J. Math. Fund. Sci.,Vol. 53, No. 1, 2021, 49-66

Coefficient Estimates for Bi-univalent Functions Defined by (P, Q) Analogue of the Salagean Differential Operator Related to the Chebyshev Polynomials

Susanta Kumar Mohapatra[1] & Trailokya Panigrahi[2*]

[1]Department of Mathematics, Kalinga Institute of Social Sciences (KISS) Deemed to be University, Bhubaneswar-751024, Odisha, India.
[2]Institute of Mathematics and Applications, Andharua, Bhubaneswar-751029, Odisha, India.
*E-mail: trailokyap6@gmail.com

Abstract. In the present investigation, we use the Jackson (p,q)-differential operator to introduce the extended Salagean operator denoted by $R_{p,q}^{k}$. Certain bi-univalent function classes based on operator $R_{p,q}^{k}$ related to the Chebyshev polynomials are introduced. First two coefficient bounds and Fekete-Szego inequalities for the function classes are established. A number of corollaries are developed by varying parameters involved.

Keywords: *analytic function; bi-univalent function; Chebyshev polynomial; Fekete-Szego inequalities; (p,q)-differential operator; Salagean operator.*

1 Introduction

The q-calculus has great applications in the space of geometric functions theory because of their usefulness in the area of ordinary fractional calculus and optimal control problems. Jackson (see [1,2]) developed the concept of q-integral and q-derivative and much later its geometrical interpretation was identified through studies of quantum groups. This has attracted the attention of several researchers. Researchers all over the globe have applied it to construct and investigate several classes of analytic and bi-univalent functions. For recent expository work on so called post-quantum calculus or (p,q) calculus, see [3,4]. We here recall the definition of fractional q-calculus operators of complex valued function *f(z)*.

Definition 1.1. (see [3]) The (p,q)-derivative of *f* is defined as:

$$(D_{p,q}f)(z)=\begin{cases}\frac{f(pz)-f(qz)}{(p-q)z} & (z\neq 0)\\ f'(0)\end{cases} \tag{1}$$

provided that *f* is differentiable at 0. Now $D_{p,q}z^{n}=[n]_{p,q}z^{n-1}$, where

Received August 5th, 2018, Revised June 15th, 2020, Accepted for publication March 3rd, 2021
 ISSN: 2337-5760,
DOI:10.5614/j.math.fund.sci.2021.53.1.4

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} (0 < q < p \leq 1) \tag{2}$$

refers to a twin-basic number. For *p=1*, the Jackson (p,q)-derivative reduces to the Jackson q-derivative given by:

$$\left(D_q f\right)(z) = \frac{f(z)-f(qz)}{(1-q)z} \; (z \neq 0).$$

The class of all analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ is given by:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \; (z \in U) \tag{3}$$

where $U:=\{z \in C: |z| < 1\}$ represents the open unit disk. We denote such class by A. Let S represent the class of all analytic univalent functions of the form (3) in U. Let $f, g \in A$. Then f is subordinate to g, written as $f \prec g$, if there is an analytic function w in U with w(0)=0 and |w(z)|<1 such that f(z)=g(w(z)) ($z \in U$) (see [5, 6]). "The Koebe One-Quarter-Theorem asserts that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, the inverse of $f \in S$ is a univalent analytic function on the disk $U_\rho = \{z: z \in C \; and \; |z| < \rho, \rho \geq \frac{1}{4}\}$", see [7]. For each $f \in S, f(z) = w$ has an inverse function $f^{-1}(w)$ of *f(z)* defined as:

$$g(w) = f^{-1}(w)$$
$$= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{4}$$

If both $f, f^{-1} \in S$ then f is said to be bi-univalent in U. The class of all functions *f* given by (3) is denoted by $\sum$. For a detailed history and other related properties of functions in the class $\sum$, see recent works in [8-13].

For a function *f* given by (3), a simple calculation shows that

$$D_{p,q} f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}. \tag{5}$$

The (p,q)-analogue of Salagean differential operator $R_{p,q}^k: A \to A (k \in N_0 = N \cup \{0\})$ is defined by:

$$R_{p,q}^0 f(z) = f(z)$$

$$R_{p,q}^1 f(z) = z\left(D_{p,q} f(z)\right),$$

$$\dots$$

$$R_{p,q}^k f(z) = R_{p,q}^1 (R_{p,q}^{k-1} f(z)) \tag{6}$$

Thus, for a function *f(z)* of the form (3), we have:

$$R_{p,q}^k f(z) = z + \sum_{n=2}^{\infty} [n]_{p,q}^k \, a_n z^n. \tag{7}$$

Similarly, for a function g of the form (4), we have:

$$R_{p,q}^k g(w) = w - [2]_{p,q}^k a_2 w^2 + (2a_2^2 - a_3)[3]_{p,q}^k w^3 - (5a_2^3 - 5a_2 a_3 + a_4)[4]_{p,q}^k w^4 + \cdots \tag{8}$$

From above, we observe that:

$$\lim_{p\to 1, q\to 1-} R_{p.q}^k f(z) = z + \lim_{p\to 1, q\to 1-} \sum_{n=2}^{\infty} [n]_{p,q}^k \, a_n z^n = z + \sum_{n=2}^{\infty} n^k a_n \, z^n = D^k f(z), \tag{9}$$

where D^k is the Salagean differential operator which was defined in [14] and has been studied by several authors.

Chebyshev polynomials of the first and second kind and their properties have been studied by several researchers (see, for details [15,16]). We consider

$$L(z,t) = \frac{1}{1-2tz+z^2} \, (z \in U)$$

as its generating function. Taking $t = \cos\alpha$, $\alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, we have:

$$\begin{aligned} L(z,t) &= \frac{1}{1-2\cos\alpha z + z^2} \\ &= 1 + 2\cos\alpha \, z + (3\cos^2\alpha - \sin^2\alpha) z^2 + \cdots \\ &= 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in U, t \in (-1,1)), \end{aligned} \tag{10}$$

where $U_{n-1}(t) = \frac{\sin(\cos^{-1} t)}{\sqrt{1-t^2}} (n \in N)$. Thus we have

$$\begin{aligned} &U_1(t) = 2t, && U_3(t) = 8t^3 - 4t, \\ &U_2(t) = 4t^2 - 1, && U_4(t) = 16t^4 - 12t^2 + 1, \ldots \end{aligned} \tag{11}$$

Recently, several researchers, Altinkaya and Yalcin [17-19], Bulut *et al.* [20,21] Guney *et al.* [22] and Caglar [23] (also see [24]) to mention a few, have obtained Fekete-Szego inequalities and some coefficient bounds for different subclasses of bi-univalent functions. Motivated by the above researchers, we consider two subclasses of bi-univalent functions that are obtained by using the $D_{p,q}$ operator of the Salagean type associated with the Chebyshev polynomial.

Definition 1.2. A function $f \in \sum$ defined as Eq. (3) belongs to the function class $R_{\sum,p,q}^k(\gamma, t) (0 \le \gamma \le 1)$ if the conditions

$$(1-\gamma)\frac{R_{p.q}^{k+1}f(z)}{R_{p.q}^{k}f(z)}+\gamma\frac{R_{p.q}^{k+2}f(z)}{R_{p.q}^{k+1}f(z)}\prec L(z,t)\left(\frac{1}{2}<t<1;z\in U\right), \quad (12)$$

and

$$(1-\gamma)\frac{R_{p.q}^{k+1}g(w)}{R_{p.q}^{k}g(w)}+\gamma\frac{R_{p.q}^{k+2}g(w)}{R_{p.q}^{k+1}g(w)}\prec L(w,t)\left(\frac{1}{2}<t<1;w\in U\right), \quad (13)$$

are satisfied, where g is stated in (4).

By specializing the parameters γ, p, q and k in the above definition, we obtain the various subclasses of $\sum$.

Definition 1.3. A function $f\in\sum$ belongs to the function class $T_{\sum,p,q}^{k}(\beta,t)$ if

$$(1-\beta)\frac{R_{p.q}^{k}f(z)}{z}+\beta(R_{p,q}^{k}f(z))'\prec L(z,t), \quad (14)$$

and

$$(1-\beta)\frac{R_{p.q}^{k}g(w)}{w}+\beta(R_{p,q}^{k}g(w))'\prec L(w,t) \quad (15)$$

$\left(0\le\beta\le1,\frac{1}{2}<t<1;\ z,w\in U\right)$, hold where $R_{p,q}^{k}f(z)$ and $R_{p,q}^{k}g(w)$ are given by Eq. (7) and Eq. (8) respectively.

Remark 1.4. For $p\to1, q\to1^{-}$, we get the class $T_{\sum,1,1^{-}}^{k}(\beta,t)=F_{\sum}^{k}(\beta,L(z,t))$ consists of function $f\in\sum$ and satisfying

$$(1-\beta)\frac{D^{k}f(z)}{z}+\beta(D^{k}f(z))'\prec L(z,t)$$

and

$$(1-\beta)\frac{D^{k}g(w)}{w}+\beta(D^{k}g(w))'\prec L(w,t).$$

This class is due to Guney *et al.*[22].

Remark 1.5. For $p\to1, q\to1^{-}$and $k=0$, we obtain the class $T_{\sum,1,1^{-}}^{0}(\beta,t)=B_{\sum}(\beta,t)$ (see[20, 21]) where $f\in\sum$ satisfying

$$(1-\beta)\frac{f(z)}{z}+\beta(f(z))'\prec L(z,t)$$

and

$$(1-\beta)\frac{g(w)}{w}+\beta(g(w))'\prec L(w,t).$$

In this work, we investigate the first two coefficient bounds and Fekte-Szego inequalities in the above newly constructed function classes by using the Chebyshev polynomial.

2 Coefficient Bounds

In the following theorems, we establish Chebyshev polynomial bounds $|a_2|$ and $|a_3|$ for the function classes $R^k_{\Sigma,p,q}(\gamma,t)$ and $T^k_{\Sigma,p,q}(\beta,t)$.

Theorem 2.1. Assume that $f\in\sum$ defined as Eq. (3) is in the class $\mathrm{R}^{\mathrm{k}}_{\Sigma,\mathrm{p,q}}(\gamma,\mathrm{t})\left(\frac{1}{2}<t<1\right)$. Then

$$|a_2|\le\frac{2t\sqrt{2t}}{\sqrt{|(A_1-A_2)4t^2+A_2|}}, \tag{16}$$

and

$$|a_3|\le\frac{4t^2}{[2]^{2k}_{p,q}([2]_{p,q}-1)^2(1+\gamma([2]_{p,q}-1))^2}+\frac{2t}{[3]^{k}_{p,q}([3]_{p,q}-1)(1+\gamma([3]_{p,q}-1))}, \tag{17}$$

where

$$A_1=[3]^k_{p,q}(1+\gamma([3]_{p,q}-1))([3]_{p,q}-1)-[2]^{2k}_{p,q}(1+\gamma([2]^2_{p,q}-1))([2]_{p,q}-1), \tag{18}$$

and

$$A_2=[2]^{2k}_{p,q}(1+\gamma([2]_{p,q}-1))^2([2]_{p,q}-1)^2. \tag{19}$$

Proof: Assume that $f\in\mathrm{R}^{\mathrm{k}}_{\Sigma,\mathrm{p,q}}(\gamma,\mathrm{t})$. Definition 1.2 yields:

$$(1-\gamma)\frac{R^{k+1}_{p,q}f(z)}{R^{k}_{p,q}f(z)}+\gamma\frac{R^{k+2}_{p,q}f(z)}{R^{k+1}_{p,q}f(z)}1+U_1(t)r(z)+U_2(t)r^2(z)+\cdots \tag{20}$$

and

$$(1-\gamma)\frac{R^{k+1}_{p,q}g(w)}{R^{k}_{p,q}g(w)}+\gamma\frac{R^{k+2}_{p,q}g(w)}{R^{k+1}_{p,q}g(w)}=1+U_1(t)s(w)+U_2(t)s^2(w)+\cdots \tag{21}$$

where r(z) and s(w) are analytic functions given by

$$r(z)=c_1z+c_2z^2+c_3z^3+\cdots, \tag{22}$$
$$s(w)=d_1w+d_2w^2+d_3w^3+\cdots, \tag{23}$$

where $\mathrm{r}(0)=\mathrm{s}(0)=0, |\mathrm{r}(\mathrm{z})|<1, |s(w)|<1\ (z,w\in U)$. If $|\mathrm{r}(\mathrm{z})|<1$ and $|\mathrm{s}(\mathrm{w})|<1$, then

$$|c_i| \le 1 \; and \, |d_i| < 1 \text{ for all } i \in N. \tag{24}$$

Making use of Eq. (22) in Eq. (20) and Eq. (23) in Eq. (21), we get

$$(1-\gamma)\frac{R_{p,q}^{k+1}f(z)}{R_{p,q}^{k}f(z)} + \gamma\frac{R_{p,q}^{k+2}f(z)}{R_{p,q}^{k+1}f(z)}$$
$$= 1 + U_1(t)c_1 z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \cdots \tag{25}$$

and

$$(1-\gamma)\frac{R_{p,q}^{k+1}g(w)}{R_{p,q}^{k}g(w)} + \gamma\frac{R_{p,q}^{k+2}g(w)}{R_{p,q}^{k+1}g(w)}$$
$$= 1 + U_1(t)d_1 w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + \cdots \tag{26}$$

It follows from Eq. (7) and Eq. (8) that

$$(1-\gamma)\frac{R_{p,q}^{k+1}f(z)}{R_{p,q}^{k}f(z)} + \gamma\frac{R_{p,q}^{k+2}f(z)}{R_{p,q}^{k+1}f(z)}$$
$$= 1 + [2]_{p,q}^{k}(1+\gamma([2]_{p,q}-1))([2]_{p,q}-1)a_2 z + ([3]_{p,q} - 1)\{[3]_{p,q}^{k}(1+\gamma([3]_{p,q}-1))a_3 - ([2]_{p,q}-1)[2]_{p,q}^{k}(1+\gamma([2]_{p,q}^{2} - 1))a_2^2\}z^2 + \cdots \tag{27}$$

and

$$(1-\gamma)\frac{R_{p,q}^{k+1}g(w)}{R_{p,q}^{k}g(w)} + \gamma\frac{R_{p,q}^{k+2}g(w)}{R_{p,q}^{k+1}g(w)} = 1 - [2]_{p,q}^{k}(1+\gamma([2]_{p,q} - 1))([2]_{p,q}-1)a_2 w + [\{2([3]_{p,q}-1)[3]_{p,q}^{k}(1+\gamma([3]_{p,q}-1)) - ([2]_{p,q}-1)[2]_{p,q}^{k}(1+\gamma([2]_{p,q}^{2}-1))\}a_2^2 - [3]_{p,q}^{k}(1+\gamma([3]_{p,q} - 1))([3]_{p,q}-1)a_3]w^2 + \cdots \tag{28}$$

Using Eq. (27) in Eq. (25) and Eq. (28) in Eq. (26), we obtain:

$$1 + [2]_{p,q}^{k}\left(1+\gamma\big([2]_{p,q}-1\big)\right)\big([2]_{p,q}-1\big)a_2 z + \Big[\big([3]_{p,q} - 1\big)[3]_{p,q}^{k}\left(1+\gamma\big([3]_{p,q}-1\big)\right)a_3 - \big([2]_{p,q}-1\big)[2]_{p,q}^{2k}\left(1 + \gamma\big([2]_{p,q}^{2}-1\big)\right)a_2^2\Big]z^2 + \cdots = 1 + U_1(t)c_1 z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \cdots \tag{29}$$

and

$$1-\left([2]_{p,q}-1\right)[2]_{p,q}^{k}\left(1+\gamma\left([2]_{p,q}-1\right)\right)a_2\omega+\Big[\Big\{2\left([3]_{p,q}-1\right)[3]_{p,q}^{k}\left(1+\gamma\left([3]_{p,q}-1\right)\right)-\left(1+\gamma\left([2]_{p,q}^{2}-1\right)\right)\left([2]_{p,q}-1\right)[2]_{p,q}^{2k}\Big\}a_2^2-\left([3]_{p,q}-1\right)[3]_{p,q}^{k}\left(1+\gamma\left([3]_{p,q}-1\right)\right)a_3\Big]\omega^2+\cdots$$
$$=1+U_1(t)d_1\omega+[U_1(t)d_2+U_2(t)d_1^2]\omega^2+\cdots \tag{30}$$

Equating the coefficients in Eq. (29) and Eq. (30), we get:

$$\left([2]_{p,q}-1\right)[2]_{p,q}^{k}\left(1+\gamma\left([2]_{p,q}-1\right)\right)a_2=U_1(t)c_1, \tag{31}$$

$$-\left([2]_{p,q}-1\right)[2]_{p,q}^{2k}\left(1+\gamma\left([2]_{p,q}^{2}-1\right)\right)a_2^2+\left([3]_{p,q}-1\right)[3]_{p,q}^{k}\left(1+\gamma\left([3]_{p,q}-1\right)\right)a_3=U_1(t)c_2+U_2(t)c_1^2, \tag{32}$$

and

$$-\left([2]_{p,q}-1\right)[2]_{p,q}^{k}\left(1+\gamma\left([2]_{p,q}-1\right)\right)a_2=U_1(t)d_1, \tag{33}$$

and

$$\left\{2\left([3]_{p,q}-1\right)[3]_{p,q}^{k}\left(1+\gamma\left([3]_{p,q}-1\right)\right)-\left([2]_{p,q}-1\right)[2]_{p,q}^{2k}\left(1+\gamma\left([2]_{p,q}^{2}-1\right)\right)\right\}a_2^2-\left([3]_{p,q}-1\right)[3]_{p,q}^{k}\left(1+\gamma\left([3]_{p,q}-1\right)\right)a_3=U_1(t)d_2+U_2(t)d_1^2. \tag{34}$$

From Eq. (31) and Eq. (33), we obtain:

$$c_1=-d_1, \tag{35}$$

and

$$2\left([2]_{p,q}-1\right)^2[2]_{p,q}^{2k}\left(1+\gamma\left([2]_{p,q}-1\right)\right)^2a_2^2$$
$$=U_1^2(t)(c_1^2+d_1^2). \tag{36}$$

Adding Eq. (32) and Eq. (34) and using Eq. (36) in the resulting equation, we obtain:

$$\Big[2\left([3]_{p,q}-1\right)[3]_{p,q}^{k}\left(1+\gamma\left([3]_{p,q}-1\right)\right)-2\left([2]_{p,q}-1\right)[2]_{p,q}^{2k}\left(1+\gamma\left([2]_{p,q}^{2}-1\right)\right)-\frac{U_2(t)}{U_1^2(t)}2[2]_{p,q}^{2k}\left([2]_{p,q}-1\right)^2\left[1+\gamma\left([2]_{p,q}-1\right)\right]^2\Big]a_2^2=U_1(t)(c_2+d_2), \tag{37}$$

which gives:

$$a_2^2 = \frac{(c_2+d_2)U_1^3(t)}{2[A_1U_1^2(t)-A_2U_2(t)]}, \tag{38}$$

where A_1 and A_2 are given in Eq. (18) and Eq. (19) respectively. Applying Eq. (24) to the coefficients c_2 and d_2 and using Eq. (11) in Eq. (38), we get the desire estimate for $|a_2|$.

Subtracting Eq. (34) from Eq. (32) and using Eq. (35) and Eq. (36) in the resulting equation yields:

$$a_3 = \frac{(c_1^2+d_1^2)U_1^2(t)}{2[2]_{p,q}^{2k}\left([2]_{p,q}-1\right)^2\left[1+\gamma\left([2]_{p,q}-1\right)\right]^2} + \frac{(c_2-d_2)U_1(t)}{2[3]_{p,q}^{k}\left([3]_{p,q}-1\right)\left[1+\gamma\left([3]_{p,q}-1\right)\right]} . \tag{39}$$

Taking the coefficient inequalities for $c_1, c_2, d_1$and d_2 from Eq. (24) and making use of Eq. (11) in Eq. (39) we get the estimate for $|a_3|$ as stated in Eq. (17). This proves the Theorem 2.1.

Letting $p \to 1$ and q→ 1^- in Theorem 2.1, we get the result for the class $R_{\Sigma,1,1^{-1}}^k(\gamma, t) \equiv M_\Sigma^k(\gamma, L(z,t))$ due to Guney *et al.* [22] as follows:

Corollary 2.2 (see [22]): Let $f \in M_\Sigma^k\big(\gamma, L(z,t)\big)$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[2(1+2\gamma)3^k-(\gamma(\gamma+5)+2)2^{2k}]4t^2+2^{2k}(1+\gamma)^2|}}$$

and

$$|a_3| \leq \frac{4t^2}{(1+\gamma)^2 2^{2k}} + \frac{t}{(1+2\gamma)3^k} .$$

Letting $\gamma = 0$ in Theorem 2.1, the following result for the function class $R_{\Sigma,p,q}^k(0,t) \equiv N_{\Sigma,p,q}^k(t)$ is obtained.

Corollary 2.3. If $f \in N_{\Sigma,p,q}^k(t)$, then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[([3]_{p,q}-1)[3]_{p,q}^k-([2]_{p,q}-1)[2]_{p,q}^{2k+1}]4t^2+([2]_{p,q}-1)^2[2]_{p,q}^{2k}|}}$$

and

$$|a_3| \leq \frac{2t}{[3]_{p,q}^k([3]_{p,q}-1)} + \frac{4t^2}{[2]_{p,q}^{2k}([2]_{p,q}-1)^2} .$$

Letting $p \to 1$ and $q \to 1^-$ in the above corollary, we get the following result for the class $R_{\Sigma,1,1^-}^k(0,t) \equiv N_\Sigma^k(t)$.

Corollary 2.4. Let $f \in N_{\Sigma}^{k}(t)$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(3^k-2^{2k})8t^2+2^{2k}|}},$$

and

$$|a_3| \leq \frac{t}{3^k} + \frac{4t^2}{2^{2k}}.$$

Putting $\gamma = 1$ in Theorem 2.1, the result for the class $R_{\Sigma,p,q}^{k}(1,t) \equiv U_{\Sigma.p.q}^{k}(t)$ is as follows:

Corollary 2.5. Let $f \in U_{\Sigma,p,q}^{k}(t)$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[([3]_{p,q}-1)[3]_{p,q}^{k+1}-([2]_{p,q}-1)[2]_{p,q}^{2k+3}]4t^2+[2]_{p,q}^{2k+2}([2]_{p,q}-1)^2|}},$$

and

$$|a_3| \leq \frac{4t^2}{[2]_{p,q}^{2k+2}([2]_{p,q}-1)^2} + \frac{2t}{([3]_{p,q}-1)[3]_{p,q}^{k+1}}.$$

Taking $p \to 1, q \to 1^-$and $\gamma = 1$ in the above theorem, the result for the class $R_{\Sigma,1,1^-}^{k}(1,t) \equiv K_{\Sigma}^{k}(L(z,t))$ is obtained.

Corollary 2.6 (see [22]): If $f \in K_{\Sigma}^{k}(L(z,t))$, then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|[3^{k+1}-2^{2(k+1)}]2t^2+2^{2k}|}},$$

and

$$|a_3| \leq \frac{t^2}{2^{2k}} + \frac{t}{3^{k+1}}.$$

Theorem 2.1 for $k=0$ gives

Corollary 2.7. Let the function $f \in V_{\Sigma,p,q}(\gamma,t)(\equiv R_{\Sigma,p,q}^{0}(\gamma,t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|4M_1t^2+M_2|}},$$

and

$$|a_3| \leq \frac{4t^2}{(1+\gamma([2]_{p,q}-1))^2([2]_{p,q}-1)^2} + \frac{2t}{(1+\gamma([3]_{p,q}-1))([3]_{p,q}-1)},$$

where

$$M_1 = (1+\gamma([3]_{p,q}-1))([3]_{p,q}-1) - ([2]_{p,q}-1)\{1+\gamma([2]_{p,q}^2 - 1) + ([2]_{p,q}-1)(1+\gamma([2]_{p,q}-1))^2\},$$

and

$$M_2 = (1+\gamma([2]_{p,q}-1))^2([2]_{p,q}-1)^2.$$

Letting $p \to 1$ and $q \to 1^-$in the above result, we get

Corollary 2.8. Let $f \in V_{\Sigma}(\gamma, t)(\equiv R^0_{\Sigma,1,1^-}(\gamma, t))$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(1+\gamma)^2-(\gamma+\gamma^2)4t^2|}},$$

and

$$|a_3| \leq \frac{4t^2}{(1+\gamma)^2} + \frac{t}{1+2\gamma}.$$

Putting $\gamma = 0$ in the above corollary gives the following.

Corollary 2.9. Let $f \in V_{\Sigma}(t)(\equiv V_{\Sigma}(0, t)$. Then

$$|a_2| \leq 2t\sqrt{2t},$$

and

$$|a_3| \leq t + 4t^2.$$

Putting $\gamma = 1$ in Corollary 2.8 gives:

Corollary 2.10. Let $f \in Q_{\Sigma}(t)(\equiv V_{\Sigma}(1, t))$. For $t \neq \frac{1}{\sqrt{2}}$, we have

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-2t^2|}},$$

and

$$|a_3| \leq t^2 + \frac{t}{3}.$$

Theorem 2.11. Let $f \in T^k_{\Sigma,p,q}(\beta, t)$.Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|[[3]^k_{p,q}(1+2\beta)-[2]^k_{p,q}(1+\beta)^2]4t^2+[2]^{2k}_{p,q}(1+\beta)^2|}}, \quad (40)$$

and

$$|a_3| \leq \frac{2t}{(1+2\beta)[3]_{p,q}^k} + \frac{4t^2}{(1+\beta)^2[2]_{p,q}^{2k}}. \quad (41)$$

Proof: Let $f \in T_{\Sigma,p,q}^k(\beta,t)$. Proceeding as before, we have

$$(1+\beta)[2]_{p,q}^k a_2 = U_1(t)c_1, \quad (42)$$

$$(1+2\beta)[3]_{p,q}^k a_3 = U_1(t)c_2 + U_2(t)c_1^2 \quad (43)$$

and

$$-(1+\beta)[2]_{p,q}^k a_2 = U_1(t)d_1, \quad (44)$$

$$(1+2\beta)(2a_2^2 - a_3)[3]_{p,q}^k = U_1(t)d_2 + U_2(t)d_1^2, \quad (45)$$

It follows from Eq. (42) and Eq. (44) that

$$c_1 = -d_1, \quad (46)$$

$$2(1+\beta)^2[2]_{p,q}^{2k} a_2^2 = U_1^2(t)(c_1^2 + d_1^2). \quad (47)$$

Similarly, from Eq. (43) and Eq. (45) we have:

$$2(1+2\beta)[3]_{p,q}^k a_2^2 = U_1(t)(c_2 + d_2) + U_2(t)(c_1^2 + d_1^2). \quad (48)$$

Using Eq. (47) in Eq. (48) and simplifying we get:

$$a_2^2 = \frac{(c_2+d_2)U_1^2(t)}{2[(1+2\beta)[3]_{p,q}^k U_1^2(t) - (1+\beta)^2[2]_{p,q}^{2k} U_2(t)]}. \quad (49)$$

Putting the values of $U_1(t)$, $U_2(t)$ from Eq. (11) and using Eq. (24) in Eq. (49) we get the desire estimate for $|a_2|$ as given by Eq. (40).

Subtracting Eq. (45) from Eq. (43) and making use of Eq. (46) and Eq. (47) in the resulting equation and simplifying, we get:

$$a_3 = \frac{(c_2-d_2)U_1(t)}{2(1+2\beta)[3]_{p,q}^k} + \frac{(c_2{}^2-d_2{}^2)U_1{}^2(t)}{2(1+2\beta)^2[3]_{p,q}^{2k}} \quad (50)$$

Using Eq. (11) and Eq. (24) in Eq. (50) we get the bounds for $|a_3|$. The proof of Theorem 2.11 is completed.

Taking $q \to 1^-, p \to 1$ in Theorem 2.11, the result for the class $T_{\Sigma,1,1^-}^k(\beta,t)(\equiv F_\Sigma^k(\beta,L(z,t))$ is obtained.

Corollary 2.12. Let $f \in \Sigma$ given by Eq. (3) be in the class $F_\Sigma^k(\beta,L(z,t))$. Then

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|[(1+2\beta)3^k-(1+\beta)^2 2^{2k}]4t^2+(1+\beta)^2 2^{2k}|}},$$

and

$$|a_3| \le \frac{2t}{(1+2\beta)3^k} + \frac{4t^2}{(1+\beta)^2 2^{2k}}.$$

Putting $\beta = 0$ in Corollary 2.12 we get the result for the function class $T^k_{\Sigma,1,1^-}(0,t) \equiv F^k_\Sigma(L(z,t))$ as follows:

Corollary 2.13 (see [22]): Let $f \in F^k_\Sigma(L(z,t))$. Then

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|[3^k-2^{2k}]4t^2+2^{2k}|}},$$

and

$$|a_3| \le \frac{2t}{3^k} + \frac{4t^2}{2^{2k}} \ .$$

Corollary 2.13 for $\beta = 1$ yields the result for the class $T^k_{\Sigma,1,1^-}(1,t) \equiv H^k_\Sigma(L(z,t))$ as below.

Corollary 2.14. Let $f \in H^k_\Sigma(L(z,t))$. Then

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|[3^{k+1}-2^{2(k+1)}]4t^2+2^{2(k+1)}|}},$$

and

$$|a_3| \le \frac{2t}{3^{k+1}} + \frac{t^2}{2^{2k}}.$$

Corollary 2.12 for $k = 0$ gives the result for the class $T^0_{\Sigma,1,1^-}(\beta,t) \equiv F_\Sigma(\beta, L(z,t))$ as below.

Corollary 2. 15. Let $f \in F_\Sigma(\beta, L(z,t))$. Then

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{|(\beta+1)^2-4\beta^2 t^2|}},$$

and

$$|a_3| \le \frac{2t}{1+2\beta} + \frac{4t^2}{(1+\beta)^2}.$$

Putting $\beta = 0$ in Corollary 2.15 gives the result for the function class $T^{0}_{\sum,1,1^{-}}(0,t) \equiv T_{\sum}(t)$.

Corollary 2. 16. Let $f \in T_{\sum}(t)$. Then

$$|a_2| \leq 2t\sqrt{2t},$$

and

$$|a_3| \leq 2t + 4t^2.$$

Letting $\beta = 1$ in Corollary 2.15, we get the result for the class $F_{\sum}(1, L(z,t)) \equiv F_{\sum}(L(z,t))$.

Corollary 2.17. Let $f \in F_{\sum}(L(z,t))$. Then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{1-t^2}},$$

and

$$|a_3| \leq t^2 + \frac{2}{3}t \ .$$

3 Fekete-Szego Inequalities

In the following section, we obtain the Fekete-Szego problems for the function class $R^{k}_{\sum,p,q}(\gamma,t)$ and $T^{k}_{\sum,p,q}(\beta,t)$ as follows:

Theorem 3.1. Let $f \in R^{k}_{\sum,p,q}(\gamma,t)$. Then

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2t}{([3]_{p,q}-1)[3]^{k}_{p,q}(1+\gamma([3]_{p,q}-1))} & |\eta - 1| \leq \left|\frac{\frac{A_2}{4t^2}+M_3}{M_5}\right| \\ \frac{8t^3|1-\eta|}{|[M_3]4t^2+A_2|} & |\eta - 1| \geq \left|\frac{\frac{A_4}{4t^2}+M_3}{M_5}\right|, \end{cases} \tag{51}$$

where

$$M_3 = M_5 - M_4 - A_2, \tag{52}$$

$$M_4 = ([2]_{p,q} - 1)[2]^{2k}_{p,q}(1 + \gamma([2]^{2}_{p,q} - 1)) \tag{53}$$

$$M_5 = ([3]_{p,q} - 1)[3]^{k}_{p,q}(1 + \gamma([3]_{p,q} - 1)) \tag{54}$$

and A_2 is defined in Eq. (19).

Proof: It follows from Eq. (32) and Eq. (34) that

$$a_3 - \eta a_2^2 = (1-\eta)\frac{(c_2+d_2)U_1^2(t)}{2[(M_5-M_4)U_1^2(t)-A_2U_2(t)]} + \frac{(c_2-d_2)U_1(t)}{2M_5}$$
$$= U_1(t)\left[\left(g(\eta)+\frac{1}{2M_5}\right)c_2 + \left(g(\eta)-\frac{1}{2M_5}\right)d_2\right], \quad (55)$$

where

$$g(\eta) = \frac{(1-\eta)U_1^2(t)}{2[(M_5-M_4)U_1^2(t)-A_2U_2(t)]}. \quad (56)$$

Taking the values of $U_1(t)$ and $U_2(t)$ from Eq. (11) and substituting it in Eq. (56) we conclude that

$$|a_3 - \eta a_2^2| \le \begin{cases} \frac{2t}{M_5} & 0 \le |g(\eta)| \le \frac{1}{2M_5} \\ 4t|g(\eta)| & |g(\eta)| \ge \frac{1}{2M_5}. \end{cases} \quad (57)$$

The estimate Eq. (51) follows from Eq. (57). The proof of Theorem 3.1 is thus completed.

Taking $p \to 1$ and $q \to 1^-$ in Theorem 3.1 yields:

Corollary 3.2. Let $f \in M_\Sigma^k(\gamma, L(z,t))(\equiv R_{\Sigma,1,1^-}^k(\gamma,t))$. Then

$$|a_3 - \eta a_2^2| \le \begin{cases} \frac{1}{3^k(1+2\gamma)}, |\eta-1| \le \frac{\left|\frac{(1+\gamma)^2 2^{2k}}{4t^2}+2(1+2\gamma)3^k-(\gamma^2+5\gamma+2)2^{2k}\right|}{2(1+2\gamma)3^k} \\ \frac{8|1-\eta|t^3}{\left|(2(1+2\gamma)3^k-(\gamma^2+5\gamma+2)2^{2k})4t^2+(1+\gamma)^2 2^{2k}\right|}, \\ |\eta-1| \ge \frac{\left|\frac{(1+\gamma)^2 2^{2k}}{4t^2}+2(1+2\gamma)3^k-(\gamma^2+5\gamma+2)2^{2k}\right|}{2(1+2\gamma)3^k} \end{cases}$$

Theorem 3.1 for $\eta = 1$ gives the following:

Corollary 3.3. Let $f \in R_{\Sigma,p,q}^k(\gamma,t)$. We have

$$|a_3 - a_2^2| \le \frac{2t}{M_5}.$$

Letting $\eta = 1$ in Corollary 3.2 we have:

Corollary 3.4 (see [22]): Let $f \in M_\Sigma^k(\gamma, L(z,t))$. We have

$$|a_3 - a_2^2| \le \frac{t}{(1+2\gamma)3^k}.$$

Theorem 3.1 for $\gamma = 0$ gives

Corollary 3.5. Let $f \in N_{\Sigma,p,q}^{k}(t)(\equiv R_{\Sigma,p,q}^{k}(0,t)$. Then

$$\left|a_3 - \eta a_2^2\right| \leq$$

$$\begin{cases} \dfrac{2t}{[3]_{p,q}^{k}([3]_{p,q}-1)}, \; |\eta - 1| \leq \dfrac{\left|\frac{[2]_{p,q}^{2k}([2]p,q-1)^2}{4t^2}+[3]_{p,q}^{k}([3]_{p,q}-1)-[2]_{p,q}^{2k+1}([2]_{p,q}-1)\right|}{[3]_{p,q}^{k}([3]_{p,q}-1)} \\ \dfrac{8t^3|1-\eta|}{\left|[[3]_{p,q}^{k}([3]_{p,q}-1)-[2]_{p,q}^{2k+1}([2]_{p,q}-1)]4t^2+[2]_{p,q}^{2k}([2]_{p,q}-1)^2\right|}, \\ |\eta - 1 \geq \dfrac{\left|\frac{[2]_{p,q}^{2k}([2]p,q-1)^2}{4t^2}+[3]_{p,q}^{k}([3]_{p,q}-1)-[2]_{p,q}^{2k+1}([2]_{p,q}-1)\right|}{[3]_{p,q}^{k}([3]_{p,q}-1)} \end{cases}$$

Taking $p \to 1$ and $q \to 1^-$ in Corollary 3.5, the result for the class $N_{\Sigma}^{k}(t) \equiv N_{\Sigma,1,1^-}^{k}(t)$ is obtained.

Corollary 3.6. Let $f \in N_{\Sigma}^{k}(t)$. Then for any real number η,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{t}{3^k} & |\eta - 1| \leq \left|\dfrac{\frac{2^{2k}}{8t^2}+3^k-2^{2k}}{3^k}\right| \\ \dfrac{8t^3|1-\eta|}{|(3^k-2^{2k})8t^2+2^{2k}|} & |\eta - 1| \geq \left|\dfrac{\frac{2^{2k}}{8t^2}+3^k-2^{2k}}{3^k}\right| \end{cases}$$

Taking $\eta = 1$ and $k = 0$ in Corollary 3.6 we get the estimate for the class $N_{\Sigma}(t) \equiv N_{\Sigma}^{0}(t)$.

Corollary 3.7. Let $f \in \Sigma$ given by Eq. (3) be in the class $N_{\Sigma}(t)$. Then

$$|a_3 - a_2^2| \leq t.$$

Theorem 3.8. Let $f \in \Sigma$ given by Eq. (3) be in the class $T_{\Sigma,p,q}^{k}(\beta,t)$. Then for any $\eta \in R$, we have:

$$\left|a_3-\eta a_2^2\right| \leq$$
$$\begin{cases} \frac{2t}{(1+2\beta)[3]_{p,q}^k}, & |\eta-1| \leq \frac{(1+2\beta)[3]_{p,q}^k-(1+\beta)^2[2]_{p,q}^k+\frac{(1+\beta)^2[2]_{p,q}^{2k}}{4t^2}}{(1+2\beta)[3]_{p,q}^k} \\ \frac{8|1-\eta|t^3}{|((1+2\beta)[3]_{p,q}^k-(1+\beta)^2[2]_{p,q}^{2k})4t^2+(1+\beta)^2[2]_{p,q}^k|}, & |\eta-1| \geq \frac{(1+2\beta)[3]_{p,q}^k-(1+\beta)^2[2]_{p,q}^k+\frac{(1+\beta)^2[2]_{p,q}^{2k}}{4t^2}}{(1+2\beta)[3]_{p,q}^k}. \end{cases}$$

Proof: From Eq. (43) and Eq. (45) we have:

$$a_3-\eta a_2^2=(1-\eta)\frac{U_1^3(t)(c_2+d_2)}{2[(1+2\beta)[3]_{p,q}^k U_1^2(t)-(1+\beta)^2[2]_{p,q}^{2k}U_2(t)]}+\frac{U_1(t)(c_2-d_2)}{2(1+2\beta)[3]_{p,q}^k}$$

$$=U_1(t)\left\{\left[s(\eta)+\frac{1}{2(1+2\beta)[3]_{p,q}^k}\right]c_2+\left[s(\eta)-\frac{1}{2(1+2\beta)[3]_{p,q}^k}\right]d_2\right\}, \quad (58)$$

where

$$s(\eta)=\frac{(1-\eta)U_1^2(t)}{2[(1+2\beta)[3]_{p,q}^k U_1^2(t)-(1+\beta)^2[2]_{p,q}^{2k}U_2(t)]}. \quad (59)$$

In view of (11), we obtain:

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{2t}{(1+2\beta)[3]_{p,q}^k} & 0 \leq |s(\eta)| \leq \frac{1}{2(1+2\beta)[3]_{p,q}^k} \\ 4t|s(\eta)| & |s(\eta)| \geq \frac{1}{2(1+2\beta)[3]_{p,q}^k}. \end{cases} \quad (60)$$

The estimates of Theorem 3.8 follow from Eq. (60). This completes the proof.

Remark 3.9. Many corollaries will be generated by varying parameters involved in Theorem 3.8.

4 Conclusion

A good amount of literature is available for the first few coefficients and the Fekete-Szego problem for different subclasses of univalent and bi-univalent analytic functions by making use of the class of Caratheodory functions. In the present investigation, the authors have introduced newly constructed bi-univalent analytic function classes $R_{\Sigma,p,q}^k(\gamma,t)$ and $T_{\Sigma,p,q}^k(\beta,t)$ associated with the Chebyshev polynomials by using the Salagean (p,q)-differential operator and obtained initial coefficients and Fekete-Szego problems for the above mentioned classes. The generalization of some of the previous results studied by various researchers was obtained. The sigmoid function and Faber polynomial can be used to derive similar results for the classes studied.

Acknowledgement

The authors express their thanks to the editor and anonymous referees for their comments and suggestions to improve the contents. Further, the present investigation of the second-named author was supported by CSIR research project scheme no. 25(0278)/17/EMR-II, New Delhi, India.

References

[1] Jackson, F.H., *Q-Functions and a Certain Difference Operator*, Trans. Royal Soc., Edinburgh, **46**, pp. 253-281, 1908.

[2] Jackson, F.H., *On Q-Definite Integrals*, Quarterly J. Pure Appl. Math., **41**, pp. 193-203, 1910.

[3] Chakrabarti, R. & Jagannathan, R., *A (P, Q)-Oscillator Realization of Two-parameter Quantum Algebras*, J. Phys. A, **24**, pp. 1711-1718, 1991.

[4] Arial, A., Gupta, V. & Agarawal, R.P., *Application of Q-Calculus in Operator Theory*, Springer, New York, 2013.

[5] Bulboaca, T., *Differential Subordinations and Superordinations: Recent Results*, House of Book Publication, Cluj-Napoca, 2005.

[6] Miller, S. S. & Mocanu, P. T., *Differential Subordinations: Theory and Applications*, in: Monographs and Textbooks in Pure and Applied Mathematics, **225**, Marcel Dekker, New York, 2000.

[7] Duren, P.L., *Univalent Functions, Grundlehren der Mathematischen Wissenschaften 259*, Springer-Verlag, New York, Berlin, Heidelbeg, Tokyo, 1983.

[8] Ali, R.M., Lee, S. K., Ravichandran, V. & Supramanian, S., *Coefficient Estimates for Bi-univalent Ma-Minda Starlike and Convex Functions*, Appl. Math. Lett., **25**, pp. 344-351, 2012.

[9] Caglar, M., Orhan, H. & Yagmur, N., C*oefficient Bounds for New Subclasses of Bi-univalent Functions*, Filomat, **27**, pp. 1165-1171, 2013.

[10] Orhan, H., Magesh, N. & Balaji, V.K., *Initial Coefficient Bounds for a General Class of Bi-univalent Functions*, Filomat, **29**, pp. 1259-1267, 2015.

[11] Panigrahi, T. & Murugusundaramoorthy, G., *Coefficient Bounds for Bi-univalent Analytic Functions Associated with Hohlov Operator*, Proced. Jangjeon Math. Soc., **16**(1), pp. 91-100, 2013.

[12] Srivastava, H.M., Mishra, A.K. & Gochhayat, P., *Certain Subclasses of Analytic and Bi-univalent Functions*, Appl. Math. Lett., **23**, pp. 1188-1192, 2010.

[13] Gui, Y.C., Xu, Q.H., & Srivastava, H.M., *Coefficient Estimates for a Certain Subclass of Analytic and Bi-univalent Functions*, Appl. Math, Lett., **25**, pp. 990-994, 2012.

[14] Salagean, G.S., *Subclasses of Univalent Functions, Complex Analysis*, Fifth Romanian-Finnish Seminar, Bucharest, **1**, pp. 562-372, 1983.

[15] Doha, E.H., *The First and Second Kind Chebyshev Coefficients of the Moments of the General Order Derivative of an Infinitely Differentiable Function*, Int. J. Comput. Math., **51**, pp. 21-35, 1994.

[16] Mason, J.C., *Chebyshev Polynomials Approximations for the L-membrane Eignvalue Problem*, SIAM J. Appl. Math., **15**, pp. 172-186, 1967.

[17] Altinkaya, S. & Yalcin, S., *Coefficient Bounds for a Subclass of Bi-univalent Functions*, TWMS J. Pure Appl. Math., **6**(2), pp. 180-185, 2015.

[18] Altinkaya, S. & Yalcin, S., *On the Chebyshev Polynomial Coefficient Problem of Some Subclasses of Bi-univalent Functions*, Gulf J. Math., **5**(3), pp. 34-40, 2017.

[19] Altinkaya, S. and Yalcin, S., *Estimates on Coefficients of a General Subclass of Bi-univalent Functions Associated with Symmetric Q-derivative Operator by Means of the Chebyshev Polynomials*, Asia Pacific J. Math., **4**(2), pp. 90-99, 2017.

[20] Bulut, S., Magesh, N. & Abirami, C., *A Comprehensive Class of Analytic Bi-univalent Functions by Means of Chebyshev Polynomials*, J. Frac. Cal. Appl., **8**(2), pp. 32-39, 2017.

[21] Bulut, S., Magesh, N. & Balaji, V.K., *Initial Bounds for Analytic and Bi-univalent Functions by Means of Chebyshev Polynomials*, J. Classical Anal., **11**(1), pp. 83-89, 2017.

[22] Guney, H.O., Murugusundaramoorthy, G. & Vijaya, K., *Coefficient Bounds for Subclasses of Bi-univalent Functions Associated with the Chebyshev Polynomials*, J. Complex Anal., **2017**, pp. 1-7, 2017. DOI:10.1155/2017/4150210

[23] Caglar, M., *Chebyshev Polynomial Coefficient Bounds for a Subclass of Bi-univalent Functions*, C.R. Acad. Bulg. Sci., **72**(12), pp. 1608-1615, 2019.

[24] Caglar, M. & Deniz, E., *Initial Coefficients for a Subclass of Bi-univalent Functions Defined by Salagean Differential Operator*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **66**(1), pp. 85-91, 2016.