Stability in Functional Integro-differential Equations of Second Order with Variable Delay

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Abstract. In this paper, we investigate the stability of the zero solution of an integro-differential equation of the second order with variable delay. By means of the fixed point theory and an exponential weighted metric, we find sufficient conditions under which the zero solution of the equation considered is stable.

Keywords: *fixed point; integro-differential equation; non-linear; second order; stability.*

AMS (MOS) 2010 Subject Classifications: 34K20, 45J05.

1 Introduction

Burton [1] investigated the convergence of solutions of nonlinear differential equations of the second order with constant delay, L :

$$\ddot{x}+f(t,x,\dot{x})\dot{x}+b(t)g(x(t-L))=0.$$

The author found sufficient conditions which guarantee that the solutions of the former equation satisfy $(x(t),\dot{x}(t))\to 0$ as $t\to\infty$.

Later, Pi [2,3] discussed the stability of the zero solution of the differential equation of the second order with variable delay, $\tau(t)$:

$$\ddot{x}+f(t,x,\dot{x})\dot{x}+b(t)g(x(t-\tau(t)))=0.$$

Using the fixed point theory and an exponential weighted metric, the author established sufficient conditions which guarantee that the zero solution of the equation considered is stable and asymptotic stable. Pi [4] is concerned with the integro-differential equation of the second order with variable delay, $r(t)$:

$$\ddot{x}+f(t,x,\dot{x})\dot{x}+\int_{t-r(t)}^{t}a(t,s)g(x(s))ds=0.$$

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The author investigated the stability of the zero solution of this equation by means of the fixed point theory and an exponential weighted metric. In addition, for some related works on the qualitative behavior solutions of functional differential and integro-differential equations of the second order, we refer to the reader to the papers and books of Abdollahpour, *et al.* [5], Ardjouni and Djoudi [6], Burton [7,8], Graef and Tunç [9], Hale [10], Jin and Luo [11], Korkmaz and Tunç [12], Levin and Nohel [13,14], Pi [15], Tunç and Biçer [16], Tunç [17], Tunç and Tunç [18], Zhao, *et al.* [19], Zhao and Yuan [20] and the references therein.

Motivated by the works mentioned, we consider the integro-differential equation of the second order with variable delay, $r(t) \geq 0$:

$$\ddot{x} + a_0(t)f(t,x,\dot{x})\dot{x} + b(t)h(x(t-r(t))) + \int_{t-r(t)}^{t} a(t,s)g(x(s))ds = 0, \quad (1)$$

where $b:\Re^+ \to \Re$ is a bounded and continuous function, $a:[0,\infty)\times[-r(0),\infty)\to\Re$, $g:\Re\to\Re$, $f:\Re^+\times\Re\times\Re\to\Re^+$ and a_0, $r:\Re^+\to\Re$, $\Re^+ = [0,\infty)$, are continuous functions.

We can state Eq. (1) as the following system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -a_0(t)f(t,x,y)y - b(t)h(x(t-r(t))) - \int_{t-r(t)}^{t} a(t,s)g(x(s))ds. \end{cases} \quad (2)$$

For each $t_0 \geq 0$, we define $m(t_0) = \inf\{s - r(s) : s \geq t_0\}$. Let $C(t_0) = C([m(t_0),t_0],\Re)$ with the continuous function norm $\|.\|$, where $\|\psi\| = \sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}$. We use $\|\varphi\|$ as the supremum on $[m(t_0),\infty)$.

It is well known that for a given continuous function ϕ and a number y_0, there exists a solution of system Eq. (2) on an interval $[t_0,T)$, and if the solution remains bounded, then $T=\infty$. By $(x(t), y(t))$ we denote the solution $(x(t,t_0,\varphi), y(t,t_0,\varphi))$.

Let $A(t) = a_0(t)f(t,x(t),y(t))$. Then, from the system in Eq. (2), it follows that:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -A(t)y - b(t)h(x(t-r(t)) - \int\limits_{t-r(t)}^{t} a(t,s)g(x(s))ds. \end{cases} \tag{3}$$

Throughout this paper it is assumed that $t - r(t)$ is strictly increasing and $\lim\limits_{t\to\infty}(t - r(t)) = \infty.$ Let $p(t)$ be the inverse of $t - r(t).$ Suppose that $0 \le b(t) \le M_1$ for some constant $M_1 > 0$ and let $G(t,t) = \int\limits_{t}^{p(t)} a(u,t)du$ and $G(t,s) = \int\limits_{t}^{p(s)} a(u,s)du$. There exists a constant $M_2 > 0$ such that $|G(t,t)| \le M_2.$

The following theorem is our main result.

2 Main result

Theorem 2.1 We assume that the following assumptions hold:

(i) There exists a constant $l > 0, (l = \max\{l_1, l_2\})$ such that $h(x)$ and $g(x)$ satisfy the Lipschitz condition on $[-l_1, l_1]$ and $[-l_2, l_2]$ respectively. The function $h(x)$ is odd and strictly increasing on $[-l_1, l_1]$ the function $g(x)$ is odd and strictly increasing on $[-l_2, l_2]$, and $x - h(x)$ is non-decreasing on $[0, l_1], x - g(x)$ is non-decreasing on $[0, l_2].$

(ii) There exist positive constants $\alpha_1, \alpha_2 \in (0,1)$ and a continuous function $\alpha_1(t):[0,\infty) \to [0,\infty)$ such that $a_0(t) f(t,x,y) \ge a_1(t)$ for $t \ge 0.$ For all $t \ge 0,$ $\int\limits_{t-r(t)}^{t} |G(t,v)|dv$ is increasing with respect to $t,$ and $\int\limits_{t-r(t)}^{t} |a(t,v)|dv$ is bounded for $t \ge 0$ and $x, y \in \Re$

$$2\sup_{t\ge 0} \int\limits_{t}^{p(t)} \int\limits_{0}^{\infty} e^{-\int\limits_{s}^{w+s} a_1(v)dv} b(s)dwds + 2\sup_{t\ge 0} \int\limits_{0}^{t} \int\limits_{t-s}^{\infty} e^{-\int\limits_{s}^{w+s} a_1(v)dv} b(s)dwds \le \alpha_1,$$

$$2\int\limits_{t}^{p(t)} \int\limits_{0}^{\infty} e^{-\int\limits_{s}^{w+s} a_1(v)dv} G(s,s)dwds + 2\int\limits_{0}^{t} \int\limits_{t-s}^{\infty} e^{-\int\limits_{s}^{w+s} a_1(v)dv} G(s,s)dwds$$

$$+2\int_0^t e^{-\int_s^t a_1(v)dv} \int_{t-s}^{\infty} |G(s,v)|\, dvds \le \alpha_2.$$

(iii) There exist constants $\alpha_2 > 0$ and $Q > 0$ such that, for each $t \ge 0$, if $J \ge Q$, then:

$$\int_t^{t+J} a_1(s)ds \ge \alpha_2 J.$$

Then there exists $\delta \in (0,1)$ such that for each initial function $\psi : [m(t_0), t_0] \to \Re$ and $\dot{x}(t_0)$ satisfying $|\dot{x}(t_0)| + \|\psi\| \le \delta$, there is a unique continuous function $x : [m(t_0), \infty) \to \Re$ satisfying $x(t) = \psi(t)$, $t \in [m(t_0), t_0]$, which is a solution of Eq. (1) on $[t_0, \infty)$. Moreover, the zero solution of Eq. (1) is stable.

Before giving the proof of the former theorem, we need the following two lemmas.

Lemma 2.1 (Pi [14]) Let the function $p : [-r(0), \infty) \to [0, \infty)$ denote the inverse of $t - r(t)$. Then:

$$\dot{x}(t) = \int_{t-r(t)}^{t} a(t,s) g(x(s)) ds$$

is equivalent to

$$\dot{x}(t) = -G(t,t) g(x(t)) + \frac{d}{dt} \int_{t-r(t)}^{t} G(t,s) g(x(s)) ds.$$

Lemma 2.2 Let $\psi : [m(t_0), t_0] \to \Re$ be a given continuous function. If $(x(t), y(t))$ is the solution of system Eq. (2) satisfying $x(t) = \psi(t), t \in [m(t_0), t_0]$ and $y(t) = \dot{x}(t)$, then $x(t)$ is the solution of the integral equation:

$$x(t) = \psi(t_0) e^{-\int_{t_0}^t K(s)ds} + \int_{t_0}^t e^{-\int_u^t K(s)ds} B(u)du + \int_{t_0}^t e^{-\int_u^t K(s)ds} \hat{R}(u)[x(u) - h(x(u))]du$$

$$+ \int_{t-r(t)}^{t} \hat{R}(s) h(x(s)) ds - e^{-\int_{t_0}^t K(s)ds} \int_{t_0 - r(t_0)}^{t_0} \hat{R}(s) h(x(s)) ds$$

$$-\int_{t_0}^{t}[\int_{u-r(u)}^{u}\hat{R}(s)h(x(s))ds]K(u)e^{-\int_u^t K(s)ds}du+\int_{t_0}^{t}N(t,s)h(x(s-r(s)))ds$$

$$-\int_{t_0}^{t}[\int_{t_0}^{u}N(u,s)h(x(s-r(s)))ds]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}H(u)[x(u)-g(x(u))]du+\int_{t-r(t)}^{t}H(s)g(x(s))ds$$

$$-e^{-\int_{t_0}^t K(s)ds}\int_{t_0-r(t_0)}^{t_0}H(s)g(x(s))ds$$

$$-\int_{t_0}^{t}[\int_{u-r(u)}^{u}H(s)g(x(s))ds]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t}E(t,s)g(x(s-r(s)))ds-\int_{t_0}^{t}[\int_{t_0}^{u}E(u,s)g(x(s-r(s))ds]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}[\int_{u-r(u)}^{u}G(u,v)g(x(v))dv]du$$

$$-\int_{t_0}^{t}e^{-\int_u^t K(s)ds}[e^{-\int_{t_0}^u A(v)dv}\int_{t_0-r(t_0)}^{t_0}G(t_0,v)g(x(v))dv]du$$

$$-\int_{t_0}^{t}e^{-\int_u^t K(s)ds}[\int_{t_0}^{u}[\int_{s-r(s)}^{s}G(s,v)g(x(v))dv]A(s)e^{-\int_s^u A(v)dv}ds]du. \tag{4}$$

Throughout the proof of this lemma, we use the following notations:

$$C(t,u)=e^{-\int_u^t A(s)ds}G(u,u),\ D(t)=\int_{t_0}^{\infty}C(u+t-t_0,t)du,$$

$$\frac{D(t)}{1-\dot{r}(t)}=\tilde{D}(t),\ \tilde{D}(p(t))=H(t),$$

$$E(t,s)=\int_{t_0+t-s}^{\infty}C(u+s-t_0,s)du,\ M(t,u)=e^{-\int_u^t A(s)ds}b(u),$$

$$N(t,s)=\int_{t-s+t_0}^{\infty} M(u+s-t_0,s)du,$$

$$\int_{t_0}^{\infty} M(u+t-t_0,t)du=R(t), \frac{R(t)}{1-\dot{r}(t)}=\tilde{R}(t), \tilde{R}(p(t))=\hat{R}(t).$$

Proof. Let:

$$B(t)=\dot{x}(t_0)\exp(-\int_{t_0}^{t} A(s)ds). \tag{5}$$

It can be written from Eq. (3) and Eq. (5) that:

$$\dot{x}(t)=B(t)-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)h(x(u-r(u)))du$$
$$-\int_{t_0}^{t} e^{-\int_s^t A(v)dv} (\int_{s-r(s)}^{s} a(s,v)g(x(v))dv)ds. \tag{6}$$

In view of Lemma 2.1 and Eq. (6), it follows that:

$$\dot{x}(t)=B(t)-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)h(x(u-r(u)))du$$
$$-\int_{t_0}^{t} e^{-\int_s^t A(v)dv} [G(s,s)g(x(s))-\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))dv]ds. \tag{7}$$

Hence:

$$\dot{x}(t)=B(t)-\int_{t_0}^{t} e^{-\int_u^t A(s)ds} b(u)h(x(u-r(u)))du-\int_{t_0}^{t} C(t,s)g(x(s))ds$$
$$+\int_{t_0}^{t} e^{-\int_s^t A(v)dv} (\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))dv)ds. \tag{8}$$

By $\left|G(t,t)\right|\le M_2$, $s\le t$, we get:

$$\left|\int_{t-s+t_0}^{\infty} C(u+s-t_0,s)du\right| = \left|\int_{t-s+t_0}^{\infty} e^{-\int_s^{u+s-t_0} A(v)dv} G(s,s)du\right|$$

$$\leq \int_{t-s}^{\infty} e^{-\int_s^{w+s} A(v)dv} |G(s,s)| dw$$

$$= \int_{t-s}^{Q} e^{-\int_s^{w+s} A(v)dv} |G(s,s)| dw + \int_{Q}^{\infty} e^{-\int_s^{w+s} A(v)dv} |G(s,s)| dw,$$

$$\int_{Q}^{\infty} e^{-\int_s^{w+s} A(v)dv} |G(s,s)| dw \leq \int_{Q}^{\infty} e^{-\int_s^{w+s} A(v)dv} M_2 dw \leq M_2 \frac{e^{-a_2 Q}}{a_2}.$$

The former first relation implies that the integral $\int_{t_0+t-s}^{\infty} C(u+s-t_0,s)du$ is convergent. Set $u-t_0=w,\ s\leq t.$ Then, in view of the assumptions of the theorem, we have:

$$\int_{t-s+t_0}^{\infty} M(u+s-t_0,s)du = \int_{t-s}^{\infty} e^{-\int_s^{w+s} A(s)ds} dw \leq \int_{t-s}^{\infty} e^{-\int_s^{w+s} a_1(s)ds} b(s)dw$$

$$\leq \int_0^{\infty} e^{-a_2 w} b(s)dw.$$

Since the function b is bounded, the integral $\int_{t-s+t_0}^{\infty} M(u+s-t_0,s)du$ exists. Hence, Eq. (8) can be written as:

$$\begin{aligned}\dot{x}(t) &= B(t) - h(x(t-r(t)))\int_{t_0}^{\infty} M(u+t-t_0,t)du \\ &+ \frac{d}{dt}\int_{t_0}^{t} N(t,s)h(x(s-r(s)))ds - g(x(t-r(t)))D(t) \\ &+ \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-r(s)))ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))dv\right)ds. \qquad (9)\end{aligned}$$

Then, we have:

$$\dot{x}(t) = B(t) - \tilde{R}(p(t))h(x(t)) + \frac{d}{dt}\int_{t-r(t)}^{t} \tilde{R}(p(s))h(x(s))ds$$

$$+\frac{d}{dt}\int_{t_0}^{t} N(t,s)h(x(s-r(s)))ds - \tilde{D}(p(t))g(x(t))$$

$$+\frac{d}{dt}\int_{t-r(t)}^{t} \tilde{D}(p(s))g(x(s))ds + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-r(s)))ds$$

$$+\int_{t_0}^{t} e^{-\int_s^t A(v)dv}\left(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))dv\right)ds. \tag{10}$$

Therefore, from Eq. (9) and Eq. (10), it follows that:

$$\dot{x}(t) = B(t) - \hat{R}(t)x(t) + \hat{R}(t)[x(t) - h(x(t))] + \frac{d}{dt}\int_{t-r(t)}^{t} \hat{R}(s)h(x(s))ds$$

$$+\frac{d}{dt}\int_{t_0}^{t} N(t,s)h(x(s-r(s)))ds - H(t)x(t) + H(t)[x(t) - g(x(t))]$$

$$+\frac{d}{dt}\int_{t-r(t)}^{t} H(s)g(x(s))ds + \frac{d}{dt}\int_{t_0}^{t} E(t,s)g(x(s-r(s)))ds.$$

For all $t \in [t_0, T_1]$ by the variation of parameters formula, we obtain:

$$x(t) = x(t_0)e^{-\int_{t_0}^t K(s)ds} + \int_{t_0}^{t} e^{-\int_u^t K(s)ds} B(u)du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds} \hat{R}(u)[x(u) - h(x(u))]du + \int_{t_0}^{t} e^{-\int_u^t K(s)ds} H(u)[x(u) - g(x(u))]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\left[\frac{d}{du}\int_{u-r(u)}^{u} \hat{R}(s)h(x(s))ds\right]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\left[\frac{d}{du}\int_{t_0}^{u} N(u,s)h(x(s-r(s)))ds\right]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\left[\frac{d}{du}\int_{u-r(u)}^{u} H(s)g(x(s))ds\right]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[\frac{d}{du}\int_{t_0}^{u} E(u,s)g(x(s-r(s)))ds\Big]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[\int_{t_0}^{u} e^{-\int_s^u A(v)dv}\Big(\frac{d}{ds}\int_{s-r(s)}^{s} G(s,v)g(x(v))dv\Big)ds\Big]du. \tag{11}$$

If we use the integration by parts for the last five terms in Eq. (11), then we have:

$$x(t)=\psi(t_0)e^{-\int_{t_0}^t K(s)ds}+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}B(u)du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\hat{R}(u)[x(u)-h(x(u))]du$$

$$+\int_{t-r(t)}^{t}\hat{R}(s)h(x(s))ds-e^{-\int_{t_0}^t K(s)ds}\int_{t_0-r(t_0)}^{t_0}\hat{R}(s)h(x(s))ds$$

$$-\int_{t_0}^{t}\Big[\int_{u-r(u)}^{u}\hat{R}(s)h(x(s))ds\Big]K(u)e^{-\int_u^t K(s)ds}du+\int_{t_0}^{t} N(t,s)h(x(s-r(s)))ds$$

$$-\int_{t_0}^{t}\Big[\int_{t_0}^{u} N(u,s)h(x(s-r(s)))ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}H(u)[x(u)-g(x(u))]du+\int_{t-r(t)}^{t} H(s)g(x(s))ds$$

$$-e^{-\int_{t_0}^t K(s)ds}\int_{t_0-r(t_0)}^{t_0} H(s)g(x(s))ds-\int_{t_0}^{t}\Big[\int_{u-r(u)}^{u} H(s)g(x(s))ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} E(t,s)g(x(s-r(s)))ds-\int_{t_0}^{t}\Big[\int_{t_0}^{u} E(u,s)g(x(s-r(s)))ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[\int_{u-r(u)}^{u} G(u,v)g(x(v))dv\Big]du$$

$$-\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[e^{-\int_{t_0}^{u} A(v)dv}\int_{t_0-r(t_0)}^{t_0} G(t_0,v)g(x(v))dv)]du$$
$$-\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[\int_{t_0}^{u}[\int_{s-r(s)}^{s} G(s,v)g(x(v))dv)]A(s)e^{-\int_s^u A(v)dv}ds]du.$$

Conversely, suppose that a continuous function $x(t)=\psi(t), t\in[m(t_0),t_0]$ satisfies the integral equation given on $[t_0,T]$. Then this function is differentiable on $[t_0,T_2]$. We just need to differentiate the integral equation and then we obtain the relation Eq. (4). This completes the proof of Lemma 2.2.

Let $(C,\|.\|)$ be the Banach space of bounded continuous functions on $[m(t_0),\infty)$ with the supremum norm. For a given continuous initial function $\psi:[m(t_0),t_0]\to\Re$, define the set $C_\psi \subset C$ by:

$$C_\psi=\{\varphi:[m(t_0),\infty)\to R\,|\,\varphi\in C,\varphi(t)=\psi(t),t\in[m(t_0),t_0]\},$$

$$C_\psi^l=\{\varphi:[m(t_0),\infty)\to R\,|\,\varphi\in C,\varphi(t)=\psi(t),t\in[m(t_0),t_0],|\varphi(t)|\le l,t\ge m(t_0)\},$$

where $\psi:[m(t_0),t_0]\to[-l,l]$ is given as initial function and l is a positive constant. We also use $\|.\|$ to denote the supremum norm of the initial function. Let P_1 be a mapping defined on C_ψ^1 as follows: for $\varphi\in C_\psi^l$, if $t\in[m(t_0),t_0],$, then $(P_1\varphi)(t)=\psi(t)$. In addition, if $t>t_0$, then:

$$(P_1\varphi)(t)=\psi(t_0)e^{-\int_{t_0}^{t}K(s)ds}+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}B(u)du+\int_{t_0}^{t}e^{-\int_u^t K(s)ds}\hat{R}(u)[\varphi(u)-h(\varphi(u))]du$$
$$+\int_{t-r(t)}^{t}\hat{R}(s)h(\varphi(s))ds-e^{-\int_{t_0}^{t}K(s)ds}\int_{t_0-r(t_0)}^{t_0}\hat{R}(s)h(\psi(s))ds$$
$$-\int_{t_0}^{t}[\int_{u-r(u)}^{u}\hat{R}(s)h(\varphi(s))ds]K(u)e^{-\int_u^t K(s)ds}du+\int_{t_0}^{t}N(t,s)h(\varphi(s-r(s)))ds$$
$$-\int_{t_0}^{t}[\int_{t_0}^{u}N(u,s)h(\varphi(s-r(s)))ds]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds} H(u)[\varphi(u)-g(\varphi(u))]du+\int_{t-r(t)}^{t} H(s)g(\varphi(s))ds$$

$$-e^{-\int_{t_0}^t K(s)ds}\int_{t_0-r(t_0)}^{t_0} H(s)g(\psi(s))ds-\int_{t_0}^{t}[\int_{u-r(u)}^{u} H(s)g(\varphi(s))ds]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} E(t,s)g(\varphi(s-r(s)))ds-\int_{t_0}^{t}[\int_{t_0}^{u} E(u,s)g(\varphi(s-r(s)))ds]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[\int_{u-r(u)}^{u} G(u,v)g(\varphi(v))dv)]du$$

$$-\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[e^{-\int_{t_0}^u A(v)dv}\int_{t_0-r(t_0)}^{t_0} G(t_0,v)g(\psi(v))dv)]du$$

$$-\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[\int_{t_0}^{u}[\int_{s-r(s)}^{s} G(s,v)g(\varphi(v))dv)]A(s)e^{-\int_s^u A(v)dv}ds]du.$$

Note that P_1 may not be a contraction mapping. We solve this problem in Lemma 2.3 by introducing an exponential weighted metric.

Lemma 2.3 Suppose that there exist constants $l_1>0, l_2>0, l=\max(l_1,l_2)$ such that $h(x)$ and $g(x)$ satisfy the Lipschitz condition on $[-l_1,l_1]$ and $[-l_2,l_2]$ respectively. Then there exists a metric d on C_ψ^l such that:

(i) The metric space (C_ψ^l,d) is complete;

(ii) P_2 is a contraction mapping on (C_ψ^l,d) if $P_2:C_\psi^l\to C_\psi^l$.

Proof .

(i) We change the supremum norm to an exponentially weighted norm $|\phi|_f$, which is defined on C_ψ^l. Let S be the space of all continuous functions $\varphi:[m(t_0),\infty)\to R$ such that:

$$|\varphi|_f=\sup\{|\varphi(t)|e^{-f(t)}:t\in[m(t_0),\infty)\}<\infty,$$

where $f(t)=kL\int_{t_0}^{t}[\hat{R}(s)+R(s)+H(s)+D(s)+\int_{s-r(s)}^{s}|G(s,v)|dv]ds,$ k is a constant and $k>12,$ and L_1 is the common Lipschitz constant for $x-h(x)$ and $h(x),$ L_2 is the common Lipschitz constant for $x-g(x)$ and $g(x),$ $L=\max(L_1,L_2).$ Then $(S,|.|_f)$ is a Banach space, Thus, (S,d) is a complete metric space, where d denotes the induced metric: $d(\varphi,\eta)=|\varphi-\eta|_f,$ where $\varphi,\eta\in S.$ Under this metric, the space C_ψ^l is a closed subset of $S.$ Therefore, the metric space (C_ψ^l,d) is complete.

(ii) Suppose that $P_1:C_\psi^l\to C_\psi^l.$ For $\varphi,\eta\in C_\psi^l,$ since $\tilde{R}(t)\geq 0,$ $K(t)\geq 0, N(t,s)\geq 0$ and $E(t,s)\geq 0,$ then:

$$
\begin{aligned}
|(P_1\varphi)(t)-(P_1\eta)(t)| \leq & \int_{t_0}^{t} e^{-\int_u^t K(s)ds}\hat{R}(u)|[\varphi(u)-h(\varphi(u))]-[\eta(u)-h(\eta(u))]|du \\
& + \int_{t-r(t)}^{t}\hat{R}(s)|h(\varphi(s))-h(\eta(s))|ds \\
& + \int_{t_0}^{t}K(u)e^{-\int_u^t K(s)ds}[\int_{u-r(u)}^{u}\hat{R}(s)|h(\varphi(s))-h(\eta(s))|ds]du \\
& + \int_{t_0}^{t}N(t,s)|h(\varphi(s-r(s)))-h(\eta(s-r(s)))|ds \\
& + \int_{t_0}^{t}K(u)e^{-\int_u^t K(s)ds}[\int_{t_0}^{u}N(u,s)\times|h(\varphi(s-r(s)))-h(\eta(s-r(s)))|ds]du \\
& + \int_{t_0}^{t}e^{-\int_u^t K(s)ds}H(u)|[(\varphi(u)-g(\varphi(u))]-[\eta(u)-g(\eta(u))]|du \\
& + \int_{t-r(t)}^{t}H(s)|g(\varphi(s))-g(\eta(s))|ds \\
& + \int_{t_0}^{t}K(u)e^{-\int_u^t K(s)ds}[\int_{u-r(u)}^{u}H(s)|g(\varphi(s))-g(\eta(s))|ds]du
\end{aligned}
$$

$$+\int_{t_0}^{t} E(t,s)\left|g(\varphi(s-r(s)))-g(\eta(s-r(s)))\right|ds$$

$$+\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds}\left[\int_{t_0}^{u} E(u,s)\times\left|g(\varphi(s-r(s)))-g(\eta(s-r(s)))\right|ds\right]du$$

$$+\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds}\left[\int_{u-r(u)}^{u} H(s)\left|g(\varphi(s))-g(\eta(s))\right|ds\right]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\left[\int_{u-r(u)}^{u} \left|G(u,v)\right|\left|g(\varphi(v))-g(\eta(v))\right|dv\right]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\left[\int_{t_0}^{u} A(s)e^{-\int_s^u A(v)dv}\times\left(\int_{s-r(s)}^{s} \left|G(s,v)\right|\left|g(\varphi(v))-g(\eta(v))\right|dv\right)ds\right]du.$$

If we use the relation $\hat{R}(t)\le K(t), H(t)\le K(t)\,(\hat{R}(t)+H(t)=K(t)),$ then we have:

$$\left|(P_1\varphi)(t)-(P_1\eta)(t)\right|e^{-f(t)}\le e^{-f(t)}\Big\{\int_{t_0}^{t} e^{-\int_u^t \hat{R}(s)ds}\hat{R}(u)$$

$$\times\left|[\varphi(u)-h(\varphi(u))]-[\eta(u)-h(\eta(u))]\right| + \int_{t-r(t)}^{t} \hat{R}(s)\left|h(\varphi(s))-h(\eta(s))\right|ds$$

$$+\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds}\left[\int_{u-r(u)}^{u} \hat{R}(s)\left|h(\varphi(s)-h(\eta(s))\right|ds\right]du$$

$$+\int_{t_0}^{t} N(t,s)\left|h(\varphi(s-\tau(s)))-h(\eta(s-r(s)))\right|ds$$

$$+\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds}\left[\int_{t_0}^{u} N(u,s)\times\left|h(\varphi(s-r(s)))-h(\eta(s-r(s)))\right|ds\right]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t H(s)ds}H(u)\times\left|[\varphi(u)-g(\varphi(u))]-[\eta(u)-g(\eta(u))]\right|du$$

$$+\int_{t-r(t)}^{t} H(s)\left|g(\varphi(s))-g(\eta(s))\right|ds$$

$$+\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds}[\int_{u-r(u)}^{u} H(s)\left|g(\varphi(s)-g(\eta(s))\right|ds]du$$

$$+\int_{t_0}^{t} E(t,s)\left|g(\varphi(s-r(s)))-g(\eta(s-r(s)))\right|ds$$

$$+\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds}[\int_{t_0}^{u} E(u,s)\times\left|g(\varphi(s-r(s)))-g(\eta(s-r(s)))\right|ds]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[\int_{u-r(u)}^{u}\left|G(u,v)\right|\left|g(\varphi(v))-g(\eta(v)))\right|dv]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}[\int_{t_0}^{u} A(s)e^{-\int_s^u A(v)dv}$$

$$\times\int_{s-r(s)}^{s}\left|G(s,v)\right|\left|g(\varphi(v))-g(\eta(v)))\right|dv)ds]du\}. \qquad (12)$$

For $u \le t, v \le t$ since $D(t), H(t) \ge 0,$ we have:

$$f(u)-f(t) = -kL\int_u^t [\hat{R}(s)+R(s)+H(s)+D(s)+\int_{s-r(s)}^{s}\left|G(s,v)\right|dv]ds,$$

$$f(u)-f(t) \le -kL\int_u^t \hat{R}(s)ds,\ f(u)-f(t) \le -kL\int_u^t R(s)ds,$$

$$f(u)-f(t) \le -kL\int_u^t H(s)ds,\ f(u)-f(t) \le -kL\int_u^t D(s)ds,$$

$$f(u)-f(t) \le -kL\int_u^t (\hat{R}(s)+H(s))ds = -kL\int_u^t K(s)ds,$$

$$f(u)-f(t) \le -kL\int_u^t \int_{s-r(s)}^{s}\left|G(s,v)\right|dvds.$$

For $s \le t,$ it is clear that:

$$f(s-\tau(s))-f(t) \le -kL\int_s^t R(u)du,\ f(s-r(s))-f(t) \le -kL\int_s^t D(u)du.$$

By $N(t,s)\geq 0,$ we have:

$$N(t,s)=\int_{t-s+t_0}^{\infty} M(u+s-t_0,s)du \leq \int_{t_0}^{\infty} M(u+s-t_0,s)du = R(s) \Rightarrow N(t,s)\leq R(s).$$

Since $E(t,s)\geq 0,$ then:

$$E(t,s)=\int_{t-s+t_0}^{\infty} C(u+s-t_0,s)du \leq \int_{t_0}^{\infty} C(u+s-t_0,s)du = D(s) \Rightarrow E(t,s)\leq D(s).$$

For $s\leq u,$ it is also clear that $\int_{s-r(s)}^{s} |G(s,v)|\, dv \leq \int_{u-r(u)}^{u} |G(s,v)|\, dv.$ Hence, for the terms included in Eq. (12), we find:

$$\int_{t_0}^{t} e^{-\int_u^t \hat{R}(s)ds} \hat{R}(u)\left|[\varphi(u)-h(\varphi(u))]-[\eta(u)-h(\eta(u))]\right| e^{-f(t)}du$$

$$\leq \int_{t_0}^{t} e^{-\int_u^t \hat{R}(s)ds} \hat{R}(u)L_1 \left|\varphi(u)-\eta(u)\right| e^{-f(u)}e^{f(u)-f(t)}du$$

$$\leq \int_{t_0}^{t} e^{-\int_u^t \hat{R}(s)ds} \hat{R}(u)e^{-kL\int_u^t \hat{R}(s)ds} duL_1 \left|\varphi-\eta\right|_f \leq \frac{1}{kL} L_1 \left|\varphi-\eta\right|_f ,$$

$$\int_{t-r(t)}^{t} \hat{R}(s)\left|h(\varphi(s)-h(\eta(s))\right| e^{-f(t)}ds$$

$$\leq \int_{t-r(t)}^{t} \hat{R}(s)L_1 \left|\varphi(s)-\eta(s)\right| e^{-f(s)}e^{f(s)-f(t)}ds \leq \frac{1}{kL} L_1 \left|\varphi-\eta\right|_f ,$$

$$\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds} [\int_{u-r(u)}^{u} \hat{R}(s)\left|h(\varphi(s))-h(\eta(s))\right| ds] e^{-f(t)}du$$

$$\leq \int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds} [\frac{1}{kL}e^{-kL\int_u^t \hat{R}(u)du} - \frac{1}{kL}e^{-kL\int_{u-r(u)}^{t} \hat{R}(u)du}]duL_1 \left|\varphi-\eta\right|_f$$

$$\leq \frac{1}{kL}\int_{t_0}^{t} K(u)e^{-\int_u^t K(s)ds} duL_1 \left|\varphi-\eta\right|_f \leq \frac{1}{kL} L_1 \left|\varphi-\eta\right|_f ,$$

$$\int_{t_0}^{t} N(t,s)\left|h(\varphi(s-r(s)))-h(\eta(s-r(s)))\right| e^{-f(t)}ds$$

$$\leq \int_{t_0}^{t} R(s) e^{-kL\int_s^t R(u)du} ds L_1 |\varphi - \eta|_f \leq \frac{1}{kL} L_1 |\varphi - \eta|_f ,$$

$$\int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} [\int_{t_0}^{u} N(t,s) |h(\varphi(s - r(s))) - h(\eta(s - r(s)))| ds] e^{-f(t)} du$$

$$\leq \frac{1}{kL} \int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} du L_1 |\varphi - \eta|_f \leq \frac{1}{kL} L_1 |\varphi - \eta|_f ,$$

$$\int_{t_0}^{t} e^{-\int_u^t H(s)ds} H(u) |[\varphi(u) - g(\varphi(u))] - [(\eta(u) - g(\eta(u))]| e^{-f(t)} du$$

$$\leq \int_{t_0}^{t} e^{-(kL+1)\int_u^t H(s)ds} H(u) du L_2 |\varphi - \eta|_f = [\frac{1}{kL+1} - \frac{1}{kL} e^{-(kL+1)\int_{t_0}^t H(s)ds}] L_2 |\varphi - \eta|_f$$

$$\leq \frac{1}{kL} L_2 |\varphi - \eta|_f ,$$

$$\int_{t-r(t)}^{t} H(s) |g(\varphi(s)) - g(\eta(s))| e^{-f(t)} ds \leq \frac{1}{kL} L_2 |\varphi - \eta|_f ,$$

$$\int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} [\int_{u-r(u)}^{u} H(s) |g(\varphi(s)) - g(\eta(s))| ds] e^{-f(t)} du$$

$$\leq \int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} [\int_{u-r(u)}^{u} H(s) L_2 |\varphi(s) - \eta(s)| e^{-f(s)} e^{f(s)-f(t)} ds] du$$

$$\leq \frac{1}{kL} \int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} du L_2 |\varphi - \eta|_f \leq \frac{1}{kL} L_2 |\varphi - \eta|_f ,$$

$$\int_{t_0}^{t} E(t,s) |g(\varphi(s - r(s)) - g(\eta(s - r(s)))| e^{-f(t)} ds$$

$$\leq \int_{t_0}^{t} D(s) e^{-kL\int_s^t D(u)du} ds L_2 |\varphi - \eta|_f \leq \frac{1}{kL} L_2 |\varphi - \eta|_f ,$$

$$\int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} [\int_{t_0}^{u} E(u,s) |g(\varphi(s - r(s))) - g(\eta(s - r(s)))| ds] e^{-f(t)} du$$

$$\le \int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} [\int_{t_0}^{u} D(s) e^{-kL\int_s^t D(u)du} ds] du L_2 \left|\varphi-\eta\right|_f$$

$$\le \frac{1}{kL}\int_{t_0}^{t} K(u) e^{-\int_u^t K(s)ds} du L_2 \left|\varphi-\eta\right|_f \le \frac{1}{kL} L_2 \left|\varphi-\eta\right|_f ,$$

$$\int_{t_0}^{t} e^{-\int_u^t K(s)ds} [\int_{u-r(u)}^{u} \left|G(u,v)\right|\left|g(\varphi(v))-g(\eta(v)))\right| dv] e^{-f(t)} du$$

$$\le \frac{1}{kL}\int_{t_0}^{t} e^{-\int_u^t K(s)ds} d(e^{-kL\int_u^t \int_{s-r(s)}^{s} \left|G(s,w)\right| dwds}) du L_2 \left|\varphi-\eta\right|_f \le \frac{1}{kL} L_2 \left|\varphi-\eta\right|_f ,$$

$$\int_{t_0}^{t} e^{-\int_u^t K(s)ds} [\int_{t_0}^{u} A(s) e^{-\int_s^u A(v)dv} (\int_{s-r(s)}^{s} \left|G(s,v)\right|\left|g(\varphi(v))-g(\eta(v)))\right| dv) ds] e^{-f(t)} du$$

$$\le \int_{t_0}^{t} e^{-\int_u^t K(s)ds} e^{f(u)-f(t)} (\int_{t_0}^{u} A(s) e^{-\int_s^u A(v)dv} (\int_{u-r(u)}^{u} \left|G(u,v)\right| dv) du L_2 \left|\varphi-\eta\right|_f$$

$$\le \int_{t_0}^{t} e^{-\int_u^t K(s)ds} [\int_{u-r(u)}^{u} \left|G(u,v)\right| dv] e^{-kL\int_u^t \int_{s-r(s)}^{s} \left|G(s,w)\right| dwds} du L_2 \left|\varphi-\eta\right|_f$$

$$\le \frac{1}{kL}\int_{t_0}^{t} e^{-\int_u^t K(s)ds} d(e^{-kL\int_u^t \int_{s-r(s)}^{s} \left|G(s,w)\right| dwds}) du L_2 \left|\varphi-\eta\right|_f$$

$$\le \{\frac{1}{kL} - \frac{1}{kL} e^{-\int_{t_0}^t K(s)ds} e^{-kL\int_{t_0}^t \int_{s-r(s)}^{s} \left|G(s,w)\right| dwds}\} L_2 \left|\varphi-\eta\right|_f$$

$$-\frac{1}{kL}\int_{t_0}^{t} e^{-\int_u^t K(s)ds} K(u) e^{-kL\int_u^t \int_{s-r(s)}^{s} \left|G(s,w)\right| dwds} du L_2 \left|\varphi-\eta\right|_f \le \frac{1}{kL} L_2 \left|\varphi-\eta\right|_f ,$$

In view of the discussion made, by an easy calculation, we get:

$$\left|(P_1\varphi)(t)-(P_1\eta)(t)\right| e^{-f(t)} \le \frac{5}{kL} L_1 \left|\varphi-\eta\right|_f + \frac{7}{kL} L_2 \left|\varphi-\eta\right|_f$$

$$\le \frac{12}{kL}\left|\varphi-\eta\right|_f , \; t > t_0 .$$

For $t \in [m(t_0), t_0]$, we have $(P_1\varphi)(t) = (P_1\eta)(t) = \psi(t)$. Hence, $d(P_1\varphi - P_1\eta) \le \frac{12}{k} d(\varphi - \eta)$, $k > 12$. Thus, P_1 is a contraction mapping on (C_ψ^l, d).

We continue to prove the theorem. Choose $\psi : [m(t_0), t_0] \to R$ and $\dot{x}(t_0)$ such that:

$$(Q + \frac{e^{-a_2 Q}}{a_2})\left|\dot{x}(t_0)\right| + \delta + h(\delta) \int_{t_0 - r(t_0)}^{t_0} \hat{R}(s)ds \le (1 - \alpha_1)h(l),$$

$$(Q + \frac{e^{-a_2 Q}}{a_2})\delta + g(\delta) \int_{t_0 - r(t_0)}^{t_0} H(s)ds \le (1 - \alpha_2)g(l) - l. \tag{13}$$

Since (i) implies that $g(0)=h(0)=0$ the $g(l) \le l$, $h(l) \le l$. In addition, since $g(x)$ and $h(x)$ satisfy the Lipschitz condition on $[-l,l]$, then $g(x)$ and $h(x)$ are continuous on $[-l,l]$ such that there exists a δ with $\delta < 1$. By the expression for $(P_2\varphi)$ (t) we have:

$$(P_2\varphi)(t) = \psi(t_0)e^{-\int_{t_0}^{t} K(s)ds} + \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} B(u)du + \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} \hat{R}(u)[\varphi(u) - h(\varphi(u))]du$$

$$+ \int_{t-r(t)}^{t} \hat{R}(s)h(\varphi(s))ds - e^{-\int_{t_0}^{t} K(s)ds} \int_{t_0 - r(t_0)}^{t_0} \hat{R}(s)h(\psi(s))ds$$

$$- \int_{t_0}^{t} [\int_{u-r(u)}^{u} \hat{R}(s)h(\varphi(s))ds]K(u)e^{-\int_{u}^{t} K(s)ds} du + \int_{t_0}^{t} N(t,s)h(\varphi(s - r(s)))ds$$

$$- \int_{t_0}^{t} [\int_{t_0}^{u} N(u,s)h(\varphi(s - r(s)))ds]K(u)e^{-\int_{u}^{t} K(s)ds} du$$

$$+ \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} H(u)[\varphi(u) - g(\varphi(u))]du + \int_{t-r(t)}^{t} H(s)g(\varphi(s))ds$$

$$- e^{-\int_{t_0}^{t} K(s)ds} \int_{t_0 - r(t_0)}^{t_0} H(s)g(\psi(s))ds - \int_{t_0}^{t} [\int_{u-r(u)}^{u} H(s)g(\varphi(s))ds]K(u)e^{-\int_{u}^{t} K(s)ds} du$$

$$+ \int_{t_0}^{t} E(t,s)g(\varphi(s - r(s)))ds - \int_{t_0}^{t} [\int_{t_0}^{u} E(u,s)g(\varphi(s - r(s)))ds]K(u)e^{-\int_{u}^{t} K(s)ds} du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[\int_{u-r(u)}^{u} G(u,v)g(\varphi(v))dv\Big)\Big]du$$

$$-\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[e^{-\int_{t_0}^{u} A(v)dv}\int_{t_0-r(t_0)}^{t_0} G(t_0,v)g(\psi(v))dv\Big)\Big]du$$

$$-\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[\int_{t_0}^{u}\Big[\int_{s-r(s)}^{s} G(s,v)g(\varphi(v))dv\Big)\Big]A(s)e^{-\int_s^u A(v)dv}ds\Big]du.$$

Thus,

$$|(P_1\varphi)(t)| \le \delta + \int_{t_0}^{t} e^{-\int_u^t K(s)ds}|\dot{x}(t_0)|e^{-\int_{t_0}^{u} A(s)ds}du + \int_{t_0}^{t} e^{-\int_u^t K(s)ds}\hat{R}(u)[l_1 - h(l_1)]du$$

$$+\int_{t-r(t)}^{t}\hat{R}(s)h(l_1)ds + \int_{t_0-r(t_0)}^{t_0}\hat{R}(s)h(\delta)ds$$

$$+\int_{t_0}^{t}\Big[\int_{u-r(u)}^{u}\hat{R}(s)h(l_1)ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} N(t,s)h(l_1)ds + \int_{t_0}^{t}\Big[\int_{t_0}^{u} N(t,s)h(l_1)ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}H(u)[l_2 - g(l_2)]du + \int_{t-r(t)}^{t} H(s)g(l_2)ds$$

$$+\int_{t_0-r(t_0)}^{t_0} H(s)g(\delta)ds + \int_{t_0}^{t}\Big[\int_{u-r(u)}^{u} H(s)g(l_2)ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} E(t,s)g(l_2)ds + \int_{t_0}^{t}\Big[\int_{t_0}^{u} E(u,s)g(l_2)ds\Big]K(u)e^{-\int_u^t K(s)ds}du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[\int_{u-r(u)}^{u} G(u,v)g(l_2)dv\Big]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds}\Big[e^{-\int_{t_0}^{u} A(v)dv}\int_{t_0-r(t_0)}^{t_0} G(t_0,v)g(\delta)dv\Big]du$$

$$+\int_{t_0}^{t} e^{-\int_u^t K(s)ds} [\int_{t_0}^{u} [\int_{s-r(s)}^{s} G(s,v)g(l_2)dv]A(s)e^{-\int_s^u A(v)dv} ds]du. \tag{14}$$

If we do the necessary calculations, for some terms included in Eq. (14), we get:

$$\int_{t_0}^{t} N(t,s)ds = \int_{t_0}^{t}\int_{t_0+t-s}^{\infty} M(u+s-t_0,s)duds = \int_{t_0}^{s}\int_{t_0+t-s}^{\infty} e^{-\int_s^{u+s-t_0} A(v)dv} b(s)duds$$

$$\leq \sup_{t\geq 0}\int_0^t\int_{t-s}^{\infty} e^{-\int_s^{u+s} a_1(v)dv} b(s)duds,$$

$$\int_{t-r(t)}^{t} \hat{R}(s)ds = \int_{t-r(t)}^{t} \tilde{R}(p(s))ds = \int_{t-r(t)}^{t} \frac{R(p(s))}{1-\dot{r}(s)}ds = \int_{t}^{p(t)} R(s)ds$$

$$\leq \sup_{t\geq 0}\int_t^{p(t)}\int_0^{\infty} e^{-\int_s^{u+s} a_1(v)dv} b(s)duds,$$

$$\int_{t_0}^{t} e^{-\int_u^t K(s)ds} \hat{R}(u)(l_1-h(l_1))du \leq \int_{t_0}^{t} e^{-\int_u^t K(s)ds} K(u)(l_1-h(l_1))du,\ (K(t)=\hat{R}(t)+H(t))$$

$$= (l_1-h(l_1))(1-e^{-\int_{t_0}^t K(s)ds})$$

$$\leq (l_1-h(l_1)),$$

$$\int_{t_0}^{t} e^{-\int_u^t K(s)ds} |\dot{x}(t_0)| e^{-\int_{t_0}^u A(s)ds} du \leq \int_{t_0}^{t} |\dot{x}(t_0)| e^{-\int_{t_0}^u A(s)ds} du,$$

$$\int_{t_0}^{t} e^{-\int_{t_0}^u A(s)ds} du = \int_{t_0}^{t_0+Q} e^{-\int_{t_0}^u A(s)ds} du + \int_{t_0+Q}^{t} e^{-\int_{t_0}^u A(s)ds} du \leq Q + \frac{e^{-a_2 Q}}{a_2},$$

$$\int_{t_0}^{t} e^{-\int_u^t K(s)ds} H(u)[l_2-g(l_2)]du \leq \int_{t_0}^{t} e^{-\int_u^t K(s)ds} K(u)(l_2-g(l_2))du,$$

$$= (l_2-g(l_2))\int_{t_0}^{t} \frac{d}{du}[e^{-\int_u^t K(s)ds}]du \leq (l_2-g(l_2)),$$

$$\int_{t-r(t)}^{t} H(s)ds = \int_{t-r(t)}^{t} \tilde{D}(p_2(s))ds = \int_{t-r(t)}^{t} \frac{D(p_2(s))}{1-\dot{r}(s)}ds = \int_{t}^{p(t)} D(s)ds$$

$$\leq \int_{t}^{p(t)} \int_{0}^{\infty} e^{-\int_{s}^{w+s} a_1(v)dv} G(s,s)dwds,$$

$$\int_{t_0}^{t} E(t,s)ds = \int_{t_0}^{t} \int_{t-s+t_0}^{\infty} C(u+s-t_0, s)duds = \int_{t_0}^{t} \int_{t-s+t_0}^{\infty} e^{-\int_{s}^{u+s-t_0} A(s)ds} G(s,s)duds$$

$$\leq \int_{0}^{t} \int_{t-s}^{\infty} e^{-\int_{s}^{w+s} a_1(v)dv} G(s,s)dwds,$$

$$\int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} [e^{-\int_{t_0}^{u} A(v)dv} \int_{t_0-r(t_0)}^{t_0} G(t_0,v)g(\psi(\delta))dv]du \leq \delta \int_{t_0}^{t} e^{-\int_{t_0}^{u} A(v)dv} du$$

$$= \delta(\int_{t_0}^{t_0+Q} e^{-\int_{t_0}^{u} A(s)ds} du + \int_{t_0+Q}^{t} e^{-\int_{t_0}^{u} A(s)ds} du) \delta(Q + \frac{e^{-a_2 Q}}{a_2}),$$

$$\int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} [\int_{u-r(u)}^{u} G(u,v)g(l_2)dv]du$$

$$+ \int_{t_0}^{t} e^{-\int_{u}^{t} K(s)ds} [\int_{t_0}^{u} [\int_{s-r(s)}^{s} G(s,v)g(l_2)dv] A(s) e^{-\int_{s}^{u} A(v)dv} ds]du$$

$$\leq 2g(l_2) \int_{0}^{t} e^{-\int_{u}^{t} a(v)dv} \int_{s-r(s)}^{s} \left|G(s,v)\right| dvdu.$$

So, we find:

$$\left|(P_2\varphi)(t)\right| \leq \delta + (Q + \frac{e^{-a_2 Q}}{a_2}) \left|\dot{x}(t_0)\right| + (l_1 - h(l_1)) + 2h(l_1)$$

$$\times \sup_{t \geq 0} \int_{t}^{p(t)} \int_{0}^{\infty} e^{-\int_{s}^{u+s} a(v)dv} b(s)duds + h(\delta) \int_{t_0-r(t_0)}^{t_0} \hat{R}(s)ds$$

$$+ 2h(l_1) \sup_{t \geq 0} \int_{0}^{t} \int_{t-s}^{\infty} e^{-\int_{s}^{u+s} a(v)dv} b(s)duds + (l_2 - g(l_2))$$

$$+ 2g(l_2) \int_{t}^{p(t)} \int_{0}^{\infty} e^{-\int_{s}^{w+s} a(v)dv} G(s,s)dwds + g(\delta) \int_{t_0-r(t_0)}^{t_0} H(s)ds$$

$$+2g(l_2)\int_0^t\int_{t-s}^{\infty} e^{-\int_s^{w+s} a(v)dv} G(s,s)dwds \ +2g(l_2)\int_{t_0}^t e^{-\int_u^t a(v)dv} \int_{s-r(s)}^{s} |G(s,v)|dvdu$$

$$+\delta(Q+\frac{e^{-a_2Q}}{a_2}).$$

In view of assumption (ii), we can obtain:

$$|(P_2\varphi)(t)| \le \delta + (Q+\frac{e^{-a_2Q}}{a_2})|\dot{x}(t_0)| + (l_1 - h(l_1)) + \alpha_1 h(l_1) + h(\delta)\int_{t_0-r(t_0)}^{t_0} \hat{R}(s)ds$$

$$+(l_2 - g(l_2)) + \alpha_2 g(l_2) + g(\delta)\int_{t_0-r(t_0)}^{t_0} H(s)ds + \delta(Q+\frac{e^{-a_2Q}}{a_2}).$$

Hence,

$$|(P_2\varphi)(t)| \le (1-\alpha_1)h(l) + (l-h(l)) + \alpha_1 h(l)$$

$$+(l-g(l)) + \alpha_2 g(l) + (1-\alpha_2)g(l) - l = l.$$

Observe that if $t\in[m(t_0),t_0]$, then $(P_2\varphi)(t)=\psi(t)$. We can obtain $|(P_2\varphi)(t)|\le l,\ t\in[m(t_0),\infty)$. Thus, $P_2\varphi: C_\psi^l \to C_\psi^l$. We have proved P_2 is a contraction mapping, hence P_2 has a unique fixed point $x(t)$ and $|x(t)|\le l$.

From Eq. (6), we have:

$$|y(t)| \le |\dot{x}(t)| e^{-\int_{t_0}^t A(s)ds} + \int_{t_0}^t e^{-\int_u^t A(s)ds} b(u)|h(x(u-r(u)))|du$$

$$+\int_{t_0}^t e^{-\int_s^t A(v)dv} (\int_{s-r(s)}^{s} |a(s,v)||g(x(v))|dv)ds.$$

Since for $t\in[0,\infty)$, $0\le b(t)\le M_1$ and $\int_{t-r(t)}^{t}|a(t,v)|dv$ is bounded, and there exists a constant $N>0$ such that $\int_{t-r(t)}^{t}|a(t,v)|dv \le N$, then:

$$|y(t)| \le |\dot{x}(t)| + M_1\int_{t_0}^t e^{-\int_u^t A(s)ds} |h(x(u-r(u)))|du + NLl\int_{t_0}^t e^{-\int_s^t A(v)dv} ds$$

$$\leq \left(1 + (M_1 + NL)(Q + \frac{\exp(-a_2 Q)}{a_2}) \right).$$

Thus, it follows that:

$$|x(t)| + |y(t)| \leq l \left(2 + (M_1 + NL)(Q + \frac{e^{-a_2 Q}}{a_2}) \right).$$

To show the stability of zero solution, let $\forall \varepsilon > 0$ be given; we only need to replace ε by l. This completes the proof of the theorem.

3 Conclusion

A functional integro-differential equation of the second order with variable delay was considered. The stability of the zero solution of this equation was discussed by the fixed point theory subject to an exponential weighted metric. Our result improves and includes some results found in the literature.

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