g- Inverses of Interval Valued Fuzzy Matrices

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Abstract. In this paper, we have discussed the g-Inverses of Interval Valued Fuzzy Matrices (IVFM) as a generalization of g- inverses of regular fuzzy matrices. The existence and construction of g-inverses, {1, 2} inverses, {1, 3} inverses and {1, 4} inverses of Interval valued fuzzy matrix are determined in terms of the row and column spaces.

Keywords: *g-Inverses of fuzzy matrix; g-inverses of Interval valued fuzzy matrix.*

1 Introduction

A fuzzy matrix is a matrix over the max-min fuzzy algebra $\mathcal{F}$ =[0,1] with operations defined as a+b = max{a,b} and a·b = min{a,b} for all a,b$\in\mathcal{F}$ and the standard order $\geq$ of real numbers over $\mathcal{F}$. A matrix $A\in\mathcal{F}_{mn}$ is said to be regular if there exists $X\in\mathcal{F}_{mn}$ such that AXA = A. X is called a generalized inverse of A and is denoted by A^{-}. In [1], Thomason has studied the convergence of powers of a fuzzy matrix. In [2], Kim and Roush have developed a theory for fuzzy matrices analogous to that for Boolean matrices [3]. A finite fuzzy relational equation can be expressed in the form of a fuzzy matrix equation as x.A = b for some fuzzy coefficient matrix A. If A is regular, then x.A=b is consistent and bX. is a solution for some g-inverse X of A [4]. For more details on fuzzy matrices one may refer to [5, 6]. Recently, the concept of the interval valued fuzzy matrix (IVFM) as a generalization of fuzzy matrix has been introduced and developed by Shyamal and Pal [7]. In earlier work, we have studied the regularity of IVFM [8] and analogous to that for complex matrices [9].

In this paper, we discuss the g-inverses of interval valued fuzzy matrices (IVFM) as a generalization of the g-inverses of regular fuzzy matrices studied in [2, 6], and as an extension of the regularity of the IVFM discussed in [8]. In section 2, we present the basic definition, notation of the IVFM and required results of g-inverses of regular fuzzy matrices. In Section 3, the existence and construction of g-inverses, {1, 2} inverses, {1, 3} inverses and {1, 4} inverses of interval-valued fuzzy matrices are determined in terms of the row and column spaces of IVFM.

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2 Preliminaries

In this section, some basic definitions and results needed are given. Let IVFM denote the set of all interval-valued fuzzy matrices, that is, fuzzy matrices whose entries are all subintervals of the interval [0, 1].

Definition 2.1. For a pair of fuzzy matrices $E=(e_{ij})$ and $F=(f_{ij})$ in $\mathcal{F}_{mn}$ such that $E \leq F$, the interval valued fuzzy matrix $[E, F] = ([e_{ij}, f_{ij}])$, is the matrix, whose ij^{th} entry is the interval with lower limit e_{ij} and upper limit f_{ij}.

In particular for $E = F$, IVFM $[E,E]$ reduces to the fuzzy matrix $E \in \mathcal{F}_{mn}$.

For $A = (a_{ij}) = ([a_{ijL}, a_{ijU}]) \in (IVFM)_{mxn}$, let us define $A_L = (a_{ijL})$ and $A_U=(a_{ijU})$. Clearly, the fuzzy matrices A_L and A_U belong to $\mathcal{F}_{mn}$ such that $A_L \leq A_U$. Therefore, by Definition (2.1), A can be written as

$$A = [A_L, A_U] \tag{1}$$

where A_L and A_U are called the lower and upper limits of A respectively.

Here we shall follow the basic operation on IVFM as given in [8].

For $A=(a_{ij}) = ([a_{ijL}, a_{ijU}])$ and $B=(b_{ij}) = ([b_{ijL}, b_{ijU}])$ of order mxn, their sum, denoted as A+B, is defined as

$$A+B = (a_{ij}+b_{ij}) = [(a_{ijL}+b_{ijL}), (a_{ijU}+b_{ijU})] \tag{2}$$

For $A = (a_{ij})_{mxn}$ and $B = (b_{ij})_{nxp}$ their product, denoted as AB, is defined as

$$AB = (C_{ij}) = [\, \Sigma^{n}_{k=1} a_{ik}\, b_{kj} \,] \qquad i=1,2,\ldots \ldots m \text{ and } j=1,2,\ldots \ldots p$$
$$= [\, \Sigma^{n}_{k=1} (a_{ikL} . b_{kjL}), \Sigma^{n}_{k=1} (a_{ikU} . b_{kjU}) \,]$$

If $A = [A_L, A_U]$ and $B = [B_L, B_U]$ then $A+B = [A_L + B_L, A_U + B_U]$

$$AB = [A_L B_L, A_U B_U] \tag{3}$$

$A \geq B$ if and only if $a_{ijL} \geq b_{ijL}$ and

$$a_{ijU} \geq b_{ijU} \text{ if and only if } A+B = A \tag{4}$$

In particular if $a_{ijL} = a_{ijU}$ and $b_{ijL} = b_{ijU}$ then by Eq. (3) reduces to the standard max. min. composition of fuzzy matrices [2, 6].

For $A \in (IVFM)_{mn}$, A^T, $\mathcal{R}(A)$, $\mathcal{C}(A)$, A^-, $A\{1\}$ denotes the transpose, row space, column space, g-inverses and set of all g-inverses of A, respectively.

Lemma 2.2. (Lemma 2 [5]) For $A, B \in \mathcal{F}_{mn}$, if A is regular, then

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = BA^-A$ for each $A^- \in A\{1\}$

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AA^{-}B$ for each $A^{-} \in A\{1\}$.

Lemma 2.3. If $A \in \mathcal{F}_{mn}$ with $\mathcal{R}(A) = \mathcal{R}(A^{T}A)$, then $A^{T}A$ is regular fuzzy matrix if and only if A is a regular fuzzy matrix. If $A \in \mathcal{F}_{mn}$ with $\mathcal{C}(A) = \mathcal{C}(AA^{T})$, then AA^{T} is a regular fuzzy matrix if and only if A is a regular fuzzy matrix.

In the following, we will make use of the following results proved in our earlier work [8]. For the sake of completeness we will provide the proof.

Lemma 2.4. (Theorem 3.3 [8])

Let $A = [A_L, A_U] \in (IVFM)_{mn}$
Then the following holds:
(i) A is regular IVFM $\Leftrightarrow$ A_L and $A_U \in \mathcal{F}_{mn}$ are regular
(ii) $\mathcal{R}(A) = [\mathcal{R}(A_L), \mathcal{R}(A_U)]$ and $\mathcal{C}(A) = [\mathcal{C}(A_L), \mathcal{C}(A_U)]$.

Proof.
(i) Since $A \in (IVFM)_{mn}$, any vector $x \in R(A)$ is of the form $x = y.A$ for some $y \in (IVFM)_{1n}$, that is, x is an interval valued vector with n components.

Let us compute $x \in R(A)$ as follows:

$$\text{x is a linear combination of the rows of A} \Rightarrow x = \sum_{i=1}^{m} \alpha_i . A_{i*}$$

where A_{i*} is the i^{th} row of A. Equating the j^{th} component on both sides yields

$$x_j = \sum_{i=1}^{m} \alpha_i . a_{ij}.$$

Since, $a_{ij} = [a_{ijL}, a_{ijU}]$

$$\begin{aligned} x_j &= \sum_{i=1}^{m} \alpha_i . [a_{ijL}, a_{ijU}] \\ &= \sum_{i=1}^{m} [\alpha_i a_{ijL}, \alpha_i a_{ijU}] \\ &= \left[\sum_{i=1}^{m} (\alpha_i . a_{ijL}) \; , \; \sum_{i=1}^{m} (\alpha_i . a_{ijU}) \right] \\ &= [x_{jL}, x_{jU}]. \end{aligned}$$

x_{jL} is the j^{th} component of $x_L \in R(A_L)$ and x_{jU} is the j^{th} component of $x_U \in R(A_U)$. Hence $x = [x_L, x_U]$. Therefore, $R(A) = [R(A_L), R(A_U)]$

(ii) For $A = [A_L, A_U]$, the transpose of A is $A^T = [A_L^T, A_U^T]$. By using (i) we get, $C(A) = R(A^T) = [R(A_L^T), R(A_U^T)] = [C(A_L), C(A_U)]$.

Lemma 2.5. (Theorem 3.7 [8])

For A and $B \in (IVFM)_{mn}$

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow B = XA$ for some $X \in (IVFM)_m$

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A) \Leftrightarrow B = AY$ for some $Y \in (IVFM)_n$

Proof.

(i) Let $A = [A_L, A_U]$ and $B = [B_L, B_U]$. Since, $B = XA$, for some $X \in$ (IVFM), put $X = [X_L, X_U]$. Then, by Equation (3), $B_L = X_L A_L$ and $B_U = X_U A_U$. Hence, by(Lemma (2.2)), $\mathcal{R}(B_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(B_U) \subseteq \mathcal{R}(A_U)$

By Lemma (2.4)(ii), $\mathcal{R}(B) = [R(B_L), \mathcal{R}(B_U)] \subseteq [\mathcal{R}(A_L), \mathcal{R}(A_U)] = \mathcal{R}(A)$.Thus $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Conversely, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

$\Rightarrow \mathcal{R}(B_L) \subseteq \mathcal{R}(A_L)$ and $\mathcal{R}(B_U) \subseteq \mathcal{R}(A_U)$ (By Lemma (2.4) (ii))

$\Rightarrow B_L = YA_L$ and $B_U = ZA_U$ (By Lemma (2.2))

Then $B = [B_L, B_U]$

$= [YA_L, ZA_U]$

$= [Y,Z]\,[A_L, A_U]$ (By Eq. (3))

$= X[A_L, A_U]$, where $X = [Y, Z] \in (IVFM)_{mn}$

$= XA$

$B = XA$

(ii) This can be proved along the same lines as that of (i) and hence omitted.

3 g- Inverses of Interval Valued Fuzzy Matrices

In this section, we will discuss the g-inverses of an IVFM and their relations in terms of the row and column spaces of the matrix as a generalization of the results available in the literature on fuzzy matrices [2, 6] as a development of our earlier work [8] on regular IVFMs and analogous to that for complex matrices [9].

Definition 3.1. For $A\in(IVFM)_{mn}$ if there exists $X\in(IVFM)_{nm}$ such that

(1) $AXA = A$
(2) $XAX = X$
(3) $(AX)^T = (AX)$
(4) $(XA)^T = (XA)$, *then X is called a g-inverse of A.*

X is said to be a λ- inverse of A and $X\in A\{\lambda\}$ if X satisfies λ equation where λ is a subset of {1, 2, 3, 4}. A $\{\lambda\}$ denotes the set of all λ- inverses of A. In particular if λ = {1, 2, 3, 4} then X unique and is called the Moore Penrose inverse of A, denoted as $A^{\dagger}$.

Remark 3.2. From Definition (3.1) of λ-inverses for $A\in$(IVFM), by applying Eq. (3) for $A = [A_L, A_U]$ and $X = [X_L, X_U]$ it can be verified that the existence and construction of $\{\lambda\}$-inverses of $A\in(IVFM)_{mn}$ reduces to that of the $\{\lambda\}$-inverses of $A_L, A_U\in \boldsymbol{F}_{mn}$.

Theorem 3.3. Let $A\in(IVFM)_{mn}$ and $X\in A\{1\}$, then $X\in A\{2\}$if and only if $\mathcal{R}(AX) = \mathcal{R}(X)$

Proof.

Since $A = [A_L, A_U]$ and $X = [X_L, X_U]$

$X\in A\{2\}\Rightarrow XAX = X$, then by Eq. (3),

$\Rightarrow X_LA_LX_L = X_L$ and $X_UA_UX_U = X_U$; $X_L\in A_L\{2\}$and $X_U\in A_U\{2\}$

$\Rightarrow A_L\in X_L\{1\}$ and $A_U\in X_U\{1\}$

$\Rightarrow \mathcal{R}(X_L) = \mathcal{R}(A_LX_L)$ and $\mathcal{R}(X_U) = \mathcal{R}(A_UX_U)$

$\Rightarrow \mathcal{R}(AX) = \mathcal{R}(X)$. (By Lemma (2.4))

Conversely,

Let $\mathcal{R}(AX) = \mathcal{R}(X)$, then by Lemma (2.4), $\mathcal{R}(X) \subseteq \mathcal{R}(AX)$ implies $X = YAX$ for some $Y\in(IVFM)_m$.

$$\begin{aligned} X(AX) &= (YAX)(AX) \\ XAX &= Y(AXA)X \\ &= YAX \quad \text{(By Definition (3.1))} \\ &= X \end{aligned}$$

Thus $X\in A\{2\}$.

Remark 3.4. In the above Theorem (3.3), the condition $X \in A\{1\}$ is essential. This is illustrated in the following example.

Example 3.5.

Let $A = \begin{pmatrix} [0,1] & [1,1] \\ [1,1] & [0,0] \end{pmatrix}$, $X = \begin{pmatrix} [1,1] & [0,1] \\ [0,0] & [0,1] \end{pmatrix}$

Then by representation (1) we have, $A_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$X_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X_U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$A_LX_LA_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq A_L$ implies $X_L \notin A_L\{1\}$ and $A_UX_UA_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq A_U$ implies $X_U \notin A_U\{1\}$

$A_LX_L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $A_UX_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

But $X_LA_LX_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq X_L$. and $X_UA_UX_U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq X_U$.

Hence $X_L \notin A_L\{2\}$ and $X_U \notin A_U\{2\}$. Then by Eq. (3) we have, $AXA \neq A$, therefore $X \notin A\{1\}$. Here $\mathcal{R}(X_L) = \mathcal{R}(A_LX_L)$ and $\mathcal{R}(X_U) = \mathcal{R}(A_UX_U)$. Therefore by Lemma (2.4), $\mathcal{R}(X) = \mathcal{R}(AX)$, but $XAX \neq X$. Hence $X \notin A\{2\}$.

Theorem 3.6. For $A \in (IVFM)_{mn}$, A has a {1, 3}inverse if and only if A^TA is a regular IVFM and $\mathcal{R}(A^TA) = \mathcal{R}(A)$.

Proof. Since A is regular, Lemma (2.4), A_L and A_U are regular. Let A has a {1, 3} inverse X (say) then by Eq. (3), A_L has a {1, 3} inverse X_L and A_U has a {1, 3} inverse X_U.

Then $A_LX_LA_L = A_L$ and $(A_LX_L)^T = A_LX_L$

$$A_L^T(A_LX_LA_L) = A_L^TA_L$$

$$(A_L^TA_LX_L)A_L = A_L^TA_L$$

$$\mathcal{R}(A_L^TA_L) \subseteq \mathcal{R}(A_L) \qquad \text{(By Lemma (2.2))}$$

Similarly, $\mathcal{R}(A_U^TA_U) \subseteq \mathcal{R}(A_U)$

Therefore by Equation (3) we have, $\mathcal{R}(A^TA) \subseteq \mathcal{R}(A)$

Also $(A_LX_L)^TA_L = A_LX_LA_L$

$$\Rightarrow X_L^TA_L^TA_L = A_L$$

$$\Rightarrow X_L^T(A_L^TA_L) = A_L$$

$$\mathcal{R}(A_L) \subseteq \mathcal{R}(A_L^TA_L) \qquad \text{(By Lemma (2.2))}$$

Similarly, $\mathcal{R}(A_U) \subseteq \mathcal{R}(A_U^TA_U)$. By Equation (3) we have, $\mathcal{R}(A) \subseteq \mathcal{R}(A^TA)$.Thus, $\mathcal{R}(A) = \mathcal{R}(A^TA)$. Since $X \in A\{1\}$, $\mathcal{R}(A) = \mathcal{R}(XA)$. Hence, $\mathcal{R}(A^TA) = \mathcal{R}(A) = \mathcal{R}(XA)$. Since $\mathcal{R}(A^TA) \supseteq \mathcal{R}(XA)$ (By Lemma (2.5)),

$YA^TA = XA$ let $Y = [Y_L, Y_U]$ then, $A_L^TA_L(Y_LA_L^TA_L) = A_L^TA_L(X_LA_L)$

$$(A_L^TA_L)Y_L(A_L^TA_L) = A_L^T(A_LX_LA_L)$$

$$= A_L^TA_L$$

Similarly, $A_U^TA_U(Y_UA_U^TA_U) = A_U^TA_U$. By (3) we have, $A^TA(YA^TA) = A^TA$

Thus A^TA is a regular interval valued fuzzy matrix. Conversely, let A^TA be a regular interval-valued fuzzy matrix and $\mathcal{R}(A) = \mathcal{R}(A^TA)$. By Lemma (2.3), A is a regular IVFM. Let us take $Y = (A^T)^- A^T \in$(IVFM). We claim that $Y \in A\{1, 3\}$.

$\mathcal{R}(A) = \mathcal{R}(A^TA)$ and A^TA is regular, by Lemma (2.3) $A = A(A^TA)^- A^TA) = AYA$, $Y \in A\{1\}$ and since $\mathcal{R}(A) = \mathcal{R}(A^TA)$, $A = XA^TA$, by Lemma (2.4), $A_L = X_LA_L{}^TA_L$ and $A_U = X_U A_U{}^TA_U$. Let $Y= [Y_L, Y_U]$.

Then, $A_LY_L = X_LA_L{}^TA_L (A_L{}^TA_L)^- A_L{}^T$

$= X_LA_L{}^TA_L (A_L{}^TA_L)^- A_L{}^TA_LX_L{}^T$

$= X_L (A_L{}^TA_L)(A_L{}^TA_L)^- (A_L{}^TA_L)X_L{}^T$

$= X_L (A_L{}^TA_LX_L{}^T)$

$= X_LA_L{}^T$

Similarly, $A_UY_U = X_UA_U{}^T$. Then by Eq. (3) we have, $AY = XA^T$

$(A_LY_L)^T = (X_LA_L{}^T)^T$

$= A_LX_L{}^T$

$= X_LA_L{}^TA_LX_L{}^T$

$= X_LA_L{}^T = A_LY_L$

Similarly, $(A_UY_U)^T = X_UA_U{}^T = A_UY_U$. Then by Equation (3) we have, $(AY)^T = AY$, $Y \in A\{3\}$. Since $\mathcal{R}(A) = \mathcal{R}(A^TA)$ by Lemma (2.4) and regularity of A^TA we get

$A = A(A^TA)^- (A^TA) = AYA$, $Y \in A\{1\}$. Thus A has a {1, 3} inverse.

Theorem 3.7. For $A \in (\text{IVFM})_{mn}$, A has {1, 4} inverse if and only if AA^T is regular and $\mathcal{C}(AA^T) = \mathcal{C}(A)$.

Proof. This can be proved in the same manner as that of Theorem (3.6).

Corollary 3.8. Let $A \in (\text{IVFM})_{mn}$ be a regular IVFM with A^TA is a regular IVFM and $\mathcal{R}(A^TA) = \mathcal{R}(A)$, then $Y = (A^TA)^- A^T \in A\{1, 2, 3\}$.

Proof. $Y \in A\{1, 3\}$ follows from Theorem (3.6), it is enough verify $Y = [Y_L, Y_U] \in A\{2\}$ that is, $Y_LA_LY_L = Y_L$ and $Y_UA_UY_U = Y_U$.

$Y_LA_LY_L = Y_L (X_L{}^TA_L{}^TA_L) (A_L{}^TA_L)^- A_L{}^T$

$= Y_L X_L^T (A_L^T A_L)(A_L^T A_L)^- (A_L^T A_L X_L)$

$= Y_L X_L^T (A_L^T A_L)(A_L^T A_L)^- (A_L^T A_L) X_L$

$= Y_L X_L^T A_L^T A_L X_L$

$= Y_L A_L X_L$

$= [(A_L^T A_L)^- A_L^T] A_L X_L$

$= (A_L^T A_L)^- (A_L^T A_L X_L)$

$= (A_L^T A_L)^- A_L^T$

$= Y_L$

Similarly, $Y_U A_U Y_U = Y_U$. Then by Eq. (3), $YAY = Y$.

Thus $Y \in A\{1,2,3\}$.

Theorem 3.9. Let $A \in (IVFM)_{mn}$ be a regular IVFM with AA^T is a regular IVFM and $\mathcal{R}(A^T) = \mathcal{R}(AA^T)$ then $Z = A^T (AA^T)^- \in A\{1, 2, 4\}$.

Proof. Similar to the proof of Theorem (3.7) and Corollary (3.8) hence omitted.

4 Conclusion

The main results of the present paper are the generalization of the results on g-inverses of regular fuzzy matrices found in [2, 6] and the extension of our earlier work on regular IVFMs [8].

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