Auslander Reiten Quiver of Nakayama Algebra type Dynkin Graph A_n

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Abstract. In this paper we will determine Auslander Reiten quiver of Nakayama algebra with quiver type Dynkin graph A_n for all natural number n ≥ 2. The AR-quiver is a visualization of module category of finite dimensional algebras. From the AR-quiver of an algebra A we may know all the isomorphism classes of indecomposable modules in mod A and the homomorphism between them. Once we get the general shape of the AR-quiver of this algebra, we will use it to compute a tilting module of this algebra.

Keywords: *AR-quiver; Nakayama algebra; quiver; Dynkin type A.*

1   Introduction

Representation theory of finite dimensional algebras provides a way to describe an algebra and a module over an algebra using directed graph or quiver. Not only for the algebra and module, the module category of an algebra can also be represented by quiver. The way to do this is by using the theory of almost split sequences and irreducible morphisms. These were introduced by Auslander [1,2] and Auslander and Reiten [3]. Quiver representation of this module category called Auslander-Reiten quiver or simply AR-quiver. In this paper we will determine the AR-quiver of Nakayama algebra with quiver of type A_n for any natural number $n \geq 2$. The method for this construction has been explained in [3]. In this paper we provide an explicit technique which uses almost split sequences of Nakayama algebra given in [4], [3], and also uses the relationship between Nakayama algebra of type A_{n-1} with type A_n to give the construction by induction.

Tilting theory is one of essential tools in the representation theory of algebra. Tilting theory firstly appeared in the study of reflection functors ([5-7]). Tilting theory is a method for constructing a new algebra B from original algebra A such that the module categories are "close" to each other. The method is to construct an A-module T, called a tilting module and set a new algebra $B = \text{End } T_A$. In this paper, as an application of the main theorem, we will construct a

tilting module of Nakayama algebra with quiver type Dynkin graph A_n for all natural number n ≥ 2.

2   Auslander Reiten Quiver

Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field *K*. For an algebra *A*, we denote by mod *A* the category of finitely generated right *A*-modules. An algebra *A* is said to be representation finite if the number of the isomorphism classes of indecomposable right *A*-module is finite. A *K*-algebra *A* is called representation-infinite if *A* is not representation-finite. An *A*-homomorphism module is a section (or a retraction) whenever it admits a left inverse (or a right inverse, respectively).

To construct the AR-quiver of an algebra we need some tools. These tools are the notion of almost split sequences and irreducible morphisms. Before we introduce this theory we give some definitions and theorems from [4] and [2].

Definition 1. Let *L*, *M* be modules in mod *A*. An *A*-module homomorphism

f : *L* → *M* is called **left minimal almost split** if

(a) every $h \in$ End *M* such that $hf = f$ is an automorphism,
(b) *f* is not section,
(c) for every *A*-homomorphism *u*: *L* → U that is not section there exists *u'*: *L* → *U* such that $u'f = u$.

A right minimal almost split homomorphism is defined dually.

Definition 2. A homomorphism *f* : *X* → *Y* in mod *A* is said to be **irreducible** if

(a) *f* is neither a section nor a retraction and
(b) if $f=f_1 f_2$, either f_1 is a retraction or f_2 is a section.

We may think an irreducible morphism as a component of a right (left) minimal almost split by the following theorem.

Theorem 1. (a) Let *f* :*L* → *M* be left minimal almost split in mod *A*. Then *f* is irreducible. Moreover, a homomorphism *f'*:*L*→*M'*of *A*-modules is irreducible if and only if $M' \neq 0$ and there exists a direct sum decomposition $M \cong M' \oplus M''$ and a homomorphism *f''*: *L*→*M''* such that $\begin{bmatrix} f' \\ f'' \end{bmatrix}$: *L* → $M' \oplus M''$ is left minimal almost split. (b) Let *g*: *M* → *N* be right minimal almost split. Then *g* is irreducible. Moreover, a homomorphism *g'*: *M'*→ *N* of *A*-modules is irreducible

if and only if $M'\neq 0$ and there exists decomposition $M \cong M'\oplus M''$ and a homomorphism g'': $M''\to N$ such that $[g'g'']$: $M'\oplus M''\to N$ is right minimal almost split.

Proof. See [2, section IV.1].

Theorem 2. (a) Let S be a simple projective noninjective module in mod A. If f: $S \to M$, then M is projective. (b) Let S be a simple injective nonprojective module in mod A. If g: $M \to S$ is irreducible, then M is injective.

Proof. See [2, section IV.3].

Let X and Y be modules in mod A, define rad $_A(X, Y) = \{h \in \text{Hom}(X, Y) : 1_X - gh$ is invertible for any $g \in \text{Hom}(Y, X)\}$.

We define rad $_A^2$ (X, Y) to consist of all A-module homomorphism of the form gf, where $f \in \text{rad}_A(X, Z)$ and $g \in \text{rad}_A(Z, Y)$.

Definition 3. A short exact sequence in mod A

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{1}$$

is called an **almost split sequence** if f is left minimal almost split.

We have several equivalent characterization of almost split sequences.

Theorem 3. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in mod A. The following assertions are equivalent:

(a) The given sequence is almost split.
(b) f is left minimal almost split.
(c) g is right minimal almost split
(d) L and N are indecomposable, and f and g are irreducible.

Proof. See [2, section IV.1].

The following propositions are used to get the right (left) minimal almost split ending (or starting) at an indecomposable projective (or injective, respectively) module.

Proposition 1. (a) Let P be an indecomposable projective module in mod A. An A-module homomorphism g: $M \to P$ is right minimal almost split if and only if g is a monomorphism with image equal to rad P. (b) Let I be an

indecomposable injective module in mod A. An A-module homomorphism $f : I \to M$ is left minimal almost split if and only if f is an epimorphism with kernel equal to soc I.

Proof. See [2, section IV.3].

Let M and N be indecomposable modules in mod A, then $\text{rad}_A(M, N)$ is the K-vector space of all noninvertible homomorphism from M to N. Denote $\text{Irr}(M, N) = \text{rad}_A(M,N) / \text{rad}_A^2 (M, N)$, that is the quotient of the K-vector spaces $\text{rad}_A(M, N)$ and $\text{rad}_A^2 (M, N)$.

Definition 4. Let A be a basic and connected finite dimensional K-algebra. The **Auslander Reiten-quiver $\Gamma(\text{mod } A)$** of mod A is defined as follows:

(a) The points of $\Gamma(\text{mod } A)$ are the isomorphism classes $[X]$ of indecomposable modules X in mod A.
(b) Let $[M]$, $[N]$ be the points in $\Gamma(\text{mod } A)$ corresponding to the indecomposable modules M, N in mod A. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the K-vector space $\text{Irr}(M, N)$.

One of properties of the AR-quiver of representation finite algebra is given by the following proposition.

Proposition 2. Let A be a representation-finite algebra. Then $\Gamma(\text{mod } A)$ has no multiple arrows.

Proof. See [2, section IV.4].

Now we recall a module which can be thought of as being close to the Morita progenerator.

**Definition 5.** A module T in mod A is called a tilting module if the following three conditions are satisfied:

(T1)   the projective dimension of T is at most one,
(T2) $\text{Ext}_A^1 (T,T) = 0$,
(T3)   there exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T', T'' $\in$ add T.

Note that any Morita progenerator is a tilting module and any projective A-module always satisfy both (T1) and (T2). It is easy to see that every tilting module is faithful.

3   The Construction of Nakayama algebra type Dynkin Graph A_n

Now we will describe the AR-quiver of a Nakayama algebra A with a quiver of type A_n. An algebra A is called Nakayama algebra if it is both right serial and left serial. For a basic and connected algebra there are only two types of Nakayama algebra.

Theorem 4. A basic and connected algebra A is a Nakayama algebra if and only if its quiver representation Q_A is one of the following two quivers:

(a)

(b)
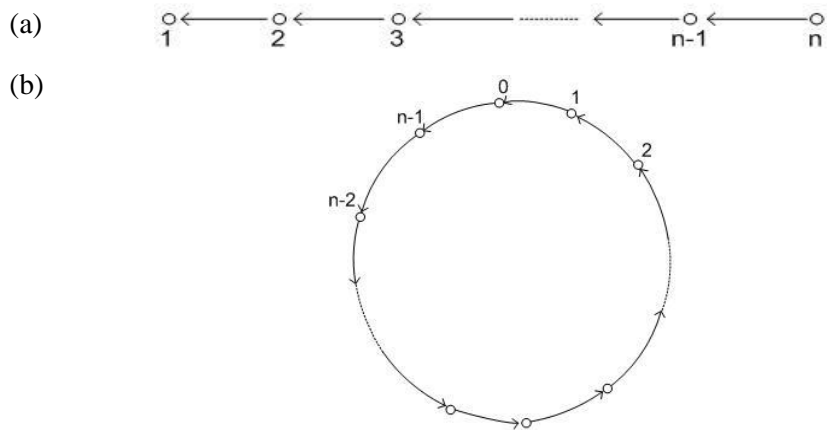
(with $n \geq 1$ points).

Proof. See [2, section V.3].

The first Nakayama algebra will be called the Nakayama algebra type Dynkin graph A_n. The following theorem and corollary are some properties of Nakayama algebras.

Theorem 5. Let A be a basic and connected Nakayama algebra, and let M be an indecomposable A-module. There exists an indecomposable projective A-module P and an integer t with $1 \leq t \leq \ell\ell(P)$ where $\ell\ell(P)$ is the Loewy length of P, such that $M \cong P/\mathrm{rad}^t P$. In particular, A is representation finite.

Proof. See [2, section V.3].

Corollary 1. A basic and connected algebra A is a Nakayama algebra if and only if every indecomposable A-module is uniserial.

Proof. See [2, section V.3].

Before constructing the AR-quiver, we will need the following result. Let Q be the quiver

and Q' be the quiver

Write $Q_0=\{1,2,...,n\}$, $Q'=\{1,2,..., n+1\}$, $Q_1=\{\alpha_i: i + 1 \rightarrow i \mid i \in Q_0 \backslash \{n\}\}$ and $Q'_1=Q_1 \cup \{\alpha_n: n + 1 \rightarrow n\}$. Set $A = KQ$ and $A' = KQ'$.

Lemma 1. Let $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be a module in mod A. Define $M = (M'_j, \varphi'_\alpha)_{j \in Q'_0, \alpha \in Q'_1}$ where $M'_j = M_j$ if $j \in Q_0$, $M'_{n+1} = 0$, $\varphi'_\alpha = \varphi_\alpha$ if $\alpha \in Q_1$ and $0 = \varphi'_{\alpha_n}: M'_{n+1} \rightarrow M'_n$. The A-module M is indecomposable if and only if the A'-module M' is indecomposable.

Proof. ($\Rightarrow$) Assume that M is indecomposable. Suppose that M' is decomposable, then $M' = U' \oplus V' = \left(U'_j \oplus V'_j, \begin{bmatrix} \phi'_\alpha & 0 \\ 0 & \tau'_\alpha \end{bmatrix} \right)_{j \in Q'_0, \alpha \in Q'_1}$. If $U' = \left(U'_j, \phi'_\alpha \right)_{j \in Q'_0, \alpha \in Q'_1}$ and $V' = \left(V'_j, \phi'_\alpha \right)_{j \in Q'_0, \alpha \in Q'_1}$ then $U'_{n+1} = 0 = V'_{n+1}$. Set $U = (U_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $V = (V_i, \tau_\alpha)_{i \in Q_0, \alpha \in Q_1}$ where $U_i = U'_i$, $V_i = V'_i$ and $\phi_\alpha = \phi'_\alpha$, $\tau_\alpha = \tau'_\alpha$. By definition, we have $M = K \oplus L$. This contradiction shows that M' is indecomposable.

($\Leftarrow$) Assume that M' is indecomposable. Suppose that M is decomposable. We have $M = U \oplus V = \left(U_i \oplus V_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \tau_\alpha \end{bmatrix} \right)_{i \in Q_0, \alpha \in Q_1}$ with U and V are nonzero submodules of M. By definition, $M' = U' \oplus V'$ where $U' = \left(U'_j, \phi'_\alpha \right)_{j \in Q'_0, \alpha \in Q'_1}$ and $V' = \left(V'_j, \tau'_\alpha \right)_{j \in Q'_0, \alpha \in Q'_1}$ with $U_j = U'_j$, $V_j = V'_j$ if $j \in Q_0$, $U'_{n+1} = 0 = V'_{n+1}$ and $\phi'_\alpha = \phi_\alpha$, $\tau'_\alpha = \tau_\alpha$ if $\alpha \in Q_1$, $\phi'_{\alpha_n} = 0$, $\tau'_{\alpha_n} = 0$. This contradict that M' is indecomposable.

Lemma 2. Let M and M' be the modules in Lemma 1. The A-module M is simple if and only if the A'-module M' is simple.

Proof. Note that any simple both A-module and A'-module are 1-dimensional. It follows from the total dimension of K-vector space in linear representation of M and M', we have $\dim_K(M) = 1$ if and only if $\dim_K(M') = 1$.

Both Lemma 1 and Lemma 2 above say that there exists an inclusion between the collection of isomorphism classes of indecomposable (or simple) A-module and the collection of isomorphism classes of indecomposable (or simple) A'-module.

Lemma 3. Let $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $N = (N_i, \tau_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be modules in mod A and let M', N' be modules defined in the same way like in Lemma 1. Let $f = (f_i)_{i \in Q_0} : M \to N$ and $f' = \left(f'_j\right)_{j \in Q'_0} : M' \to N'$ where $f'_j = f_j$ if $j \in Q_0$ and $f'_{n+1} = 0 : M'_{n+1} \to N'_{n+1}$. If f is an irreducible morphism then so is f'.

Proof. It is clear that $f' = \left(f'_j\right)_{j \in Q'_0}$ is a morphism: indeed $f'_k \varphi'_{\alpha_k} = \tau'_{\alpha_k} f'_{k+1}$ for every $k \in Q_0$. We claim that f has no both right inverse and left inverse. Suppose that f' has a right inverse $g' = (g_i)_{i \in Q'_0}$. It means that $f'_i g_i = 1_{N'_i}$ for every $i \in \{1, 2, \ldots, n + 1\}$. Consequently, if $i \in Q_0$ then $f_i g_i = f'_i g_i = 1_{N'_i} = 1_{N_i}$. Define $g = (g_i)_{i \in Q_0}$, then g is a morphism in mod A, moreover $fg = 1_N$, contrary to our assumption. So f has no right inverse. In a similar way, we conclude that f has no left inverse also, claim proved.

Now, let $f' = h' g'$ and assume that h' is not a retraction. If $h' = (h_1, h_2, \ldots, h_n, h_{n+1})$ and $g' = (g_1, g_2, \ldots, g_n, g_{n+1})$ then $h_{n+1} = 0$ and $g_{n+1} = 0$. We get $f = hg$ where $h = (h_1, h_2, \ldots, h_n)$ and $g' = (g_1, g_2, \ldots, g_n)$. Because h' is not a retraction, then h is not a retraction. This means that g is a section because f is irreducible. Hence, g' is a section. This proves that f is irreducible.

Now let $2 \leq n \in \mathbb{N}$ and A be a Nakayama algebra of type A_n. To start to build the AR-quiver of A we first list all isomorphism classes of indecomposable projective A-module, indecomposable injective and simple. Each of modules will be considered as linear representation and the vertices were replaced by the dimension of its vector space. All indecomposable projective modules of A are:

$$P(1) = \underbrace{100 \ldots 00}_{n}$$

$$P(2) = 110 \ldots 00$$

$$P(3) = 111 \ldots 00$$

$$\vdots$$

$$P(i) = \underbrace{11\ldots1}_{i}0\ldots00$$

$$\vdots$$

$$P(n - 1) = 111\ldots10$$

$$P(n) = 111\ldots11$$

All indecomposable injective modules of A are:

$$I(1) = \underbrace{111\ldots11}_{n}$$

$$I(2) = 011\ldots11$$

$$I(3) = 001\ldots11$$

$$\vdots$$

$$I(i) = \underbrace{00\ldots0}_{i-1}1\ldots11$$

$$\vdots$$

$$I(n - 1) = 000\ldots011$$

$$I(n) = 000\ldots01$$

and all simple modules are:

$$S(1) = \underbrace{100\ldots00}_{n}$$

$$S(2) = 010\ldots00$$

$$S(3) = 001\ldots00$$

$$\vdots$$

$$S(i) = 000\ldots0\underbrace{1}_{i-th\ position}0\ldots00$$

$$\vdots$$

$$S(n - 1) = 000\ldots10$$

$$S(n) = 000\ldots01$$

We have $P(1) = S(1)$, $P(n) = I(1)$, $I(n) = S(n)$. Further, $I(k) = \frac{P(n)}{P(k-1)}$ for every $k \in \{1,2,\ldots, n\}$ with $P(0) = 0$. It easy to check that rad $P(k) = P(k\text{-}1)$ for every $k \in \{1,2,\ldots, n\}$. In other words $\mathrm{rad}^t P(n) = P(n - t)$, with $t \leq n$. By Proposition 1, the inclusion $i_j: P(j - 1) = \mathrm{rad}\, P(j) \rightarrow P(j)$ is a right minimal almost split for every $j \in \{2,3,\ldots,n\}$ and the canonical homomorphism $p_k: I(k) \rightarrow I(k + 1) = \frac{I(k)}{\mathrm{soc}\, I(k)}$ is a left minimal almost split for every $k \in \{1, 2,\ldots, n - 1\}$.

Lemma 4. (a) The sequence $0 \rightarrow P(1) \xrightarrow{i} P(2) \xrightarrow{p} S(2) \rightarrow 0$ is almost split, where i is the inclusion and p is the canonical homomorphism with kernel equal

to $P(1)$.(b) The sequence $0 \rightarrow S(n-1) \xrightarrow{i'} I(n-1) \xrightarrow{p'} S(n) \rightarrow 0$ is almost split, where i' is the inclusion and p' is the canonical homomorphism with kernel equal to $S(n$ - $1)$.

Proof. (a) It is clear that the sequence in (a) is exact. Because $P(2) = \text{rad } P(1)$, then $i : P(1) \rightarrow P(2)$ is a right minimal almost split (by Proposition 1(a)). We know that $P(1)$ is indecomposable, using Theorem 1(b), the inclusion i is the only irreducible morphism ending with $P(2)$. Let $h : P(1) \rightarrow M$ be a left minimal almost split. Because $P(1)$ is a simple projective noninjective module in mod A, then by Theorem 2, M is a projective module. Without loss of generality, by Proposition 2 we may assume that $M = \bigoplus\limits_{j=1}^{t} P(j)$ with the $P(j)$ indecomposable projective and pairwise nonisomorphic. This means that $h_j : P(1) \rightarrow P(j)$ is irreducible and consequently $j = 1$ and $M = P(j)$. We have a right minimal almost split g: rad $P(j) \rightarrow P(j)$. This means that $P(1)$ is a direct summand of rad $P(j)$. But rad $P(j) = P(j$-$1)$ is indecomposable, it follows that $M = P(2)$. Thus, $h = i : P(1) \rightarrow P(2)$ is a left minimal almost split. By Theorem 3(b), the given sequence is almost split, we have proved part (a).

(b) It is clear that the sequence in (b) is exact. We know that the canonical homomorphism $s: I(n$-$1) \rightarrow S(n) = I(n)$ is left minimal almost split. Since $S(n)$ is indecomposable, using Theorem 1(a), the inclusion i' is the only irreducible morphism starting with $I(n$ - $1)$. Let $f: M \rightarrow I(n)$ be right minimal almost split. Since $S(n) = I(n)$ is a simple injective nonprojective, by Theorem 2, M is injective. Without loss of generality, by Proposition 2 we may assume that $M = \bigoplus\limits_{j=1}^{t} I(j)$ with the $I(j)$ indecomposable injective and pairwise nonisomorphic. Thus, $f_j: I(j) \rightarrow I(n)$ is irreducible and implies $t \leq n$ - 1. We have a left minimal almost split $I(j) \rightarrow I(j+1) = \dfrac{I(j)}{\text{soc } I(j)}$ This means that $I(n)$ is direct summand of $I(j+1)$. Since $I(j+1)$ is indecomposable, then $M = I(n)$. Hence, $f = p: I(n$-$1) \rightarrow I(n)$ is a right minimal almost split. It follows from Theorem 3(c) this sequence is almost split.

Lemma 5. Let $2 \leq n \in \mathbb{N}$ and $A=KQ$ where Q is a quiver of Dynkin type A_n. Then for every $j \in \{1,2,\ldots,n$-$1\}$, the sequence $0 \rightarrow S(j) \xrightarrow{i} V_{j,j+1} \xrightarrow{p} S(j+1) \rightarrow 0$ is almost split where $V_{j,j+1} = 00\ldots0 \underbrace{11}_{j,j+1} 0\ldots00$.

Proof. We proceed by induction on *n*. Let *n* = 2, 3, it follows from Lemma 4 that this sequence is almost split. Now assume that n > 4, if *j* = 1 and *j* = n-1 then the proof follows from Lemma 4. If $j \in \{2,\dots,n-2\}$, it is clear the sequence $0 \to S(j) \xrightarrow{i} V_{j,j+1} \xrightarrow{p} S(j+1) \to 0$ is exact. By hypothesis the sequence $0 \to S(j) \xrightarrow{i} V_{j,j+1} \xrightarrow{p} S(j+1) \to 0$ in mod $B = KQ'$ with Q' is quiver type A_{n-1} is almost split. In other words, *i* and *p* are irreducible morphism in mod *B*. Now consider the exact sequence $0 \to S'(j) \xrightarrow{i'} V'_{j,j+1} \xrightarrow{p'} S'(j+1) \to 0$ in mod *A*. By Lemma 3, *i'* and *p'* are irreducible morphisms in mod *A*. It follows from Theorem 3 (d) that the sequence $0 \to S'(j) \xrightarrow{i'} V'_{j,j+1} \xrightarrow{p'} S'(j+1) \to 0$ is almost split, the lemma is proved.

Proposition 3. Let *P* be a nonsimple indecomposable projective-injective module, *S* = soc *P* and *R* = rad *P*. Then the sequence

$$0 \to R \xrightarrow{\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right]} \frac{R}{S} \oplus P \xrightarrow{[-j \quad p]} P/S \to 0 \tag{2}$$

is almost split, where *i*, *j* are the inclusion and *p*, *q* the projections.

Proof. See [2, section IV.3].

Corollary 2. The sequence

$$0 \to P(n-1) \xrightarrow{\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right]} \frac{P(n-1)}{P(1)} \oplus P(n) \xrightarrow{[-j \quad p]} P(n)/P(1) \to 0 \tag{3}$$

is almost split in mod *A*.

Proof. Apply Proposition 3.

Lemma 6. The sequence

$$0 \to \frac{P(n-1)}{P(k)} \xrightarrow{\left[\begin{smallmatrix} q \\ i \end{smallmatrix}\right]} \frac{P(n-1)}{P(k+1)} \oplus \frac{P(n)}{P(k)} \xrightarrow{[-j \quad p]} P(n)/P(k+1) \to 0 \tag{4}$$

is almost split in mod *A*, where $k \in \{1,2,..., n-3\}$.

Proof. It is clear that the above sequence is exact. Because $\frac{P(n-1)}{P(k)}, \frac{P(n-1)}{P(k+1)}, \frac{P(n)}{P(k)}$, $P(n)/P(k+1)$ are indecomposable and pairwise nonisomorphic, this sequence is not split. Using Theorem 3, it suffices to show that the homomorphism $g = [-jp]$ is right almost split. Clearly *g* is not a retraction. Let *V* be an indecomposable *A*-module, $M = \frac{P(n)}{P(k+1)}$ and $v : V \to M$ is not an isomorphism. We have two cases.

If v is not surjective, then $\text{Im}(v)$ is contained in the unique maximal submodule (because M is uniserial) rad M of M. We compute rad M by its linear representation, then we have rad $M = \frac{P(n-1)}{P(k+1)}$. Hence, we get a homomorphism $\begin{bmatrix} -v \\ 0 \end{bmatrix} : V \to \frac{P(n-1)}{P(k+1)} \oplus \frac{P(n)}{P(k)}$ that satisfies $g\begin{bmatrix} -v \\ 0 \end{bmatrix} = v$. If v is surjective, then $\frac{V}{\text{Ker } V} \cong M = \frac{P(n)}{P(k+1)}$. Since $\frac{V}{\text{Ker } V}$ is uniserial and v is not an isomorphism, Ker v = rad$^s V$ with $s \geq 1$ and and $\dim_K(V) > n - (k+1)$. Since V is indecomposable, by Theorem 5 we get $V \cong M = \frac{P(m)}{\text{rad}^r P(m)} = \frac{P(m)}{P(m-r)}$ with $1 \leq m \leq n$ and $1 \leq r \leq m$, thus r= $\dim_K(V) > n - (k+1)$. We have rad$^s V$ = rad$^s \left(\frac{P(m)}{P(m-r)} \right) = \frac{\text{rad}^s P(m)}{P(m-r)} = \frac{P(m-s)}{P(m-r)}$. Consequently, $\frac{V}{\text{Ker } V} = \frac{P(m)/P(m-r)}{P(m-s)/P(m-r)}$. Using its linear representation we get $m = n$: indeed $\frac{V}{\text{Ker } V}$ is indecomposable injective. Hence, $V = \frac{P(n)}{P(n-r)}$ with $n - r < k + 1$. Thus, there exists an epimorphism $v' : V \to \frac{P(n)}{P(r)}$ such that $v = pv'$. Define a homomorphism $\begin{bmatrix} 0 \\ v' \end{bmatrix} : V \to \frac{P(n-1)}{P(k+1)} \oplus \frac{P(n)}{P(k)}$, then $g.\begin{bmatrix} 0 \\ v' \end{bmatrix} = v$. Assume now that $f: N \to M$ is not a retraction and let $N = \overset{t}{\underset{k=1}{\oplus}} N(k)$, then for every k, $N_k \ncong M$. As we have seen before there exists $h_k : N_k \to \frac{P(n-1)}{P(k+1)} \oplus \frac{P(n)}{P(k)}$ such that $g[h_1 h_2 \dots h_t] = [f_1 f_2 \dots f_t]$ where $f_i: N_i \to M$ is a component of f. We conclude that g is right almost split and thus the given sequence is almost split.

In the process of proving the main result we need one fact about the AR-quiver with connected components.

Theorem 6. Assume that A is a basic and connected finite dimensional K-algebra. If $\Gamma(\text{mod } A)$ admits a connected component C whose modules are of bounded length, then C is finite and C=$\Gamma(\text{mod } A)$. In particular, A is representation finite.

Proof. See [2, section IV.5].

Now we give the main results in this paper

Theorem 7. Let Q be a quiver of type A_n with $n \in \mathbb{N}$ and n ≥ 2. Let $A = KQ$, then $\Gamma(\text{mod } A)$ is the quiver
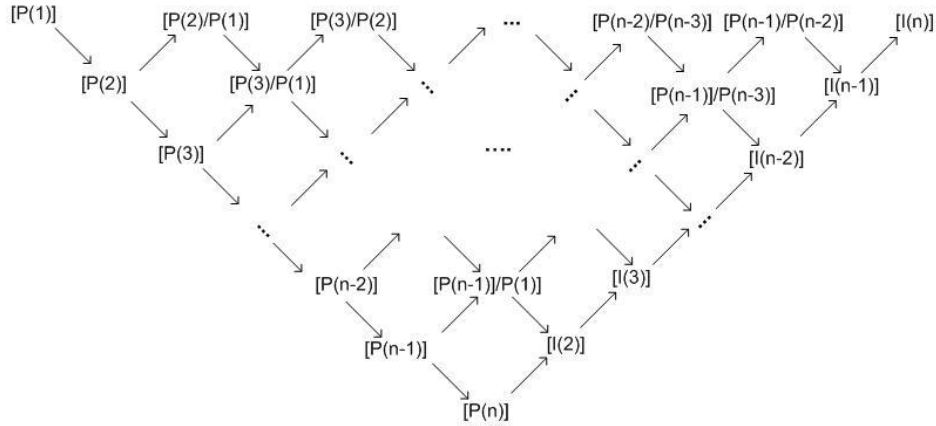
Proof. We proceed by induction on n. For $n=2, 3$ it is easy to check that its AR-quiver satisfy the given picture. Before we continue our induction process, we need the following useful fact: every subquiver of $\Gamma(\bmod A)$ of the form smallest diamond with 4 arrows, its vertices form an almost split sequence. Precisely, if we take a subquiver of $\Gamma(\bmod A)$ of the form
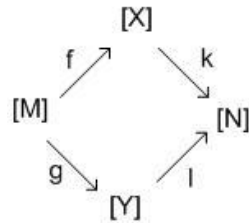

with $f: M \to X$, $g: M \to Y$, $k: X \to N$, $l: Y \to N$ are irreducible morphism then the sequence

$$0 \to M \xrightarrow{s=\begin{bmatrix} f \\ g \end{bmatrix}} X \oplus Y \xrightarrow{t=[-k \quad l]} N \to 0 \qquad (5)$$

is almost split. It can be checked that f, l are projection and g, k are inclusion. Note that this fact holds for $n = 2, 3$. Now assume that the statement is true for n-1 and denote $A'=KQ'$ with Q' quiver of Nakayama algebra type A_{n-1}. It follows that its AR-quiver is
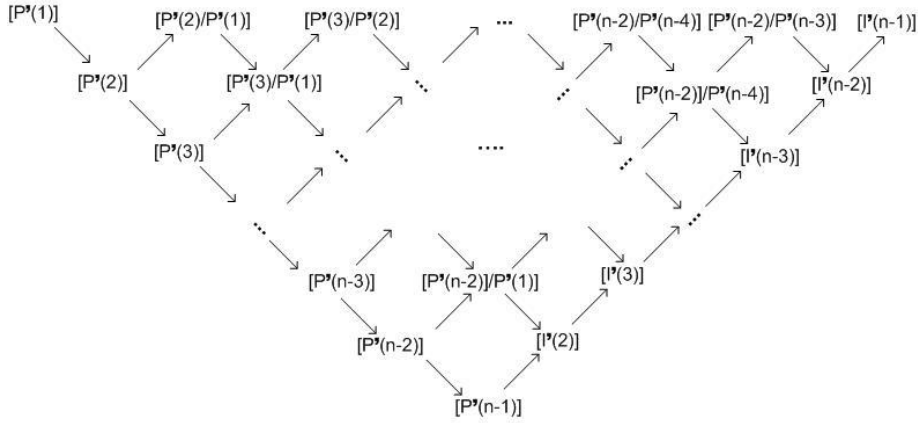
and for every smallest diamond form an almost split sequence as we say before. By Lemma 1 we get some isomorphism classes of indecomposable A-module such that the number of its is the same as the number of vertices of $\Gamma(\text{mod } A')$. By the induction hypothesis and Lemma 3, we get a subquiver of $\Gamma(\text{mod } A)$:
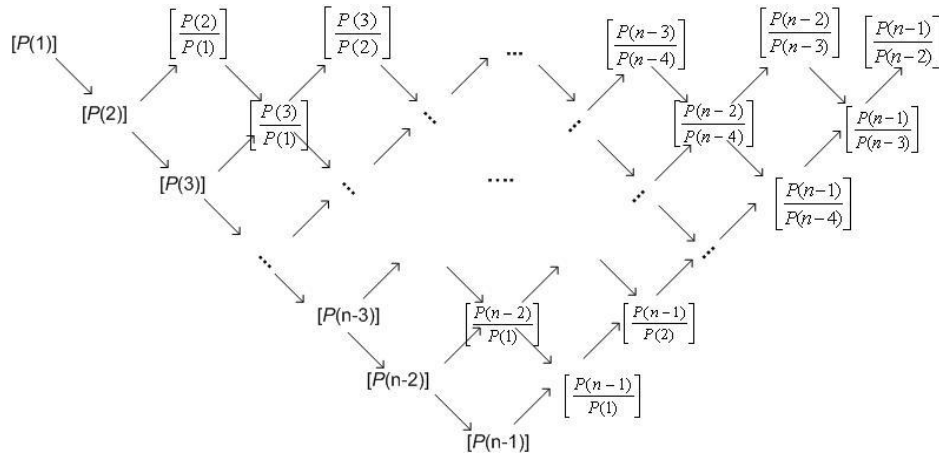

We claim that for every subquiver of the form smallest diamond in this quiver form an almost split sequence. We have the sequence

$$0 \to M \xrightarrow{s=\begin{bmatrix} f \\ g \end{bmatrix}} X \oplus Y \xrightarrow{t=[-k \quad l]} N \to 0 \tag{6}$$

by taking the smallest diamond subquiver $\Gamma(\text{mod } A')$. Let M', N', X', Y' be A-modules defined as same as modules in Lemma 1. Thus, we have an exact sequence

$$0 \to M' \xrightarrow{\; s' = \begin{bmatrix} f' \\ g' \end{bmatrix} \;} X' \oplus Y' \xrightarrow{\; t' = [-k' \quad l'] \;} N' \to 0 \tag{7}$$

where f', l' are projection and g', k' are inclusion. Because s, t are irreducible morphisms, then by Lemma 3 we have that s', t' are irreducible morphisms in mod A. It follows from Theorem 3 the latter exact sequence is almost split, and our claim proved. By Corollary 2, Lemma 5 and Lemma 6 we get a connected component of $\Gamma(\text{mod } A)$:
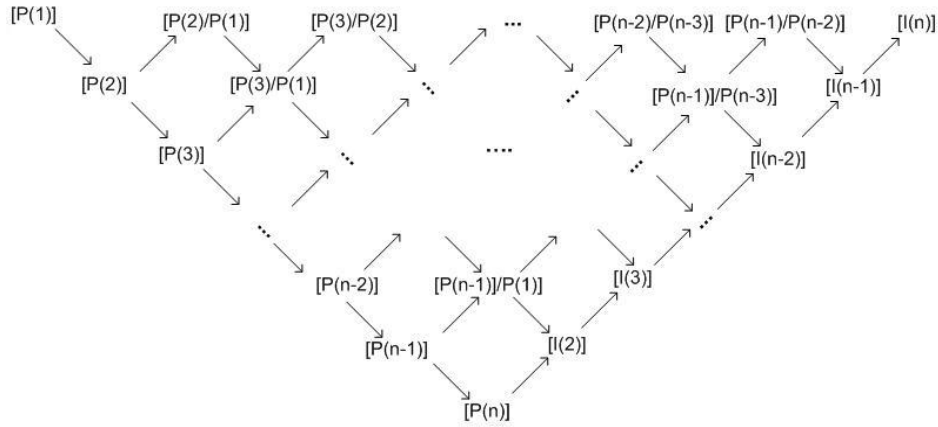
It follows from Theorem 5 the connected component is same as with the $\Gamma(\text{mod } A)$. This completes the proof.

Corollary 3. Let A be a Nakayama algebra with quiver type A_n. The number of the isomorphism classes of indecomposable A-module is $\frac{1}{2}n(n+1)$.

Proof. We proceed by induction on n. For $n=2$, 3 we can directly compute from its quiver and we get both the number of the isomorphism classes of indecomposable modules are 3 and 6 respectively. Now assume that for quiver type A_{n-1} the number of the isomorphism classes of indecomposable modules is $\frac{1}{2}(n-1)n$. We have seen that in the proof of our main theorem, $\Gamma(\text{mod } A)$ has a subquiver which isomorphic to AR-quiver of Nakayama algebra type A_{n-1}. Further, all indecomposable modules in this subquiver are noninjective. This means, the number of the isomorphism classes indecomposable noninjective of A-module is $\frac{1}{2}(n-1)n$. But, we know that A has n isomorphism classes of indecomposable injective A-modules. Hence, the number of the isomorphism classes of indecomposable A-modules is $\frac{1}{2}(n-1)n + n = \left(\frac{1}{2}(n-1)+1\right)n =$

$\frac{1}{2}n(n+1)$.

AR-quiver has an important role in studying the category of modules. It stores the information of indecomposable modules, homomorfisms between indecomposable modules and almost split sequences. Now by using AR-quiver constructed above we may give a tilting module of algebra $A=KQ$ with Q is quiver of type A_n. Let $T = I(1) \oplus P(1) \oplus I(3) \oplus I(4) \oplus \dots \oplus I(n)$, we claim that T is a tilting module. Because A is hereditary we have gl. dim $A = 1$. So projective dimension of T is at most 1. Since T is not projective the projective dimension is equal 1. Let $M = I(3) \oplus \ \dots \ \oplus I(n)$. Because $I(1) \oplus P(1)$ is projective, whereas $I(1) \oplus I(3) \oplus \ \dots \ \oplus I(n)$ is injective, we have

$$\begin{aligned}
\operatorname{Ext}_A^1(T, T) &\cong \operatorname{Ext}_A^1(M, P(1)) \cong D\operatorname{Hom}_A(P(1), \tau(M)) \\
&\cong D\operatorname{Hom}_A(P(1), \tau(I(3)) \oplus \dots \oplus \tau(I(n)))) \\
&\cong D\operatorname{Hom}_A\left(P(1), \frac{P(n-1)}{P(1)} \oplus \dots \oplus \frac{P(n-1)}{P(n-2)}\right) \\
&\cong D\operatorname{Hom}_A\left(P(1), \frac{P(n-1)}{P(1)}\right) \oplus \dots \oplus D\operatorname{Hom}_A\left(P(1), \frac{P(n-1)}{P(n-2)}\right) = 0. \quad (8)
\end{aligned}$$

Note that, we have $\tau(I(j)) = \frac{P(n-1)}{j-2}$, by looking its AR-quiver. Indeed, in [1] we know that for any indecomposable nonprojective A-module M, there exists an almost split sequence $0 \to \tau(M) \to E \to M \to 0$. This almost split sequence appears in AR-quiver in the form of a smallest diamond. To show T satisfy (T3) it suffices to prove that for any indecomposable projective A-module P, there exists a short exact sequence

$$0 \to P_A \to T'_A \to T''_A \to 0. \tag{9}$$

with T', $T'' \in$ add T. Because $P(1)$, $P(n) = I(1) \in$ add T we have short exact sequences

$$0 \to P(1) \to P(1) \oplus P(n) \to P(n) \to 0, \tag{10}$$

$$0 \to P(n) \to P(1) \oplus P(n) \to P(1) \to 0. \tag{11}$$

with $P(1) \oplus P(n) \in$ add T. On the other hand, for every i with $2 \leq i \leq n$ - 1 there exists a short exact sequence

$$0 \to P(i) \xrightarrow{f_i} P(n) \xrightarrow{g_i} I(i+1) \to 0, \tag{12}$$

where f_i is the inclusion and g_i is the projection for all i. We conclude that T is a tilting module of A. By the above reason we get the following result.

Proposition 4. Let A be the KQ algebra with Q is quiver of type A_n. Then $T = I(1) \oplus P(1) \oplus I(3) \oplus I(4) \oplus \dots \oplus I(n)$ is a tilting module.

4   Conclusion

Our main result is a generalization of AR-quiver of the first kind Nakayama algebra. This AR-quiver have a nice structure and the isomorphism classes of its indecomposable module is easy to understand. In this AR-quiver the position of all simple modules is on the top of the AR-quiver. On the left side we have all indecomposable projective and on the right side we have all indecomposable injective. Since A is a Nakayama algebra, all indecomposable module which not injective is only a quotient of indecomposable projective module. The AR-quiver of any finite representation algebra has a role in finding all homomorphisms between indecomposable modules. By using this information, we get a tilting module of Nakayama algebra of type Dynkin graph A_n. We have that, if $T = I(1) \oplus P(1) \oplus I(3) \oplus I(4) \oplus \dots \oplus I(n)$ then T is a tilting module.

Acknowledgement

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