\infty)$,

$$\alpha(t)(\varphi)=\int_0^t \varphi(\tau)\,d\tau$$

is an absolutely continuous function which is of bounded variation for t in $[0,\infty)$. Applying the complex inversion formula for scalar-valued function we have

$$\lim_{T\to\infty}\int_{1-iT}^{1+iT} e^{st}\,\frac{f(s)\ (\varphi)}{s}\,ds=\alpha(t)(\varphi)=\int_0^t \varphi(\tau)\,d\tau.$$

Therefore

$$x(\varphi)=\int_0^\infty x(\tau)\,\varphi(\tau)\,d\tau=\int_0^\infty \chi_{[0,t]}(\tau)\,\varphi(\tau)\,d\tau,$$

or $x=\chi_{[o,t]}\ \varepsilon\ ([0,\infty]$, which is absurd.

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