Riesz Representation Theorem on Bilinear Spaces of Truncated Laurent Series

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Abstract. In this study a generalization of the Riesz representation theorem on non-degenerate bilinear spaces, particularly on spaces of truncated Laurent series, was developed. It was shown that any linear functional on a non-degenerate bilinear space is representable by a unique element of the space if and only if its kernel is closed. Moreover an explicit equivalent condition can be identified for the closedness property of the kernel when the bilinear space is a space of truncated Laurent series.

Keywords: *bilinear forms; closed subspaces; non-degenerate; Riesz representation theorem; truncated Laurent series.*

1 Introduction

Bilinear forms can be viewed as a generalization of inner products, which include the class of vector spaces with an arbitrary field. Similarly, bilinear spaces can be viewed as a generalization of inner product spaces. This view inspires a big question: which notions and results that have been developed on inner product spaces can be generalized to bilinear spaces? This paper proposes a generalization of the Riesz representation theorem on bilinear spaces. It is expected that the proposed result will trigger generalizations of other results whose developments employed the theorem.

The goal of this study was investigation of the Riesz representation theorem on bilinear spaces of truncated Laurent series equipped with a bilinear developed by Fuhrmann [1]. Spaces of truncated Laurent series play an important role as the underlying spaces for the study of linear systems using a behavioral approach (see for example Fuhrmann [2]). A good understanding of this class of spaces is certainly indispensable for the development of linear systems theory using a behavioral approach. Conversely, some development of properties of spaces of truncated Laurent series is also triggered by the development of linear systems theory using a behavioral approach.

The organization of this paper is as follows. In Section 2 we develop the Riesz representation theorem on any non-degenerate bilinear space. We show that

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boundedness of the functional for the equivalent condition of the existence of the Riesz representation theorem on Hilbert spaces can be translated to closedness of the kernel in the world of bilinear spaces. In Section 3 we identify the equivalent property of boundedness of the kernel of a linear functional in a space of truncated Laurent series.

Hereafter, the notation V stands for a bilinear space with the bilinear form on V is symmetry and non-degenerate, denoted by $[-,-]$. Several notions that have been developed in inner product spaces can also be generalized to bilinear spaces. Two elements $x, y \in V$ are called orthogonal, denoted by $x \perp y$ if $[x, y] = 0$. An element x is called orthogonal to a subset $S \subset V$, written as $x \perp S$, if x is orthogonal to y for all y in S. Two subsets are called orthogonal if any element in one set is orthogonal to any element in the other set.

Let S be a subset of the bilinear space V. It is routine to check that the set consisting of all elements that are orthogonal to S forms a subspace, which is called the orthogonal complement of the set S and is written as $S^{\perp}$. Thus $S^{\perp} = \{v \in V | [v, s] = 0\ for\ all\ s \in S\}$. It is easy to check that $S \subseteq (S^{\perp})^{\perp} = S^{\perp\perp}$. A subspace S is called closed if $S^{\perp\perp} = S$.

2 Bilinear Spaces

In the world of inner product spaces, the existence of the Riesz representation theorem is guaranteed if the linear functional is bounded. The boundedness of a linear transformation is equivalent to the transformation of being continuous. Meanwhile, continuity for a linear functional can be connected to closedness of its kernel. Hence, it is natural to derive an equivalent condition for the existence of the Riesz representation theorem on bilinear spaces in terms of closedness of the kernel of the linear functionals.

Before discussing the Riesz representation theorem on bilinear spaces, we present and prove some necessary lemmas.

Lemma 2.1 Let $U \subsetneq V$ be a closed subspace of a non-degenerate bilinear space *V*. Then there exists an $x \neq 0$ in *V* such that $x \in U^{\perp}$.

Proof. Suppose such x does not exist. Since $U^{\perp}$ is a subspace, we obtain $U^{\perp} = \{0\}$. As a result $(S^{\perp})^{\perp} = S^{\perp\perp} = (\{0\})^{\perp} = V$. Hence $U \subsetneq U^{\perp\perp}$, which is a contradiction to the fact that U is closed. Thus $U^{\perp} \neq \{0\}$ or there exists an $x \neq 0$ in *V* such that $x \in U^{\perp}$.

Lemma 2.2 Let U be a subset of a non-degenerate bilinear space V. Then $U^{\perp}$ is a closed subspace.

Proof. It is obvious that $U^{\perp} \subseteq (U^{\perp})^{\perp\perp}$. Conversely, since $U \subseteq U^{\perp\perp}$ we obtain $(U^{\perp})^{\perp\perp}$=$(U^{\perp\perp})^{\perp} \subseteq U^{\perp}$. Thus $U^{\perp} = (U^{\perp})^{\perp\perp}$, which implies $U^{\perp}$ is closed.

The following is the Riesz representation theorem on any non-degenerate bilinear space.

Theorem 2.3 Let ϕ be a linear functional on a non-degenerate bilinear space V. Then there exists a unique $z_{\phi} \in V$ such that

$$\phi(v) = [v, z_{\phi}] \text{ for all } v \in V \tag{1}$$

if and only if $Ker(\phi)$ is a closed subspace.

Proof. The uniqueness element that satisfies condition Eq. (1) is guaranteed by the bilinear property of being non-degenerate. It is obvious that the theorem holds if the linear functional is the zero map; that is $z_{\phi} = 0$. Hence, we assume that the linear functional ϕ is a nonzero map. That means there exists an $x \in$ V such that $\phi(x) \neq 0$.

($\Longrightarrow$) Based on the premise that there exists a $z_{\phi} \in V$ such that Eq. (1) holds. Since ϕ is a nonzero map we obtain $z_{\phi} \neq 0$. Let $\langle z_{\phi} \rangle$ be the nonzero subspace generated by z_{ϕ}. It is easy to show that $Ker(\phi) = \langle z_{\phi} \rangle^{\perp}$ the orthogonal complement of the subspace $\langle z_{\phi} \rangle$. By Lemma 2.2. we obtain $Ker(\phi)$ is closed.

($\Longleftarrow$) Let $U = Ker(\phi)$ be a closed subspace. Since ϕ is a nonzero map there exists an $\alpha = \phi(x) \neq 0$. That means $U \subsetneq V$. By Lemma 2.1 there exists a $y \in U^{\perp}$, $y \neq 0$. Define the linear functional on the quotient space $/U$:

$$\bar{\phi} : \begin{array}{rl} V/U & \longrightarrow F \\ \bar{v} = v + U & \longmapsto \phi(v). \end{array}$$

According to the first isomorphism theorem (see for example Roman [3]) we obtain $\bar{\phi}$ is an isomorphism with $\bar{\phi}(\bar{x}) \neq 0$. Hence $V/U = \langle \bar{x} \rangle$ and $V = U \oplus \langle x \rangle$ for the above $x \in V$ with $\phi(x) = \alpha$. Since V is non-degenerate, $y \in U^{\perp}$ and $y \neq 0$ then $[x, y] \neq 0$. Let us denote $\beta = [x, y]$ and $z_{\phi} = \alpha\beta^{-1}y \in V$. We obtain that for every $v = u + \gamma x \in V$ with $u \in U$ and $\gamma \in F$, $\phi(v) = \phi(u + \gamma x) = \gamma\phi(x) = \gamma\alpha$.

$$[v, z_{\phi}] = [u + \gamma x, y] = \alpha\beta^{-1}[u, y] + \gamma\alpha\beta^{-1}[x, y] = \gamma\alpha.$$

Thus, Eq. (1) holds for $z_{\phi} = \alpha\beta^{-1}y$.

3 Spaces of Truncated Laurent Series

So far we have discussed the Riesz representation theorem on bilinear spaces. In this section we explain the Riesz representation theorem on a certain class of bilinear spaces, i.e. spaces of truncated Laurent series with the bilinear forms developed by Fuhrmann [1].

To begin with, we discuss Fuhrmann's bilinear forms on spaces of truncated Laurent series defined in [1]. A space of truncated Laurent series over the field F consists of a series with coefficients in F^n, the space of all n-column vectors with components in F, denoted by

$$F^n\big((z^{-1})\big) := \left\{ \sum_{j=-\infty}^{N_f} f_j z^j \,\middle|\, f_j \in F^n, N_f \in \mathbb{Z} \right\}.$$

The space $F^n\big((z^{-1})\big)$ is equipped with the standard operation of addition and the standard action of any element in the field F on any vector in $F^n\big((z^{-1})\big)$. It is easy to see that the space $F^n\big((z^{-1})\big)$ can be decomposed as a direct sum of $F^n[z]$, the subspace of polynomials with coefficients in F^n, and $z^{-1}F^n\big[[z^{-1}]\big]$, the subspace of formal series starting with z^{-1},

$$F^n\big((z^{-1})\big) = F^n[z] \oplus z^{-1}F^n\big[[z^{-1}]\big].$$

Fuhrmann [1] defined a bilinear form on the space of truncated Laurent series as follows:

$$[f, g] = \sum_{j=-\infty}^{\infty} {g_{-j-1}}^t \; f_j$$

for any $f = \sum_{j=-\infty}^{N_f} f_j z^j$ and $g = \sum_{j=-\infty}^{N_g} g_j z^j$ in $F^n\big((z^{-1})\big)$ and g_j^t denotes the transpose of g_j. It is routine to show that this bilinear form is symmetric and non-degenerate.

The Riesz representation theorem on spaces of truncated Laurent series is given in the following theorem.

Theorem 3.1 Let $V = F^n\big((z^{-1})\big)$ be a space of truncated Laurent series and ϕ be a linear functional on V. Then the following statements are equivalent:

1. There exists a unique $g \in V$ such that $\phi(f) = [f, g]$ for all $f \in V$.
2. There exists an integer $k \in \mathbb{Z}$ such that $z^{-k}F^n\big[[z^{-1}]\big] \subseteq Ker(\phi)$.

Proof. The uniqueness of g in Statement 1 is guaranteed by the non-degenerate property of the bilinear $[-,-]$.

$(1.\Longrightarrow 2.)$ According to the assumptions, let $g = \sum_{j=-\infty}^{N_g} g_j z^j \in V$ such that

$\phi(h) = [h, g]$ for all $h \in V$.

Put $k = N_g + 2 \in \mathbb{Z}$. Then we obtain that for every $f = \sum_{j=-\infty}^{-k} f_j z^j \in z^{-k}F^n[[z^{-1}]]$,

$$\phi(f) = [f, g] = \sum_{j=-\infty}^{\infty} {g_{-j-1}}^t f_j = 0.$$

$(2.\Longrightarrow 1.)$ For $j \in \mathbb{Z}$, we define a function

$$\psi_j : F^n \longrightarrow \mathrm{F}$$

as follows: for every $v \in F^n$, $\psi_j(v) = \phi(vz^j)$. It is routine to show that ψ_j is a linear functional on the space F^n with properties $\psi_j = 0$ for all $j = -k, (-k-1), (-k-2), \cdots$. According to the Riesz representation theorem, for any $j = \{-k+1, -k+2, \cdots\}$ there exists a unique $g_{-j-1} \in F^n$ such that $\psi_j(v) = {g_{-j-1}}^t v$ for all $v \in F^n$. Define

$$g = g_{k-2}z^{k-2} + g_{k-3}z^{k-3} + \cdots \in F^n((z^{-1})).$$

We will show that for any $f \in V$, $\phi(f) = [f, g]$. Consider $V = V_1 \oplus V_2$ where $V_1 = z^{-k+1}F^n[z], V_2 = z^{-k}F^n[[z^{-1}]]$.

Let $f = \sum_{j=-\infty}^{N_f} f_j z^j \in V$. We will show $\phi(f) = [f, g]$ using two cases, namely when $N_f \le -k$ and when $N_f > -k$. The first case: if $N_f \le -k$ then we obtain $f \in V_2 \subseteq Ker(\phi)$ and also $[f, g] = 0$. The second case: suppose $N_f > -k$.

Let $f = u + v$ where $u = \sum_{j=-k+1}^{N_f} f_j z^j \in V_1$ and $v = \sum_{j=-\infty}^{-k} f_j z^j \in V_2$

Then

$$\phi(f) = \phi(u) = \sum_{j=-k+1}^{N_f} \phi(f_j z^j) = \sum_{j=-k+1}^{N_f} \psi_j(f_j) = \sum_{j=-k+1}^{N_f} {g_{-j-1}}^t f_j = [f, g].$$

Thus $\phi(f) = [f, g]$ for all $f \in V$. □

Note that by comparing Theorem 2.3 and Theorem 3.1 we can conclude that all closed subspaces of co-dimension one are of the form Point 2 in Theorem 3.1. Since the intersection of finite numbers of closed subspaces is also closed, we obtain the following corollary.

Corollary 3.2 Let S be a subspace of $F^n((z^{-1}))$ having finite co-dimension, i.e. there exists a $T \subseteq F^n((z^{-1}))$, a finite dimensional subspace such that $F^n((z^{-1})) = S \oplus T$. Then S is closed if and only if there exists an integer $k \in \mathbb{Z}$ such that $z^{-k}F^n[[z^{-1}]] \subseteq S$.

We conclude this section with the following counterexample of a linear functional that does not satisfy the condition in Theorem 3.1. This means that the linear functional is not representable by a unique element in the space. Let $V = F^n((z^{-1}))$ be a space of truncated Laurent series and $\{e_1, \cdots, e_n\}$ be the standard basis of F^n.

Let $V_1 = Span\{e_i z^j \,|\, i = 1, \cdots, n, j \in \mathbb{Z}\}$ be the subspace of V generated by the linear independent set $\{e_i z^j \,|\, i = 1, \cdots, n, j \in \mathbb{Z}\}$. Let V_2 be a subspace of V such that $V = V_1 \oplus V_2$. Define a linear function on V_1 as the following:

$$\phi : V_1 \longrightarrow \mathrm{F}$$
$$e_i z^j \longmapsto 1.$$

Then, we can expand ϕ to $\bar{\phi}$, a linear function on V, by defining $\bar{\phi}$ as in the following:

$$\bar{\phi} : V \longrightarrow \mathrm{F}$$
$$a + b \longmapsto \phi(a),$$

where $a \in V_1$ and $b \in V_2$. Suppose that $\bar{\phi}$ is representable by an element in the space V. Suppose $g = \sum_{j=-\infty}^{N_g} g_j z^j \in V$ is the unique element in V such that

$\bar{\phi}(f) = [f, g] = \sum_{j=-\infty}^{\infty} g_{-j-1}{}^t \; f_j$ for all $f \in V$.

Then we obtain a contradiction as follows:

$$1 = \phi(e_1 z^{-N_g-2}) = \bar{\phi}(e_1 z^{-N_g-2}) = [e_1 z^{-N_g-2}, g] = 0.$$

Thus Point 1 in Theorem 3.1 does not hold for the above defined linear functional $\bar{\phi}$.

Point 2 in Theorem 3.1 does not hold either. For any integer j , the element $e_1 z^j$ is not in kernel $\bar{\phi}$. As a result we obtain

$$z^{-k}F^n\left[[z^{-1}]\right] \nsubseteq Ker(\phi)$$

for all integers *k*.

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