J. Math. Fund. Sci., Vol. 51, No. 3, 2019, 265-277

Donsker's Delta Functional of Stochastic Processes with Memory

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Abstract. A class of stochastic processes with memory within the framework of the Hida calculus was studied. It was proved that the Donsker delta functionals of the processes are Hida distributions. Furthermore, the probability density function of the processes and the chaos decomposition of the Donsker delta functional were derived. As an application, the existence of the renormalized local times in an arbitrary dimension of the Riemann-Liouville fractional Brownian motion as a white noise generalized function was proved.

Keywords: *Donsker delta functional; Hida calculus; local time; stochastic process with memory; white noise.*

1 Introduction

In many disciplines, such as physics, biology, complex systems, and financial mathematics, one often needs to study phenomena that contain uncertainty (randomness) as well as memory from the past, see for example [1-5]. Since Brownian motion is Markovian, it is not suitable for modeling stochastic systems with short- or long-range dependence. Fractional Brownian motion has become a popular tool in the modeling of random systems with memory. It has the properties of being Gaussian, non-Markovian, non-Martingale, statistically persistent, and self-similar, among others. For a comprehensive account on fractional Brownian motion the reader is referred to [3] and [6], and the references therein. Bernido and Carpio-Bernido [1] studied a general class of stochastic processes with memory beyond fractional Brownian motion. In particular, they evaluated the probability density function and discussed the corresponding modified diffusion equation for different types of memory behavior. It is mentioned in that paper that tools from the Hida calculus (white noise analysis) were used. The present paper offers a more detailed treatment of some results in [1] via the Hida calculus. Moreover, our results were obtained in a more general setting of d-dimensional Euclidean space. In addition, we derived the chaos decomposition of the Donsker delta functional. Application to the local times of Riemann-Liouville fractional Brownian motion is also discussed.

Received August 6th, 2018, Revised April 23rd, 2019, Accepted for publication May 20th, 2019.
 ISSN: 2337-5760, DOI: 10.5614/j.math.fund.sci.2019.51.3.5

The main object of this paper is the continuous time stochastic process $\mathrm{X} = (\mathrm{X}_\mathrm{t})_{\mathrm{t}\geq 0}$ of the form

$$X_t = x_0 + \int_0^t m(t-s)g(s)dB_s \tag{1}$$

where $m(t-s)$, g are measurable functions from $[0,\infty)$ to R, $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, and $x_0 \coloneqq X_0$ almost surely is the initial value of the process. This corresponds to the process x in Eq. (1) in [1]. The function $m(t-s)$ and g are called the memory function and the weight function of the process, respectively. Intuitively, they are used to modulate the Gaussian white noise, dB_s, which in turn affects the history of the process X. The stochastic integral in Eq. (1) is interpreted as the Wiener integral with respect to the Brownian motion. We require that $m(t-\cdot) \in L^2_{loc}([0,\infty))$, that is $\int_A |m(t-s)g(s)|\, ds < \infty$ for any compact set $A \subset [0,\infty)$ so that the Wiener integral is well-defined.

2 Basics of the Hida Calculus

In this section we recall some basic notions and results from the Hida calculus. For a thorough discussion see for example [7-9]. Let $(S'_d(R), C, \mu)$ be the R^d-valued white noise space, i.e. $S'_d(R)$ is the space of R^d-valued tempered distributions, C is the cylindrical σ-algebra in $S'_d(R)$, and the existence of the white noise measure μ is guaranteed by the Bochner-Minlos theorem via

$$\int_{S'_d(R)} \exp(i\langle\vec{\omega},\vec{\eta}\rangle)d\mu(\vec{\omega}) = \exp\left(-\tfrac{1}{2}|\vec{\eta}|_0^2\right)$$

for all R^d-valued Schwartz test functions $\vec{\eta} \in S_d(R)$. Here $|\cdot|_0$ denotes the usual norm in the real Hilbert space of all R^d-valued Lebesgue square-integrable functions $L^2_d(R)$. The dual pairing $\langle\cdot,\cdot\rangle$ on $S'_d(R) \times S_d(R)$ is considered as the bilinear extension of the inner product on $L^2_d(R)$. It is known that in the white noise analysis setting a version of the d-dimensional Brownian motion B is represented by the stochastic vector $(B_t)_{t\geq 0}$ with

$$B_t \coloneqq \left(\langle\cdot, 1_{[0,t)}\rangle, \cdots, \langle\cdot, 1_{[0,t)}\rangle\right)$$

where 1_A denotes the indicator function on a set A.

The Hilbert space $L^2(\mu) \coloneqq L^2(S'_d(R), C, \mu)$ is unitary isomorphic to the multiple tensor product of the Fock space of a symmetric square-integrable function:

$$L^2(\mu) \cong \left(\bigoplus_{k=0}^{\infty} \hat{L}^2\left(R^k, k!\, d^k x\right)\right)^{\otimes d}$$

Then, one chaos decomposition of any $F \in L^2(\mu)$ is obtained:

$$F(\omega_1, \cdots, \omega_d) = \sum_{(n_1,\cdots,n_d)\in N_0^d} \langle :\omega_1^{\otimes n_1}:\otimes \cdots \otimes :\omega_d^{\otimes n_d}: \vec{f}_{(n_1,\cdots,n_d)}\rangle \qquad (2)$$

The term $\vec{f}_{(n_1,\cdots,n_d)}$ is called the kernel functions of the n-th chaos and is an element of the Fock space. Here: $\omega^{\otimes n}$: denotes the n-th Wick power of $\omega \in S_1'(R)$. By introducing the symbols

$$\boldsymbol{n} \coloneqq (n_1, \dots, n_d) \in N_0^d, \quad n \coloneqq \sum_{j=1}^d n_j, \quad \boldsymbol{n}! \coloneqq \prod_{j=1}^d n_j!,$$

Eq. (2) can be simplified to

$$F(\vec{\omega}) = \sum_{n\in N_0^d} \langle :\vec{\omega}^{\otimes n} : \vec{f}_{\boldsymbol{n}}\rangle, \quad \vec{\omega} \in S_d'(R).$$

Next, we consider a Gel'fand triple around the Hilbert space $L^2(\mu)$, namely $(S) \subset L^2(\mu) \subset (S)'$.

Here (S) is called the space of Hida test functions and can be constructed by taking the intersection of a collection of Hilbert subspaces of $L^2(\mu)$. It is a nuclear countably Hilbert space and equipped with a projective limit topology. The Hida distribution space $(S)'$ is defined as a topological dual space of (S).

For $\vec{\eta} \in S_d(R)$ and the corresponding Wick exponential:

$$:\exp(\langle \cdot, \vec{\eta}\ \rangle) :\coloneqq \exp\left(\langle \cdot, \vec{\eta}\rangle - \frac{1}{2}|\vec{\eta}|_0^2\right),$$

we define the S-transform of $\Phi \in (S)'$ by

$$(S\Phi)(\vec{\eta}) \coloneqq \langle\langle \Phi, :\exp(\langle \cdot, \vec{\eta}\ \rangle): \rangle\rangle, \text{ for all } \vec{\eta} \in S_d(R) \qquad (3)$$

Here $\langle\langle \cdot,\cdot \rangle\rangle$ denotes the dual pairing of $(S)'$ and (S). The multilinear expansion of Eq. (3) can be extended to the chaos decomposition of $\Phi \in (S)'$ with distribution-valued kernels $F_{\boldsymbol{n}}$, such that $\langle\langle \Phi, \varphi \rangle\rangle = \sum_{\boldsymbol{n}\in N_0^d} \boldsymbol{n}! \langle F_{\boldsymbol{n}}, \varphi_{\boldsymbol{n}}\rangle$ for every Hida test function $\varphi \in (S)$ with kernel functions $\varphi_{\boldsymbol{n}}$. Now we give an integration theorem of a family of Hida distributions.

Theorem 2.1 [10] Let (Ω, F, ν) be a measure space and $\xi \mapsto \Phi_\xi$ be a function from Ω to $(S)'$. If

(1) The function $\xi \mapsto (S\Phi_\xi)(\vec{\eta})$ is measurable for all $\vec{\eta} \in S_d(R)$, and

(2) There exist $C_1(\xi) \in L^1(\Omega, \nu)$, $C_2(\xi) \in L^\infty(\Omega, \nu)$, and a continuous norm $\|\cdot\|$ on $S_d(R)$ such that $|(S\Phi_\xi)(\overline{z\vec{\eta}})\ | \le C_1(\xi) \exp(C_2(\xi)|z|^2\|\vec{\eta}\|^2)$, for all $\vec{\eta} \in (S)$ and complex number z, then Φ_ξ is Bochner integrable with respect to some Hilbertian norm topologizing $(S)'$. Moreover, $\int_\Omega \Phi_\xi d\nu(\xi) \in (S)'$, and for any $\vec{\eta} \in (S)$ it holds

$$S\left(\int_{\Omega} \Phi_{\xi} d\nu(\xi)\right)(\vec{\eta}) = \int_{\Omega} (\mathrm{S}\Phi_{\xi})(\vec{\eta}) d\nu(\xi).$$

3 Main Results

Let the index set I be either a compact interval $[0, T]$, $0 < T < \infty$ or the nonnegative half line $[0, \infty)$. In the frame of the Hida calculus the d-dimensional stochastic process with memory X starting at $\vec{x}_0 = (x_0^1, \cdots, x_0^d) \in R^d$ is represented by the stochastic process $(X_t)_{t\in I}$ with

$$\begin{aligned} X_t &:= \vec{x}_0 + \langle \cdot, \vec{m}(t - \cdot)\vec{g}\ \rangle \\ &= \left(x_0^1 + \langle \cdot, m_1(t - \cdot)g_1\rangle, \cdots, x_0^d + \langle m_d(t - \cdot)g_d\rangle\right) \end{aligned}$$

where $m_1(t - \cdot), \ldots, m_d(t - \cdot)$, $g_1, \ldots, g_d$ are measurable functions from I into R such that $m_j(t - \cdot)g \in L^2_{loc}(I)$ for all $j = 1, \ldots, d$. It can be shown that $(X_t)_{t\in I}$ is a Gaussian process with mean vector $\vec{x}_0$ and covariance matrix $M = (m_{ij})_{i,j=1,\ldots,d}$ with

$$m_{ij} = \left(m_i(t - \cdot)g_i, m_j(t - \cdot)g_j\right)_{L^2(I)}$$

In the following, by $\left| m_j(t - \cdot)g_j \right|_0$ we mean $\left| m_j(t - \cdot)g_j \right|_{L^2(I)}$. Henceforth, we will work with a continuous version of X, which exists due to the Kolmogorov-Chentsov theorem.

In several applications, such as in the area of probability theory, quantum mechanics, and polymer physics, one needs to fix a stochastic process at some spatial point, see for example [11-13]. This motivates us to study the Donsker delta functional of the process X. Informally, it is defined as the composition of the Dirac delta function $\delta_{\vec{c}} \in S'(R^d)$ with a d-dimensional stochastic process with memory $(X_t)_{t\in I}$, that is $\delta_{\vec{c}}(X_t)$. We will make sense of this object as a Hida distribution via the Fourier-transform representation of the Dirac delta function.

Theorem 3.1 Let $X = (X_t)_{t\in I}$ be a d-dimensional stochastic process with memory and $\vec{c}_0 = (c_0, \cdots, c_d) \in R^d$. The Donsker delta functional

$$\delta_{\vec{c}}(X_t) := \left(\tfrac{1}{2}\right)^d \int_{R^d} \exp\left(i\vec{\lambda}(\vec{x}_0 + \langle \cdot, \vec{m}(t - \cdot)\vec{g}\rangle - \vec{c})\right) d\vec{\lambda}$$

is a Hida distribution. Moreover, its S-transform is of the form

$$S\delta_{\vec{c}}(X_t)(\vec{\eta})$$

$$= \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi \left| m_j(t-\cdot)g_j \right|_0^2}} \right) \exp\left(-\frac{1}{2} \sum_{j=1}^{d} \frac{\left(\langle \eta_j, m_j(t-\cdot)g_j \rangle + x_0^j - c_j \right)^2}{\left| m_j(t-\cdot)g_j \right|_0^2} \right)$$

for all $\vec{\eta} \in S_d(R)$.

Proof. We will show that $\int_{R^d} \exp\left(i\vec{\lambda}(\vec{x}_0 + \langle \cdot, \vec{m}(t-\cdot)\vec{g}\rangle - \vec{c})\right) d\vec{\lambda} \in (S)'$. For simplicity we use the following notations: $\vec{\lambda}\vec{m}(t-\cdot)\vec{g} \coloneqq \sum_{j=1}^{d} \lambda_j m_j(t-\cdot)g_j$, $\vec{\lambda}\langle \vec{\eta}, \vec{m}(t-\cdot)\vec{g}\rangle = \langle \vec{\eta}, \vec{\lambda}\,\vec{m}(t-\cdot)\vec{g}\rangle \coloneqq \sum_{j=1}^{d} \lambda_j \eta_j m_j(t-\cdot)g_j$, and $\left|\vec{\lambda}\right|^2 |\vec{m}(t-\cdot)|_0^2 \coloneqq \sum_{j=1}^{d} \lambda_j^2 \left| m_j(t-\cdot)g_j \right|_0^2$. For $\vec{\lambda} \in R^d$ let us define a mapping $F_{\vec{\lambda}}: S_d(R) \to R$ by

$$F_{\vec{\lambda}}(\vec{\eta}) \coloneqq S \exp\left(i\vec{\lambda}(\vec{x}_0 + \langle \cdot, \vec{m}(t-\cdot)\vec{g}\rangle - \vec{c})\right)(\vec{\eta}).$$

Then,

$$\begin{aligned} &F_{\vec{\lambda}}(\vec{\eta}) \\ &= \langle\langle \exp\left(i\vec{\lambda}(\vec{x}_0 + \langle \cdot, \vec{m}(t-\cdot)\vec{g}\rangle - \vec{c})\right), :\exp(\langle \cdot, \vec{\eta}\rangle):\rangle\rangle \\ &= \exp\left(-\tfrac{1}{2}|\vec{\eta}|_0^2\right) \int_{S_d'(R)} \exp\left(i\vec{\lambda}(\vec{x}_0 - \vec{c})\right) \exp\left(\langle \vec{\omega}, i\vec{\lambda}\vec{m}(t-\cdot)\vec{g} + \vec{\eta}\rangle\right) d\mu(\vec{\omega}) \\ &= \exp\left(-\tfrac{1}{2}|\vec{\eta}|_0^2\right) \exp\left(i\vec{\lambda}(\vec{x}_0 - \vec{c})\right) \exp\left(\tfrac{1}{2}\left| i\vec{\lambda}\vec{m}(t-\cdot)\vec{g} + \vec{\eta} \right|_0^2\right) \\ &= \exp\left(-\tfrac{1}{2}\left|\vec{\lambda}\right|^2 |\vec{m}(t-\cdot)\vec{g}|_0^2\right) \exp\left(i\vec{\lambda}(\langle \vec{\eta}, \vec{m}(t-\cdot)\vec{g}\rangle + \vec{x}_0 - \vec{c})\right). \end{aligned}$$

The function $\vec{\lambda} \mapsto F_{\vec{\lambda}}(\vec{\eta})$ is measurable with respect to the Lebesgue measure $d\vec{\lambda}$. Moreover, for $\vec{\eta} \in S_d(R)$ and $z \in C$ we have

$$\begin{aligned} \left|F_{\vec{\lambda}}(z\vec{\eta})\right| &\le \exp\left(-\tfrac{1}{2}\left|\vec{\lambda}\right|_0^2 |\vec{m}(t-\cdot)\vec{g}|_0^2\right) \exp\left(\left|\vec{\lambda}\right| |z| |\vec{\eta}, \vec{m}(t-\cdot)\vec{g}|\right) \\ &\le \exp\left(-\tfrac{1}{4}\left|\vec{\lambda}\right|_0^2 |\vec{m}(t-\cdot)\vec{g}|_0^2\right) \exp\left(\frac{|z|^2 |\vec{\eta}, \vec{m}(t-\cdot)\vec{g}|^2}{|\vec{m}(t-\cdot)\vec{g}|_0^2}\right) \\ &\le \exp\left(-\tfrac{1}{4}\left|\vec{\lambda}\right|_0^2 |\vec{m}(t-\cdot)\vec{g}|_0^2\right) \exp\left(|z|^2 |\vec{\eta}|_0^2\right) \end{aligned}$$

The first factor is an integrable function of $\vec{\lambda}$ and the second factor is a constant with respect to $\vec{\lambda}$. Theorem 2.1 delivers that $\delta_{\vec{c}}(X_t) \in (S)'$. To obtain the S-transform we integrate $F_{\vec{\lambda}}(\vec{\eta})$ over R^d and use the Gaussian integral formula

$$S\delta_{\vec{c}}(X_t)(\vec{\eta})$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{R^d} S \exp\left(i\vec{\lambda}(\vec{x}_0 + \langle\cdot, \vec{m}(t-\cdot)\vec{g}\rangle - \vec{c}\,)\right)(\vec{\eta})\, d\vec{\lambda}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{\mathrm{R^d}} \exp\left(-\tfrac{1}{2}\left|\vec{\lambda}\right|^2 |\vec{m}(t-\cdot)\vec{g}|_0^2\right) \exp\left(i\vec{\lambda}(\langle\vec{\eta}, \vec{m}(t-\cdot)\vec{g}\rangle + \vec{x}_0 - \vec{c})\right) d\vec{\lambda}$$

$$= \prod_{j=1}^{d} \frac{1}{2\pi} \int_R \exp\left(-\tfrac{1}{2}\left|m_j(t-\cdot)g_j\right|_0^2 \lambda_j^2 \right.$$
$$\left. + i\lambda_j\left(\langle\eta_j, m_j(t-\cdot)g_j\rangle + x_0^j - c_j\right)\right) d\lambda_j$$

$$= \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}\left|m_j(t-\cdot)g_j\right|_0}\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{d} \frac{\left((\langle\eta_j, m_j(t-\cdot)g_j\rangle + x_0^j - c_j)\right)^2}{\left|m_j(t-\cdot)g_j\right|_0^2}\right)$$

This finishes the proof.

Corollary 3.2. The transition probability function of a particle in a system described by a d-dimensional stochastic process with memory $X = (X_t)_{t\in I}$ to move from $\vec{x}_0 \in R^d$ to an endpoint $\vec{x}_T \in R^d$ at a later time $t = T$ is given by

$$p(\vec{x}_0, 0; \vec{x}_T, T)$$
$$= \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi\left|m_j(t-\cdot)g_j\right|_0^2}}\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{d} \left(\frac{x_0^j - x_T^j}{\left|m_j(T-\cdot)g_j\right|_0}\right)^2\right)$$

Proof. The generalized expectation of the Donsker delta functional can be obtained from Theorem 3.1 by evaluating the value of the S-transform at $\vec{\eta} = \vec{0}$: $E_\mu\left(\delta_{\vec{c}}(X_t)\right) = S\delta_{\vec{c}}(X_t)(\vec{0})$

$$= \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi\left|m_j(t-\cdot)g_j\right|_0^2}}\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{d} \frac{\left(x_0^j - x_T^j\right)^2}{\left|m_j(T-\cdot)g_j\right|_0^2}\right).$$

The transition probability function $p(\vec{x}_0, 0; \vec{x}_T, T)$ now follows immediately from the last expression by fixing the endtime $t = T$ with endpoint $\vec{c} = \vec{x}_T$.

Note that in the case of a one-dimensional system and $I = [0, T]$ we recover the object (8) in [1].

Example 3.3.

1. d-dimensional Riemann-Liouville fractional Brownian motion with Hurst index $H \in \left(\frac{1}{2}, 1\right)$. In this case, $m_j(t-s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)}$ and $g_j(s) = 1$, which gives
$$\left|m_j(T-\cdot)g_j\right|^2_{L^2[0,T]} = \int_0^T \left(\frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)}\right)^2 ds = \frac{T^{2H}}{2H\Gamma\left(H+\frac{1}{2}\right)^2},$$
where Γ is the usual gamma function. The probability density function is
$$p(\vec{x}_0, 0; \vec{x}_T, T) = \left(\frac{H\Gamma\left(H+\frac{1}{2}\right)^2}{\pi T^{2H}}\right)^{\frac{d}{2}} \exp\left(-\frac{H\Gamma\left(H+\frac{1}{2}\right)^2}{T^{2H}}|\vec{x}_0 - \vec{x}_T|^2\right),$$
where $|\vec{x}_0 - \vec{x}_T|$ is the Euclidean distance in R^d between $\vec{x}_0$ and $\vec{x}_T$.

2. d-dimensional exponentially-modified Brownian motion. Here, $m_j(t-s) = (t-s)^{\frac{\mu-1}{2}}, \Re(\mu) > 0$ and $g_j(s) = \frac{1}{s^{\frac{\mu+1}{2}}} \exp\left(-\frac{\beta}{2s}\right)$. Then, by using formula 3.471(3) in [14] we have
$$\left|m_j(T-\cdot)g_j\right|^2_{L^2[0,T]} = \int_0^T \frac{(T-s)^{\mu-1}\exp\left(-\frac{\beta}{s}\right)}{s^{\mu+1}} ds = \beta^{-\mu}T^{\mu-1}\Gamma(\mu).$$
The probability density function is
$$p(\vec{x}_0, 0; \vec{x}_T, T) = \left(\frac{\beta^{\mu}\exp\left(\frac{\beta}{T}\right)}{2\pi T^{\mu-1}\Gamma(\mu)}\right)^{\frac{d}{2}} \exp\left(-\frac{\beta^{\mu}\exp\left(\frac{\beta}{T}\right)}{2T^{\mu-1}\Gamma(\mu)}|\vec{x}_0 - \vec{x}_T|^2\right).$$

For a more complete example of a stochastic processes with memory (with various memory and weight functions) we refer to [1].

We can also deduce the chaos decomposition of the Donsker delta functional.

Corollary 3.4. It holds that
$$\delta_{\vec{c}}(X_t) = \sum_{\boldsymbol{n}\in N_0^d} \langle :\vec{\omega}^{\otimes \boldsymbol{n}}:, F_{\boldsymbol{n}}\rangle,$$
where the $\boldsymbol{n}$-kernel, $\boldsymbol{n} \in N_0^d$, is given by
$$F_{\boldsymbol{n}}(u_1, \cdots, u_n)$$

$$= \left(\prod_{j=1}^{d} p_{|m_j(t-\cdot)g_j|_0^2}\left(x_0^j - c_j\right)\right)\frac{1}{\boldsymbol{n}!}(-1)^n$$

$$\times\left(\prod_{i=1}^{n}\left(\prod_{j=1}^{d} H_{n_j}\left(\frac{x_0^j - c_j}{\sqrt{2\left|m_j(t-\cdot)g_j\right|_0^2}}\right)\frac{m_j(t-\cdot)g_j}{\sqrt{2\left|m_j(t-\cdot)g_j\right|_0^2}}\right)(u_i)\right).$$

Here $p_{\sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$ denotes the heat kernel and H_n denotes the Hermite polynomial of order n defined by the Rodrigues formula

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}\exp(-x^2).$$

Proof. From Theorem 3.1 and the generating function formula for the Hermite polynomial we have

$S\delta_{\vec{c}}(X_t)(\vec{\eta})$

$$= \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\left(\prod_{j=1}^{d}\frac{1}{\left|m_j(t-\cdot)g_j\right|_0}\right)\exp\left(-\frac{1}{2}\sum_{j=1}^{d}\frac{\left(\left(\langle\eta_j, m_j(t-\cdot)g_j\rangle + x_0^j - c_j\right)\right)^2}{\left|m_j(t-\cdot)g_j\right|_0^2}\right)$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{d}{2}}\left(\prod_{j=1}^{d}\frac{1}{\left|m_j(t-\cdot)g_j\right|_0}\exp\left(-\frac{1}{2}\sum_{j=1}^{d}\frac{\left(x_0^j - c_j\right)^2}{\left|m_j(t-\cdot)g_j\right|_0^2}\right)\right)$$

$$\times\prod_{j=1}^{d}\exp\left(-\frac{\langle\eta_j, m_j(t-\cdot)g_j\rangle^2}{2\left|m_j(t-\cdot)g_j\right|_0^2} - \frac{\left(x_0^j - c_j\right)^2\langle\eta_j, m_j(t-\cdot)g_j\rangle}{\left|m_j(t-\cdot)g_j\right|_0^2}\right)$$

$$= \left(\prod_{j=1}^{d}\frac{1}{\sqrt{2\pi\left|m_j(t-\cdot)g_j\right|_0^2}}\exp\left(-\frac{1}{2}\frac{\left(x_0^j - c_j\right)^2}{\left|m_j(t-\cdot)g_j\right|_0^2}\right)\right)$$

$$\times\prod_{j=1}^{d}\left(\sum_{n=1}^{\infty}\frac{1}{n!}H_n\left(-\frac{x_0^j - c_j}{\sqrt{2\pi\left|m_j(t-\cdot)g_j\right|_0^2}}\right)\left(\frac{\langle\eta_j, m_j(t-\cdot)g_j\rangle}{\sqrt{2\pi\left|m_j(t-\cdot)g_j\right|_0^2}}\right)^n\right)$$

$$= \left(\prod_{j=1}^{d} p_{|m_j(t-\cdot)g_j|_0^2} \left(x_0^j - c_j \right) \right)$$

$$\times \sum_{n=0}^{\infty} (-1)^n \sum_{n_1+\cdots+n_d=n} \frac{1}{\boldsymbol{n}!} \prod_{j=1}^{d} H_{n_j} \left(\frac{x_0^j - c_j}{\sqrt{2\pi \left| m_j(t-\cdot) g_j \right|_0^2}} \right) \langle \eta_j, \frac{m_j(t-\cdot) g_j}{\sqrt{2\pi \left| m_j(t-\cdot) g_j \right|_0^2}} \rangle^{n_j}$$

By comparing with the general form of the chaos decomposition we arrive at the desired expression for the kernel functions.

The local times of a d-dimensional stochastic process with memory X at point $c \in R^d$ over a compact interval $[0, T], 0 < T < \infty$ is defined informally via the Tanaka formula (see e.g. Section 8.6 of [15]):

$$\int_0^T \delta_{\vec{c}}(X_t)\, dt \tag{4}$$

Expression (4) is used heuristically to measure the amount of time the trajectory of X spends at a given point $\vec{c} \in R^d$ within the time horizon $[0, T]$. A priori, Expression (4) is mathematically meaningless. From the point of view of the Hida calculus, local time is nothing else but integration of the Donsker delta functional over the time horizon. The idea of studying local times using the Hida calculus was initiated, at least, in the work of Watanabe [16]. We will prove that the integral of the truncated version of the Donsker delta functional is well-defined as a Hida distribution. However, it is not possible to establish the result for the general stochastic process with memory since the computation later on very much depends on the explicit expression of the variance of the process. In other words, the study of local time has to be done for specific memory and weight functions. In the following we will prove an existence result for the local times at the origin of a d-dimensional Riemann-Liouville fractional Brownian motion $(X_t)_{t \in [0,T]}$ with Hurst index $H \in \left(\frac{1}{2}, 1\right)$. Recall that the process is given by

$$X_t = \vec{x}_0 + \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H + \frac{1}{2}\right)}\, dB_s\,,$$

or in the white-noise representation:

$$X_t = \vec{x}_0 + \langle \cdot, \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} \rangle = \left(x_0^1 + \langle \cdot, \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} \rangle, \cdots, x_0^d + \langle \cdot, \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} \rangle \right).$$

From the last expression it is understood that for independent d-tuples of white noise $\vec{\omega} = (\omega_1, \cdots, \omega_d) \in S'_d(R)$ the following holds:

$$X_t(\vec{\omega}) = \left(x_0^1 + \langle \omega_1, \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} \rangle, \cdots, x_0^d + \langle \omega_d, \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} \rangle \right).$$

Now we define the truncated Donsker's delta functional $\delta_{\vec{c}}^{(N)}(X_t) \in (S)'$ via its S-transform. For any $\vec{\eta} \in S_d(R)$

$$S\delta_{\vec{c}}^{(N)}(X_t)(\vec{\eta}) := \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi \left| m_j(t - \cdot) g_j \right|_0^2}} \right)$$

$$\times \; \exp_{(\mathrm{N})} \left(-\frac{1}{2} \sum_{j=1}^{d} \frac{\left(\left(\langle \eta_j, m_j(t - \cdot) g_j \rangle + x_0^j - c_j \right) \right)^2}{\left| m_j(t - \cdot) g_j \right|_0^2} \right),$$

where the truncated exponential function $\exp_{(\mathrm{N})}$ is given by

$$\exp_{(\mathrm{N})}(x) := \sum_{n=N}^{\infty} \frac{x^n}{n!}.$$

Since the S-transform characterizes the Hida distribution, the above definition is well-defined.

Theorem 3.5. Let $X = (X_t)_{t \in [0,T]}$ be a d-dimensional Riemann-Liouville fractional Brownian motion with Hurst index $H \in \left(\frac{1}{2}, 1\right)$ starting at $\vec{0} \in R^d$. For any integers $d \geq 1$ and $N \geq 0$ satisfying $dH + 2N(H-1) < 1$ the truncated local time at the origin

$$L_X^{(N)}(T) := \int_0^T \delta_{\vec{0}}^{(N)}(X_t) \, dt$$

is a Hida distribution.

Proof. From the definition of the truncated Donsker delta functional we have that $S\delta_{\vec{0}}^{(N)}(X_t)(\vec{\eta})$ is a measurable function of t for every $\vec{\eta} \in S_d(R)$. Moreover, for every $z \in C$ and $\vec{\eta} \in S_d(R)$, using Theorem 3.1, we obtain

$$S\delta_{\vec{0}}^{(N)}(X_t)(z\vec{\eta})$$

$$\leq \left(\prod_{j=1}^{d} \frac{1}{\sqrt{2\pi |m_j(t-\cdot)g_j|_0^2}}\right) \exp_{(N)}\left(\frac{1}{2}|z|^2 \sum_{j=1}^{d} \frac{\langle \eta_j, m_j(t-\cdot)g_j\rangle^2}{|m_j(t-\cdot)g_j|_0^2}\right)$$

$$\leq \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left(\prod_{j=1}^{d} \frac{\sqrt{2H}\Gamma\left(H+\frac{1}{2}\right)}{t^H}\right)$$

$$\times \exp_{(N)}\left(|z|^2 \frac{2H\Gamma\left(H+\frac{1}{2}\right)^2}{t^{2H}} \sum_{j=1}^{d} \left|\int_0^t \eta_j(s) \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)} ds\right|^2\right)$$

$$\leq \left(\frac{H\Gamma\left(H+\frac{1}{2}\right)^2}{\pi}\right)^{\frac{d}{2}} \frac{1}{t^{dH}} \exp_{(N)}\left(\frac{2H\mathrm{T}^{2\mathrm{H}-1}|z|^2}{t^{2H}} \sum_{j=1}^{d} \left(\int_0^t \sup_{s\in\mathrm{R}} |\eta_j(s)|\right)^2\right)$$

$$\leq \left(\frac{H\Gamma\left(H+\frac{1}{2}\right)^2}{\pi}\right)^{\frac{d}{2}} \frac{1}{t^{dH}} \exp_{(N)}\left(2H\mathrm{T}^{2\mathrm{H}-1} t^{2-2H} |z|^2 \|\vec{\eta}\|_*^2\right)$$

$$\leq \left(\frac{H\Gamma\left(H+\frac{1}{2}\right)^2}{\pi}\right)^{\frac{d}{2}} t^{2N(1-H)-dH} \exp\left(2H\mathrm{T}^{2\mathrm{H}-1}|z|^2 \|\vec{\eta}\|_*^2\right),$$

where $\|\cdot\|_*$ is a continuous norm on $S_d(R)$ defined by

$$\|\vec{\eta}\|_*^2 := \sum_{j=1}^{d} \left(\sup_{s\in\mathrm{R}} |\eta_j(s)|\right)^2.$$

Note that $t^{2N(1-H)-dH}$ is dt −integrable on $[0,T]$ if and only if $2N(1-H) - dH > -1$. Finally, the conclusion follows from Theorem 2.1.

Theorem 3.5 asserts that for one-dimensional Riemann-Liouville fractional Brownian motion the local time at zero is well-defined as a Hida distribution. For $d \geq 2$ local times at zero become well-defined only after renormalization, i.e. by omission of the divergent terms that occur in the low-order terms in the truncated Donsker delta functional. For example, for $d = 2$ or $d = 3$ it is sufficient to take $N = 1$, which means we only need to throw away the first lower term to have $L_X^{(N)}(T) \in (S)'$. The effectiveness of the renormalization method comes from the fact that the kernels of increasing order in the chaos decomposition are less singular in the sense of Lebesgue integrable functions.

4 Conclusion

This research developed a Hida calculus approach to a class of stochastic processes with memory in a general setting of d-dimensional space. In particular, we considered Donsker's delta functional of such processes together with the probability density functions and their chaos decompositions. We applied our results to the study of local time at zero of a d-dimensional Riemann-Liouville fractional Brownian motion.

Acknowledgments

The author would like to thank the anonymous reviewers for their comments which significantly improve the quality of this paper. This work was supported by a research grant from Sanata Dharma University with contract number nr. 036/Penel./LPPM-USD/V/2019.

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